p-ADIC INTEGRATION AND p-ADIC ABEL-JACOBI MAPS

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Plan

In this talk I want to explain (parts of) Coleman's theory of p-adic integration, its relation to cycle classes of zero cycles and some generalizations. The plan of this talk will be:

1. Motivation - The classical (complex) Abel map.
2. Review of some rigid analytic geometry
3. The Coleman integral
4. Cycle classes of zero cycles and p-adic periods
5. Higher dimension cycles and integration.

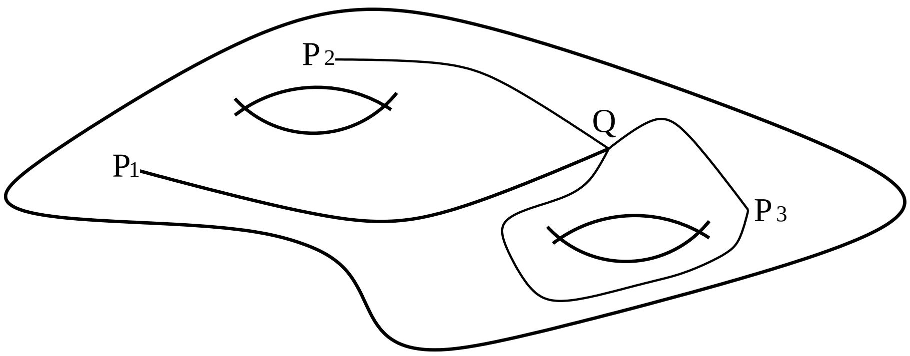
1. Motivation - The classical Abel map

$X/\mathbb{C}$ smooth projective.

$Z_0(X)$= collection of all $z = \sum n_i P_i$, $P_i \in X(\mathbb{C})$ with $\sum n_i = 0$.

We can define $\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$ and map $z \in Z_0(X)$ to

$$\mathrm{Alb}(z)(\omega) := \sum n_i \int_Q^{P_i} \omega \text{ (mod periods)}$$



In the case that X is a complex curve, Abel's theorem tells us that $\mathrm{Alb}(z) = 0$ iff z is rationally equivalent to 0 (= is the divisor of a function).

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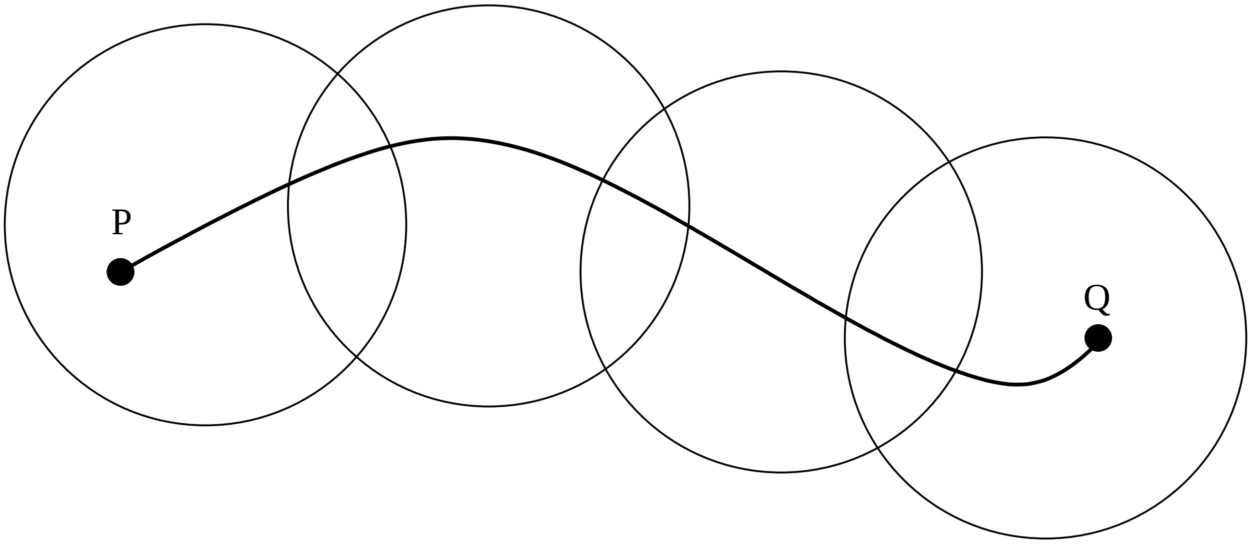
Complex integration is done via local integration and analytic continuation



This does not work over the p-adics, as we will see.

