The syntomic regulator for K_1 of surfaces

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Plan

- The problem and results over $\mathbb{C}$
- Double and triple indices
- Statement of the main result
- fp-cohomology and pushforwards in syntomic cohomology
- syntomic cohomology with compact supports
- Sketch of the proof
- Applications

The elements we consider

S/L - smooth projective surface over a field L

$H^1(S, \mathcal{K}_2)$ = H^1 of

$$K_2(L(S)) \to \oplus_{\mathrm{cod}\, C=1} L(C)^\times \to \oplus_x \mathbb{Z}$$

$$\theta = \sum (Z_i, f_i) \in H^1(S, \mathcal{K}_2)$$

$$H^1(S, \mathcal{K}_2) \otimes \mathbb{Q} \cong H^3_{\mathcal{M}}(S, \mathbb{Q}(2))$$

The regulator

$\mathrm{reg}_{\text{ét}}(\theta) \in H^1(L, H^2_{\text{ét}}(\bar{S}, \mathbb{Q}_p(2)))$

Assume: $[L : \mathbb{Q}_p] < \infty$

Bloch and Kato

$$\exp : H^{2i-j-1}_{\mathrm{dR}}(S/L)/F^i \to H^1(L, H^{2i-j-1}_{\text{ét}}(\bar{S}, \mathbb{Q}_p(i)))$$

$i = 2$, $j = 1$

$$\exp : H^2_{\mathrm{dR}}(S/L)/F^2 \to H^1(L, H^2_{\text{ét}}(\bar{S}, \mathbb{Q}_p(2)))$$

Goal

1. find $\mathrm{reg}_{\mathrm{syn}}(\theta) \in H^2_{\mathrm{dR}}(S/L)/F^2$ s.t. $\exp(\mathrm{reg}_{\mathrm{syn}}(\theta)) = \mathrm{reg}_{\text{ét}}(\theta)$
2. describe explicitly
   $\mathrm{reg}_{\mathrm{syn}}(\theta) \in H^2_{\mathrm{dR}}(S/L)/F^2 = \mathrm{Hom}(F^1 H^2_{\mathrm{dR}}(S/L), L)$

Goal 1 $\leftarrow$ integrality assumption
$\mathcal{S}$ - smooth integral model of S

$$\begin{array}{ccc} K_j(\mathcal{S}) & \longrightarrow & K_j(S) \\ \downarrow{\scriptstyle \mathrm{reg}_{\mathrm{syn}}} & & \downarrow{\scriptstyle \mathrm{reg}_{\text{ét}}} \\ H^{2i-j-1}_{\mathrm{dR}}(S/L)/F^i & \xrightarrow{\exp} & H^1(L, H^{2i-j-1}_{\text{ét}}(\bar{S}, \mathbb{Q}_p(i))) \end{array}$$

Goal 1 is achieved if θ comes from $\mathcal{S}$
Goal 2 $\leftarrow$ Strong integrality: each Z_i and f_i are integral

Classical case

$L = \mathbb{C}$
$\mathrm{reg}(\theta) \in H^3_{\mathcal{D}}(S, \mathbb{R}(2))$

Theorem (Beilinson)

$\omega \in F^1 H^2_{\mathrm{dR}}(S/\mathbb{C})$.

$$\mathrm{reg}(\theta)(\omega) = \frac{1}{2\pi\sqrt{-1}} \sum \int_{Z_i - Z_i^{\mathrm{sing}}} \omega \log |f_i| .$$

Coleman integration

$\log$ - branch of the p-adic logarithm
C/L - proper curve

Theorem

$\omega \in \Omega^1(L(C)) \Rightarrow F_\omega : C(\bar{L}) \to \bar{L}$ *(depends on the choice of* $\log$*) unique up to constant.*

Remark

Coleman integrals exist in higher dimensions

Local integration:
$A_{\log} = L((z))[\log(z)]$
$0 \to L \to A_{\log} \xrightarrow{d} A_{\log} dz \to 0$ - integration by parts
$\omega \in L((z))dz \subset A_{\log} dz \Rightarrow F_\omega \in A_{\log}$ up to constant

Double indices

$\omega, \eta \in L((z))dz$.
want a notion of $\mathrm{Res}_0 F_\omega \eta$

Problem

$\omega = a_0 dz/z + \cdots \Rightarrow F_\omega = a_0 \log(z) + \cdots$, $\mathrm{Res}_0 \log(z) dz/z = ?$.

