Variation of complex and p-adic regulators

Amnon Besser

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- The proof depends on a simple homological algebra Lemma, due to Jannsen.
- It could explain various instances of polylogarithms
- It also has (conjecturally) a p-adic analogue.

Mahler measures as regulators

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log |P(z_1, \ldots, z_n)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

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Deninger computes:

$$\{P, z_1, \ldots, z_n\} := P \wedge z_1 \wedge \cdots \wedge z_n \mapsto m(P)$$

He further shows that Mahler measures, or differences of Mahler measures, can sometimes be related to elements in the K-theory of Z, hence via the Beilinson conjectures to special values of L-functions.

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Is there an explanation for this?

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In (almost) complete generality

$$H_D^{m+1}(X,\mathbb{Z}(k)) = \frac{H^m(X,\mathbb{C})}{F^kH^m(X,\mathbb{C}) + H^m(X,(2\pi i)^k\mathbb{Z})}$$

which is also known as an "intermediate jacobian", where

$$F^kH^m(X,\mathbb{C}) = \oplus_{p\geq k} H^{p,m-p}$$

$H_D^{m+1}(X,\mathbb{R}(k))$ is the same thing with $\mathbb{R}$ instead of $\mathbb{Z}$.

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and we get $H^n(X,(2\pi i)^n\mathbb{R})$ by taking the real or imaginary value.

Families of varieties and normal functions

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In particular, for a family of polynomials P_s we get a normal function $f_P(s) = r_D(\{P_s, z_1, \ldots, z_n\})$

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To apply ∇ to $f(s)$ first lift to $\tilde{f}(s) \in \mathcal{H}^m(X/S)$, then apply ∇.
The integral part disappears and we get

$$\nabla f(s) \in \Omega^1(S, \mathcal{H}^m(X/S)/F^{k-1}\mathcal{H}^m(X/S))$$

The main theorem

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More precisely, with respect to the Leray spectral sequence

$$H^i(S, \mathcal{H}^{m+1-i}(X/S)) \Rightarrow H^{m+1}(X)$$

$ch(\alpha)$ maps to 0 in $H^0(S, \mathcal{H}^{m+1}(X/S))$ and $-\nabla f_\alpha(s)$ gives its image in $H^1(S, \mathcal{H}^m(X/S))$.

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Many people know about this in various forms: Voisin, Collino, Bloch, de Jeu, Saito ....

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- It explains the results of Rodriguez Villegas.
- Conjecturally, you can prove the same result with Deligne cohomology replaced by its p-adic analogue – Syntomic cohomology. Moreover, the original class can be reconstructed by using a Coleman integral.
- It can be used to compute regulators in many interesting cases, in particular related to many types of polylogarithms.

Idea of the proof

Homological algebra Lemma (Jannsen, B.)

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "\varepsilon"'] & Y \arrow[r, "\beta"] \arrow[d, "\phi"'] & C(X \to Y) \arrow[d] \\
Z \arrow[r, "\gamma"] \arrow[d, "\eta"'] & W \arrow[r] \arrow[d] & C(Z \to W) \arrow[d] \\
C(X \to Z) \arrow[r] & C(Y \to W) \arrow[r] & U
\end{tikzcd}$$

leading to

$$\begin{array}{ccccccc}
H^{i-1}_{X,Z} & \longrightarrow & H^{i-1}_{Y,W} & \longrightarrow & H^{i-1}_{U} & \longrightarrow & H^{i}_{X,Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{i}_{X} & \longrightarrow & H^{i}_{Y} & \longrightarrow & H^{i}_{X,Y} & \longrightarrow & H^{i+1}_{X} \\
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\end{array}$$

Theorem follows from applying this to the following diagram

$$\begin{array}{ccccc} ?? & \longrightarrow & \mathbb{Z}_{\mathcal{D}}(k)_X & \longrightarrow & \mathbb{Z}_{\mathcal{D}}(k)_{X/S} \\ \downarrow & & \downarrow & & \downarrow \\ 0 \oplus \Omega_S^1 \otimes F^k\Omega_X^\bullet[-1] & \longrightarrow & \mathbb{Z} \oplus F^k\Omega_X^\bullet & \longrightarrow & \mathbb{Z} \oplus F^k\Omega_{X/S}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_S^1 \otimes \Omega_X^\bullet[-1] & \longrightarrow & \Omega_X^\bullet & \longrightarrow & \Omega_{X/S}^\bullet \end{array}$$

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- An analogue of integration against T^n – Schnirelman integral

$$\int_{T_p^n} f(z_1,\ldots,z_n)\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} := \lim_{\substack{N\to\infty \\ (N,p)=1}} \frac{1}{N^n}\sum_{\zeta\in\mu_N^n} f(\zeta)$$

We obtain the p-adic Mahler measure

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In the case $n=1$ this gives for $P = a_m t^m + \cdots a_r t^r$

$$m_p(P) = \log_p(a_m) + \sum_{|\alpha|_p>1} \log_p(\alpha)$$

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This can sometimes be tied to special values of p-adic L-functions.