2. Review of some rigid analytic geometry

Let K be a field complete w.r.t. a non archimedean absolute value $|\ |$

Examples: $\mathbb{Q}_p$, a finite extension of $\mathbb{Q}_p$, and $\mathbb{C}_p$ = completion of $\bar{\mathbb{Q}}_p$. (Assume $K \subset \mathbb{C}_p$)

Basic object: for any field $L \supset K$,

$$B^n(L) = \{(z_1, \dots, z_n) \in L, \quad |z_i| \leq 1 \text{ for all } i\}$$

Let T^n = "algebra of rigid analytic functions on B^n" = formal power series converging in B^n

$$T^n = \left\{ \sum a_v X^v \quad a_v \in K \quad a_v \to 0 \text{ as } v \to \infty \right\}$$

with

$$v = (v_1, \dots, v_n), \quad X^v := X_1^{v_1} \cdots X_n^{v_n}$$

Definition 1. *An affinoid algebra A over K is a K algebra for which there is a surjective map $T^n \to A$.*

Definition 2. $\mathrm{Max}(A)$ *is the set of maximal ideals of A.*

Example: $\mathrm{Max}(T^n)$ is $B^n(\bar{K})$ divided by the action of $\mathrm{Gal}(\bar{K}/K)$ - $(z_1, \dots, z_n) \in B^n(\bar{K})$ defines a map $T^n \to \bar{K}$ by evaluation. the corresponding maximal ideal is its kernel.

For a general affinoid algebra A, if $x \in \operatorname{Max}(A)$ then A/x is a finite extension of K and for $f \in A$ we define $f(x)$ to be the image of f in A/x.

Example: For $A = T^2/ < X_1 \cdot X_2 - 1 >$ we have $\operatorname{Max}(A) = \{z \in B^1, \quad |z| = 1\}/\operatorname{Gal}(\bar{K}/K)$.

We call the pair $(A, X = \operatorname{Max}(A))$ an affinoid. We usually just write X and let $A = A(X)$.

The affinoid X has a reduction, denoted $\widetilde{X}$ which is an affine scheme over the residue field of K. It's defined as $\operatorname{Spec}(A^0/mA^0)$ where A^0 is the subring of functions with integral values and m is the maximal ideal of $\mathcal{O}_K$. We say X has good reduction if $\widetilde{X}$ is smooth.

There is a reduction map $\operatorname{red} : X \to \widetilde{X}$. We call $U_x := \operatorname{red}^{-1}(x)$ the reside disc around x.

In the following examples $K = \mathbb{C}_p$:

Examples: (1) for $X = B^1$, $\widetilde{X} = \mathbb{A}^1$, $U_0 = \{|z| < 1\}$.

(2) for $X = \{|z| = 1\}$, $\widetilde{X} = \mathbb{A}^1 - \{0\}$

The module of one forms of X, Ω^1_X is a completion of $\Omega^1_{X/K}$. For B^1 this will be $T^1 dz$.

3. The Coleman Integral

The problem: Suppose we are given a variety (enough to think of curves) C over $\mathbb{C}_p$, with good reduction $\tilde{C}$, and a closed holomorphic one form ω on C. How can we define

$$\int_P^Q \omega?$$

Attempt 1: (a) Cover C with affinoids (b) integrate on them (c) do analytic continuation.

(a) and (c) are OK. If you cover $\tilde{C}$ by affines U_i, C will be covered by affinoids