$\mathrm{Res}_0 \omega = 0 \Rightarrow$ no problem
$\mathrm{Res}_0 \eta = 0 \Rightarrow$ no problem: define as $-\mathrm{Res}_0 F_\eta \omega$)

General solution

double index $\langle F_\omega, F_\eta \rangle$ depending on both F_ω and F_η, bilinear, antisymmetric and equal to above when defined.

Key idea: $\langle \log(z), \log(z) \rangle = 0$.
$\langle F_\omega + C, F_\eta \rangle = \langle F_\omega, F_\eta \rangle + C \,\mathrm{Res}_0 \eta$

Global index

$\omega, \eta \in \Omega^1(L(C))$
$\sum_{x \in C} \langle F_\omega, F_\eta \rangle_x := \langle F_\omega, F_\eta \rangle_{gl}$
F_η, F_ω are Coleman integrals.

Interesting properties

- depends only on ω, η
- $= \Psi(\omega) \cup \Psi(\eta)$
  where $\Psi : \Omega^1(L(C)) \to H^1_{\mathrm{dR}}(C/L)$

Triple index

Data $F, G, H \in A_{\log}, \; dF, dG, dH \in L((z))dz$

We Want

Something like $\mathrm{Res}_0(FGdH) = ?$

Obvious easy case: $G \in L((z)) \Rightarrow \langle F, \int GdH \rangle$
So: should know $\int GdH$

Auxiliary data

$\int RdS \in A_{\log}, \; R \neq S \in \{F, G, H\}$
s.t. $\int RdS + \int SdR = RS$

Theorem

Exists and unique $(F, G, H) +$ *auxiliary data* $\to \langle F, G; H \rangle$ *which is:*

- *trilinear*
- *symmetric in* F, G
- *Triple identity* $\langle F, G; H \rangle + \langle F, H; G \rangle + \langle H, G; F \rangle = 0$
- *Reduction to double index* $\langle F, G; H \rangle = \langle F, \int G dH \rangle$ *when* $G \in L((z))$

Global index

$\langle F, G; H \rangle_{\mathrm{gl}} = \sum_{x \in C} \langle F, G; H \rangle_x$

independent of auxiliary data (chosen as Coleman integrals)

Main theorem setup

g_i : $X_i \to S$ - Normalizations of Z_i

Integrality assumption

- g_i extend to smooth integral models $g_i : \mathcal{X}_i \to \mathcal{S}$
- $\mathrm{div}(f_i)$ do not contain the special fiber

ω $\in F^1 H^1_{\mathrm{dR}}(S/L)$

η $\in H^1_{\mathrm{dR}}(S/L)$ represented by form of the second kind η on S.

μ $= \omega \cup [\eta] \in F^1 H^2_{\mathrm{dR}}(S/L)$

F_ω, F_η Coleman integrals of ω, η respectively

Statement of the main theorem

Theorem (*)

$$\mathrm{reg}_{\mathrm{syn}}(\theta)(\mu) = \sum_i \langle g_i^* F_\eta, \log(f_i); g_i^* F_\omega \rangle_{\mathrm{gl}, X_i}$$

Remark

(*) has to do with compatibility between pushforwards in syntomic and motivic cohomology

Syntomic cohomology

L - a p-adic field

$\mathcal{X}/\mathcal{O}_L$ - smooth

X - generic fiber

$\mathcal{X}_s$ - special fiber

rigid and de Rham complexes

$\mathbb{R}\Gamma_{\mathrm{dR}}(X/L)$ de Rham complex computing $H_{\mathrm{dR}}(X/L)$ with a filtration F^i

$\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L)$ rigid complex computing $H_{\mathrm{rig}}(\mathcal{X}_s/L)$ with a Frobenius ϕ of degree q

Syntomic cohomology

Definition

The modified syntomic complex of $\mathcal{X}$ is

$$\mathbb{R}\Gamma_{\mathrm{ms}}(\mathcal{X}, n) := \varinjlim_{j} MF(F^n \mathbb{R}\Gamma_{\mathrm{dR}}(X/L) \xrightarrow{1-(\phi/q^n)^j} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L))$$

Remark

If $\mathcal{X}$ proper and $2n \neq i-1$
then $1-(\phi/q^n)^j$ is invertible on $H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/L)$ hence
$H^{i-1}_{\mathrm{dR}}(X/L)/F^n \hookrightarrow H^i_{\mathrm{ms}}(\mathcal{X}, n) \twoheadrightarrow F^n H^i_{\mathrm{dR}}(X/L) \cap H^i_{\mathrm{rig}}(\mathcal{X}/L)^{\phi^j = q^{nj}}$

