Multiple zeta values

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Plan

- Define multiple zeta values and describe some of their properties
- Describe p-adic multiple zeta values (Furusho)
- Our work - relation between p-adic multiple zeta values.

Definition 1. For $\underline{k} = (k_1, \ldots, k_m)$, $k_i > 0$, $k_m > 1$ the multiple zeta value $\zeta(\underline{k})$ is defined as the (convergent) series.

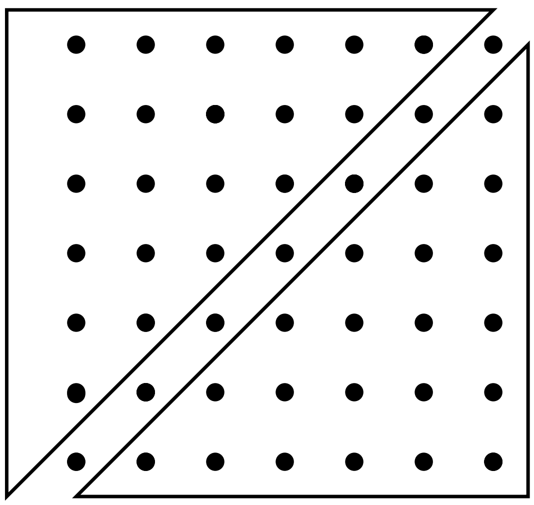
$$\zeta(\underline{k}) = \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \,. \tag{1}$$

E.g., $\underline{k} = (k)$, $\zeta(\underline{k}) = \zeta(k)$ - usual zeta value.

They were already studied by Euler.

Interesting relations.

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b) \tag{2}$$



Where they show up

- Number theory
- K-theory
- Quantum Groups (Drinfeld)
- Knot theory (Kontsevich)
- Mahler measures (Lalin)

Algebra of multiple zeta values

Weight of $\zeta(\underline{k})$, $\mathrm{wt}(\underline{k}) = \sum k_i$

$$Z_w = \langle \{\zeta(\underline{k}), \mathrm{wt}(\underline{k}) = w\} \rangle_{\mathbb{Q}} \subset \mathbb{R}$$

(2) + Similar relations $\Rightarrow$

Proposition: $Z = \oplus_{w=0}^{\infty} Z_w$ is a graded algebra

Examples:

- $Z_0 = \mathbb{Q}$
- $Z_1 = 0$
- $Z_2 = \mathbb{Q}\zeta(2) = \mathbb{Q}\pi^2$
- $Z_3 = \mathbb{Q}\zeta(3)$ $(\zeta(3) = \zeta(1,2))$

- $Z_4 = \mathbb{Q}\pi^4$ $(\zeta(1,1,2) = \zeta(4) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2))$

- $Z_5 = \langle \pi^2\zeta(3), \zeta(5) \rangle_{\mathbb{Q}}$
  Conjecture: $\dim Z_5 = 2$

conjecture (Zagier)

$$\sum \dim Z_w x^w = \frac{1}{1 - x^2 - x^3}$$

Multiple polylogarithms

$$\mathsf{Li}_{\underline{k}}(z) = \sum_{0<n_1<\cdots<n_m} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}\,. \tag{3}$$

So $\zeta(\underline{k}) = \mathsf{Li}_{\underline{k}}(1)$.

Derivative formula:

$$d\,\mathsf{Li}_{k_1,\ldots,k_m}(z) = \begin{cases} \mathsf{Li}_{k_1,\ldots,k_m-1}(z)\frac{dz}{z} & k_m \neq 1 \\ \mathsf{Li}_{k_1,\ldots,k_{m-1}}(z)\frac{dz}{1-z} & k_m = 1 \end{cases}$$

$\Rightarrow$ Li is an iterated integral

$$\mathsf{Li}_{\underline{k}}(z) = \int_0^z \frac{dt}{1-t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t} \circ \frac{dt}{1-t} \circ \cdots \circ \frac{dt}{t}$$

$$\zeta(\underline{k}) = \int_0^1 \cdots = \int_{0\leq t_1\leq t_2\leq\cdots\leq 1} \frac{dt_1}{1-t_1}\cdot\frac{dt_2}{t_2}\cdots\cdot$$

Shuffle relations

e.g.