$$X_i := \{x \in C(\mathbb{C}_p), \text{ reduction of } x \in U_i\}.$$

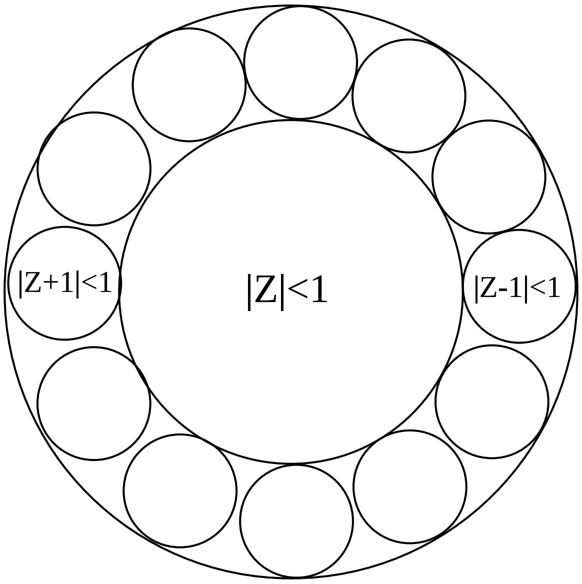
These have plenty of intersection so we can continue.

Example: $\mathbb{P}^1 = \{|z| \leq 1\} \cup \{|z| \geq 1\} \cup \{|z| = 1\}$

Problem: Can't integrate on affinoids, e.g., try to integrate dz/z on $X = \{|z| = 1\}$.

Attempt 2: Use covering by residue discs

Problem: Residue discs have no intersection



Coleman's solution: Continuation along a Frobenius.

Let X be an affinoid with good reduction $\widetilde{X}$.

Definition 3. *A Frobenius lift on X is a morphism $\phi : X \to X$ which reduces to a (relative) Frobenius on $\widetilde{X}$.*

Example: $X = B^1(\mathbb{Q}_p)$ (resp. $X = \{|z| = 1\}$). The map $z \to z^p$ is a Frobenius lift since it reduces to the Frobenius $x \to x^p$ of $\mathbb{A}^1/\mathbb{F}_p$ (resp. $\mathbb{A}^1 - \{0\}$).

A Frobenius lift always exists when the affinoid has good reduction.

Theorem 1 (Coleman). *(reference: Annals of Math 121 111-168 (1985)) Suppose there is a polynomial $P(T) \in \mathbb{C}_p[T]$ which does not vanish on any root of unity, and suppose that the form ω satisfies:*

$$P(\phi^*)(\omega) \text{ is exact,}$$

Then there is a function f_ω on $X(\mathbb{C}_p)$, which is analytic on each residue disc, with the property

$$df_\omega = \omega \text{ and } P(\phi^*)(f_\omega) \text{ is analytic.}$$

f_ω is unique up to a constant and does not depend on ϕ or P.

Example: Consider $\omega = dz/z$ on $\{|z| = 1\}$. If $|\alpha| = 1$, then for $z = \alpha + w$ with $|w| < 1$

$$\begin{aligned} f_\omega(\alpha + w) &= \log(\alpha + w) + \mathrm{Const}_\alpha \\ &= \log(1 + \frac{w}{\alpha}) + \mathrm{Const}_\alpha \end{aligned}$$

Problem: Too many constants.

However, $\phi^*\omega = p \cdot \omega$ so $f_\omega(z^p) = p \cdot f_\omega(z) + \text{Const}$. Can suppose Const $= 0$. Then $\alpha^{p^k} = \alpha$ implies $f_\omega(\alpha) = 0$. Every residue disc has such an α so f_ω is uniquely determined.

Theorem 2 (Coleman). *If $X/\mathbb{C}_p$ is a projective variety with good reduction and*

$$\omega \in H^0(X, \Omega^1_{X/\mathbb{C}_p}),$$

then there is a canonical way to associate to ω a function up to constant

$$F_\omega : X(\mathbb{C}_p) : X \to \mathbb{C}_p$$

s.t. $dF_\omega = \omega$.

One has (up to const):

1. $F_{\omega+\eta} = F_\omega + F_\eta$
2. $F_{g^*\omega} = g^*\omega$.

4. Cycle classes of zero cycles and p-adic periods

Recall: if $C/\mathbb{C}$ is a curve, $\sum n_i P_i \approx 0$ iff for any $\omega \in H^0(X, \Omega^1_{X/\mathbb{C}})$, $\sum n_i \int_Q^{P_i} \omega = \text{period}$.

Theorem 3 (Coleman). *If $C/\mathbb{C}_p$ is a curve*
$\sum n_i P_i$ is torsion mod rat equiv iff for any
$\omega \in H^0(X, \Omega^1_{X/\mathbb{C}_p})$, $\sum n_i F_\omega(P_i) = 0$.