The Syntomic regulator

$$\mathrm{reg} : H^i_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(n)) \to H^i_{\mathrm{syn}}(\mathcal{X}, n)$$

$$H^{i-1}_{\mathrm{dR}}(X/L)/F^n \hookrightarrow H^i_{\mathrm{ms}}(\mathcal{X}, n) \twoheadrightarrow F^n H^i_{\mathrm{dR}}(X/L) \cap H^i_{\mathrm{rig}}(\mathcal{X}/L)^{\phi^j = q^{nj}}$$

Fact

For $\mathcal{X}$ proper and $i \neq 2n$, $H^i_{\mathrm{ms}}(\mathcal{X}, n) \cong H^{i-1}_{\mathrm{dR}}(X/L)/F^n$

Theorem (Niziol)

$$\begin{array}{ccc} & H^i_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(n)) & \longrightarrow & H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \\ & \downarrow & & \downarrow \\ H^i(\mathcal{X}, n) = & H^{i-1}_{\mathrm{dR}}(X/L)/F^n & \xrightarrow{\exp} & H^1(L, H^{2i-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(i))) \end{array}$$

No Poincare duality

$$H^{i-1}_{\mathrm{dR}}(X/L)/F^n \hookrightarrow H^i_{\mathrm{ms}}(\mathcal{X}, n) \twoheadrightarrow F^n H^i_{\mathrm{dR}}(X/L) \cap H^i_{\mathrm{rig}}(\mathcal{X}/L)^{\phi^j = q^{nj}}$$

Problem

The sequence is non-symmetric $\Rightarrow$
No Poincare duality

Solution

Replace $1 - (\phi/q^n)^j$ *by more general polynomials*

fp cohomology

Consider polynomials $P(t)$ "of weight $2i$"

$P|Q \Rightarrow$

$$\begin{array}{ccc} F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/L) & \xrightarrow{P(\phi)} & \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L) \\ \downarrow{=} & & \downarrow{Q/P(\phi)} \\ F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/L) & \xrightarrow{Q(\phi)} & \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L) \end{array}$$

Definition

$$H^i_{\mathrm{fp}}(\mathcal{X}, n) := \varinjlim_{P} H^i(MF(F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/L) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L))$$

cup product and Poincare duality

Proposition

For $\mathcal{X}$ proper
$H^{i-1}_{\mathrm{dR}}(X/L)/F^n \hookrightarrow H^i_{\mathrm{fp}}(\mathcal{X}, n) \twoheadrightarrow F^n H^i_{\mathrm{dR}}(X/L)$

so there is a chance for Poincare duality

Proposition

There exists a cup product

$$H^i_{\mathrm{fp}}(\mathcal{X}, n) \times H^j_{\mathrm{fp}}(\mathcal{X}, m) \to H^{i+j}_{\mathrm{fp}}(\mathcal{X}, n+m)$$

compatible with the short exact sequence

Proof of existence of cup products

Idea

- In the category of $L[t]$-modules

$$C_P^\bullet := (L[t] \xrightarrow{\cdot P(t)} L[t])$$

  is a free resolution of $V_P := L[t]/P(t)$
- $\mathcal{H}om(C_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L)) = MF(\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L))$
- $V_P \otimes V_Q = V_{P*Q} \Rightarrow C_P^\bullet \otimes C_Q^\bullet$ is a resolution of $V_{P*Q} \Rightarrow C_{P*Q} \to C_P^\bullet \otimes C_Q^\bullet$

cup product is induced by

$$\mathcal{H}om(C_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L)) \times \mathcal{H}om(C_Q^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L))$$
$$\to \mathcal{H}om(C_{P*Q}^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/L)$$

Poincare duality

Proposition

For $\mathcal{X}$ proper of relative dimension d the cup product and

$$\mathrm{tr} : H^{2d+1}_{\mathrm{fp}}(\mathcal{X}, d+1) = H^{2d}_{\mathrm{dR}}(X/L) \cong L$$

gives Poincare duality

$$H^i_{\mathrm{fp}}(\mathcal{X}, n) = H^j_{\mathrm{fp}}(\mathcal{X}, m)^* \; , \; i + j = 2d+1 \; , \; m + n = d+1$$