$$\zeta(2)\cdot\zeta(2) =$$

$$\int_{0\le t_1\le t_2\le 1}\frac{dt_1}{1-t_1}\cdot\frac{dt_2}{t_2}\cdot\int_{0\le s_1\le s_2\le 1}\frac{ds_1}{1-s_1}\cdot\frac{ds_2}{s_2}$$

$$=\int_{t_1\le t_2\le s_1\le s_2}+\int_{t_1\le s_1\le t_2\le s_2}+\int_{s_1\le t_1\le t_2\le s_2}$$

$$+\text{(same but reverse } s \text{ and } t\text{)}$$

$$=2\zeta(2,2)+4\zeta(1,3).$$

$(2) \Rightarrow \zeta(2)^2 = 2\zeta(2,2)+\zeta(4)$
$\Rightarrow \zeta(4) = 4\zeta(1,3)$

p-adic multiple zeta values (Furusho)

p-adic numbers $\mathbb{Q}_p$: completion of $\mathbb{Q}$ w.r.t $|p^n \frac{r}{s}|_p = p^{-n}$ where r, s prime to p.

Fact: Series $\sum a_n$ converges in $\mathbb{Q}_p$ iff $|a_n|_p \to 0$

Problem: The series (1) does not converge in $\mathbb{Q}_p$

Furusho's solution: Define the p-adic multiple polylogarithm by analytically continuing (3) (Coleman integration) $\mathrm{Li}_{\underline{k}}(z)$ and take $z = 1$.

Still a problem: near 1

$$\mathrm{Li}_{\underline{k}}(z) = \sum a_n(z) \log^n(z-1) \tag{4}$$

not defined at 1.

Quite difficult - Furusho shows that $a_n(1) = 0$ for $n > 0$, so $\mathrm{Li}_{\underline{k}}(1) = a_0(1)$.

Main result

$$\zeta_p(a)\zeta_p(b) = \zeta_p(a,b) + \zeta_p(b,a) + \zeta_p(a+b)$$

Idea of proof: Use several variable multiple polylogarithms, where relation becomes trivial.

$$\mathsf{Li}_{\underline{k}}(\underline{x}) = \sum_{0<n_1<\cdots<n_m} \frac{x_1^{n_1}\cdots x_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}\,.$$

converging p-adically if $|x_i|_p < 1$. Obvious relation

$$\begin{aligned}\mathsf{Li}_a(x)\,\mathsf{Li}_b(y) = \mathsf{Li}_{(a,b)}(x,y) + \mathsf{Li}_{(b,a)}(y,x)\\ + \mathsf{Li}_{a+b}(xy)\end{aligned}$$

Now just substitute $x = y = 1$ ...

To make this work we need the theory of Coleman functions in several variables (B., Vologodski) - integration of closed 1-forms.

As power series:

- $$\frac{\partial}{\partial x_i}\mathrm{Li}_{\underline{k}} = \frac{1}{x_i}\mathrm{Li}_{\underline{k}-e_i}$$

- If $k_i = 0$, then $\mathrm{Li}_{\underline{k}}$ can be expressed in terms of polylogarithms of lower depth, e.g.,

$$\begin{aligned}\mathrm{Li}_{(0,k)}(x,y) &= \sum_{m=1}^{\infty}\frac{y^m}{m^k}\sum_{0<n<m}x^n \\ &= \sum_{m=1}^{\infty}\frac{y^m}{m^k}\frac{x^m-x}{x-1} \\ &= \frac{1}{x-1}\left(\mathrm{Li}_k(xy) - x\,\mathrm{Li}_k(y)\right)\ .\end{aligned}$$

These formulas allow us to define $\mathrm{Li}_{\underline{k}}$ as a Coleman function.

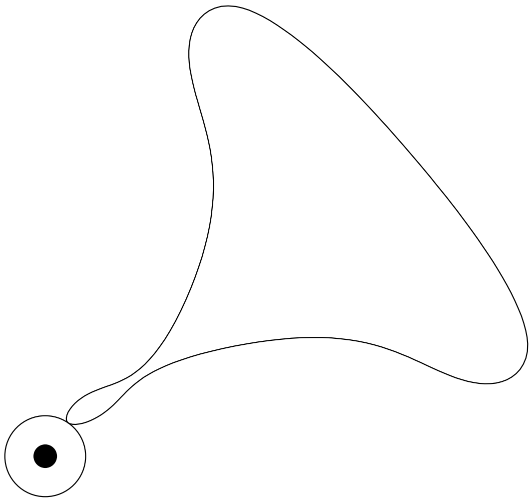
Remaining problems:

- Evaluation at $(1, 1, \ldots, 1)$.

- Comparison with the multiple zeta value.

Difficulty: 1 is a singular point.

Idea of solution: Tangential basepoints (Deligne)



There is an algebraic version of this construction.

Deligne's interpretation of the (multiple) zeta value - analytic continuation between tangent vectors at 0 and 1 $\Leftrightarrow$ in (4) *define* Li(1) as $a_0(1)$.