One can make a more precise statement when $[K : \mathbb{Q}_p] < \infty$, X/K projective + good reduction:

There are 2 maps:

A. p-adic étale Abel-Jacobi map

$$Z_0(X) \xrightarrow{\text{Alb}} \text{Alb}(X) \xrightarrow{\text{Kummer}}$$
$$H^1(K, V_p(\text{Alb}(\bar{X}))) \cong H^1(K, H^{2d-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(d)))$$

B. p-adic integration + Bloch-Kato exponential map

$$Z_0(X) \xrightarrow{\text{Integration}} (H^0(X, \Omega^1_{X/K}))^* =$$
$$(F^1 H^1_{dR}(X/K))^* \xrightarrow{\text{Poincaré duality}}$$
$$H^{2d-1}_{dR}(X/K)/F^d \xrightarrow{\text{Bloch-Kato}}$$
$$H^1(K, H^{2d-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(d)))$$

Theorem 4 (B). *Both maps are the same.*

5. Higher dimension cycles and integration

K - any complete subfield of $\mathbb{C}_p$.
$X/\mathcal{O}_K$ smooth and projective.

One defines a "cohomology theory" $H^i_{fp}(X,j)$.

Its essential properties are:

1. a short exact sequence

$$\begin{aligned} 0 &\to H^{i-1}_{dR}(X_K/K)/F^n \to H^i_{fp}(X,n) \\ &\to F^n H^i_{dR}(X_K/K) \to 0; \end{aligned}$$

2. a cup product $H^i_{fp}(X,n)\times H^j_{fp}(X,m)\to H^{i+j}_{fp}(X,n+m)$, compatible with cup products in de Rham cohomology. 3. Poincaré duality - Let d be the relative dimension of X.
By first property there is a trace map

$$H^{2d+1}_{fp}(X,d+1)\xrightarrow{\mathrm{tr}_X} K$$

and

$$\begin{aligned} &\langle\ ,\ \rangle_{fp}: H^i_{fp}(X,n)\times H^{2d+1-i}_{fp}(X,d+1-n)\xrightarrow{\cup} \\ &H^{2d+1}_{fp}(X,d+1)\xrightarrow{\mathrm{tr}_X} K \end{aligned}$$

is perfect.

Using H_{fp} one can do "integration":

$\omega\in F^{j+1}H^{2j+1}_{dR}(X_K/K)$.

Choose a pullback $\tilde{\omega}\in H^{2j+1}_{fp}(X,j+1)$.

$i_Y: Y\hookrightarrow X$ - <u>smooth</u> subscheme
Y - of relative dimension j.

Definition 4.

$$\int_Y \tilde{\omega} := \mathrm{tr}_Y(i_Y^*(\tilde{\omega})).$$

The difference between 2 lifts is in $H^{2j}_{dR}(X_K/K)$ (constants for $j=0$)

Definition 5. *Given a cycle $Z = \sum n_i Y_i$ de-Rham equivalent to 0, Y_i as in definition 4. Define*

$$\int_Z \omega := \sum n_i \int_{Y_i} \tilde{\omega}$$

for any lift $\tilde{\omega}$.

In the case $j = 0$ we recover Coleman integration:

$$\omega \in F^1 H^1_{dR}(X_{\mathbb{C}_p}/\mathbb{C}_p) = H^0(X_{\mathbb{C}_p}, \Omega^1).$$

$y_i \in X(\mathbb{C}_p)$, Let Y_i be the images of the associated sections $\mathcal{O}_{C_p} \to X$.

Theorem 5 (B). $\int_Z \omega = \sum n_i F_\omega(y_i)$.

One can define a cycle class

$$\eta : CH^i(X) \to H^{2i}_{fp}(X, i).$$

The main result in the theory is that this cycle class commutes with pushforwards on Chow groups, resp. fp-cohomology:

$\dim Y - \dim X = d$

$$\begin{array}{ccc} CH^i(X) & \xrightarrow{f_*} & CH^{i+d}(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ H^{2i}_{fp}(X, i) & \xrightarrow{f_*} & H^{2i+2d}_{fp}(Y, i+d) \end{array}$$

This implies a similar relation to étale cohomology as in the case of points.