Pushforwards

For $f : \mathcal{X} \to \mathcal{Y}$ of proper schemes, can define f_* on H_{fp} as the dual of f^*.
This is reasonably computable

Theorem (B)

f_* commutes with f_* on Chow groups via

$$CH^i(\mathcal{X}) \to H^{2i}_{\mathrm{syn}}(\mathcal{X}, i) \to H^{2i}_{\mathrm{fp}}(\mathcal{X}, i)$$

The proof is Riemann-Roch, following Gillet and Messing

What about non-proper schemes

Can't expect Poincare duality in the form above. Instead

1. Define cohomologies with compact supports and homologies, satisfying twisted Poincare duality (Bloch-Ogus) axioms
2. Define pushforwards based on homology
3. Define products between cohomologies and cohomologies with compact supports.
4. Prove a projection formula $f_*(a \cup f^* b) = f_*(a) \cup b$

1-3 are done by Chiarellotto, Ciccioni and Mazzari (for syntomic, fp is the same)

Syntomic homology

- Berthelot defines rigid complexes with compact supports $\mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/L)$ a cup product and Poincare duality
- Berthelot and Baldassarri define a map $\mathbb{R}\Gamma_{\mathrm{rig},c} \to \mathbb{R}\Gamma_{\mathrm{dR},c}$
- To get homology, dualize complexes with compact supports and write the same cone.
- Poincare duality (in Bloch-Ogus sense) follows from Poincare duality in the rigid and de Rham cases

Cohomology with compact supports

Definition

The (modified) syntomic complex with compact supports is the limit over j of the homotopy limits of the following diagram

$$
\begin{array}{ccccccc}
F^n\mathbb{R}\Gamma_{\mathrm{dR},c} & & & \mathbb{R}\Gamma_{\mathrm{rig},c} & & & MF_c \\
& \searrow & \swarrow & & \searrow & \swarrow & \\
& \mathbb{R}\Gamma_{\mathrm{dR},c} & & & \mathbb{R}\Gamma_{\mathrm{rig},c} & &
\end{array}
$$

where $MF_c = MF(1 - (\phi/q^n)^i : \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/L) \to \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/L))$

Products

Similarly one can write the modified syntomic complex as the limit of homotopy limits of

$$\begin{array}{ccccccc} F^n\mathbb{R}\Gamma_{\mathrm{dR}} & & & \mathbb{R}\Gamma_{\mathrm{dR}} & & & MF \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & \mathbb{R}\Gamma_{\mathrm{dR}} & & \mathbb{R}\Gamma_{\mathrm{rig}} & & \end{array}$$

and the terms match

$$\mathbb{R}\Gamma_{\mathrm{dR}} \times \mathbb{R}\Gamma_{\mathrm{rig},c} \to \mathbb{R}\Gamma_{\mathrm{rig}} \times \mathbb{R}\Gamma_{\mathrm{rig},c} \to \mathbb{R}\Gamma_{\mathrm{rig},c}$$

$$\mathbb{R}\Gamma_{\mathrm{fp}} \times \mathbb{R}\Gamma_{\mathrm{fp},c} \to \mathbb{R}\Gamma_{\mathrm{fp},c}$$

The projection formula is essentially formal

Sketch of the proof

- $\sum(Z_i, f_i)$ gives an element in $H^3_{\mathcal{M}}(S, \mathbb{Q}(2))$ as follows:
  - It gives an element in $H^1_{\mathcal{M}}(\cup Z_i - \cup \operatorname{div} f_i, \mathbb{Q}(1))$
  - Element is closed implies an extension to $H^1_{\mathcal{M}}(\cup Z_i, \mathbb{Q}(1))$
  - Pushforward to S
- In particular, the restriction to $S - \cup \operatorname{div} f_i$ is the sum of the pushforward from $H^1_{\mathcal{M}}(X_i - \operatorname{div} f_i, \mathbb{Q}(1))$ of the class of f_i.
- In syntomic cohomology, the analogue of the last statement holds and characterizes $\operatorname{reg}(\theta)$.
- Lift μ to an element of $H^2_{\mathrm{fp}}(S - \cup \operatorname{div} f_i)$ (Coleman integration in hiding)
- Use projection formulas to transform the computation to a cup product computation on $X_i - \operatorname{div} f_i$.

Applications

- Langer - Example of $\theta \in H^1(E \times E, \mathcal{K}_2)$,
  E a CM elliptic curve, with decomposable regulator
- p-adic analogue of Beilinson's Theorem for the self product of a modular curve?