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A generalization of Coleman's p-adic integration theory

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Summary. We associate to a scheme X smooth over a p-adic ring a kind of cohomology group $H^i_{\text{fp}}(X,j)$. For proper X this cohomology has Poincaré duality hence Gysin maps and cycle class maps which are reasonably explicit. For zero-cycles we show that the cycle class map is given by Coleman integration. The cohomology theory H_{fp} is therefore interpreted as giving a generalization of Coleman's theory. We find an embedding $H^{2i}_{\text{syn}}(X,i) \hookrightarrow H^{2i}_{\text{fp}}(X,i)$ where H_{syn} is (rigid) syntomic cohomology. Our main result is an explicit description of the syntomic Abel-Jacobi map in terms of generalized Coleman integration.

1. Introduction

In the study of cycles on algebraic varieties an important part is played by various cycle class maps. When X is a projective smooth variety of dimension d over the field $\mathbb{C}$ of complex numbers, one has the Betti cycles class map,

$$c\ell_B : A^i(X) \to H^{2i}_B(X(\mathbb{C}), \mathbb{Z}) ,$$

where $A^i(X)$ denotes the free abelian group generated by subvarieties of codimension i, and the *Abel-Jacobi* map,

$$\alpha^i_{\mathbb{C}} : A^i(X)_0 \to \left[F^{j+1} H^{2j+1}_{\text{dR}}(X/\mathbb{C})\right]^* / H_{2j+1}(X(\mathbb{C}), \mathbb{Z}) , \tag{1}$$

⋆ Present address: Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er-Sheva 84105, Israel where $A^i(X)_0$ is the kernel of $c\ell_B$ and $i + j = d$. We quickly recall that this last map is defined by sending a cycle $Z \in A^i(X)_0$ to the functional whose value on the class $[\omega]$ of a C^∞ form is

$$\alpha_{\mathbb{C}}^i(Z)([\omega]) = \int_\Gamma \omega \ , \tag{2}$$

where Γ is any topological chain bounding Z. Alternatively, the Abel-Jacobi map can be defined using extension classes in the category of mixed Hodge structures (see [Jan88]). The Betti cycle class map and the Abel-Jacobi map can be combined into a single "regulator" map,

$$A^i(X) \to H_{\mathcal{D}}^{2i}(X, \mathbb{R}(i)) \ ,$$

into Deligne-Beilinson cohomology.

When we turn our attention to the p-adic world, a similar picture emerges. It has become apparent (see for example [Nek98]) that the p-adic analogue of Deligne-Beilinson cohomology is syntomic cohomology in its many versions. Suppose X is a smooth proper scheme over the ring of integers $\mathcal{V}$ in a finite extension K of $\mathbb{Q}_p$. Then we have syntomic cohomology groups $H_{\text{syn}}^i(X, n)$ and cycle maps,

$$\eta^{\text{syn}} : A^i(X) \to H_{\text{syn}}^{2i}(X, i)$$

(see Proposition 5.4). There is also a canonical map $H_{\text{syn}}^i(X, n) \xrightarrow{\mathbf{p}} F^n H_{\text{dR}}^i(X_K/K)$, where F^n denotes the Hodge filtration. The composition of the cycle map with $\mathbf{p}$ gives the usual de Rham cycle class maps on the restriction of $A^i(X)$ to the generic fiber X_K. Restricting to cycles which are de Rham homologous to 0 gives a map into the kernel of $\mathbf{p}$. This kernel turns out to be isomorphic to $H_{\text{dR}}^{2i-1}(X_K/K)/F^i$, which is Poincaré dual to $F^{j+1} H_{\text{dR}}^{2j+1}(X_K/K)$, where $i + j = d$ as before but d denotes the relative dimension of X over $\mathcal{V}$. The resulting map we call the syntomic Abel-Jacobi map. It is the p-adic analogue of the Abel-Jacobi map. The entire picture is compatible with the p-adic étale analogues of the cycle class maps via the theory of p-adic periods. In particular, the syntomic Abel-Jacobi map is compatible with the p-adic étale Abel-Jacobi map via the exponential map of Bloch and Kato (see [Nek98] and [Bes98, Remark 9.11]).

The starting point for the present work was the hope that the syntomic Abel-Jacobi map admits a description in terms of some kind of integration similar to the one available for its complex counterpart. An integration theory suitable for our needs was available, for 1-forms, by the work of Coleman [Col85]. The part of the theory we are interested in allows one to associate to a closed 1-form ω on X_K a function $F_\omega : X(K) \to K$ which is uniquely defined up to a constant (in other words, unlike the complex case there is no multivaluedness). In particular, for a 0-cycles $Z = \sum n_l(x_l)$ the expression

$$\int_Z \omega := \sum n_l F_\omega(x_l)$$

is well defined and associates to Z a functional on the space of holomorphic 1-forms. It is in fact not so hard to show (see Proposition 9.2) that this association is exactly the syntomic Abel-Jacobi map for 0-cycles. In order to extend this result to higher dimensional cycles we develop a generalization of Coleman's theory.

Perhaps unsatisfyingly our generalization takes on a very cohomological guise. One can think of the problem of computing a cycle class as a special case of the computation of the Gysin (or pushforward) map in cohomology. This computation in a cohomology theory is made much simpler if the theory satisfies Poincaré duality in the naive sense of having a perfect pairing between cohomologies in appropriate degrees. Unfortunately syntomic cohomology (as well as Deligne-Beilinson cohomology) does not satisfy this version of Poincaré duality. Our strategy for computing Gysin maps in syntomic cohomology is to define a functorial embedding $H_{\mathrm{syn}} \hookrightarrow H_{\mathrm{fp}}$ of syntomic cohomology into a new theory, called here *finite polynomial cohomology*. This new theory, while not quite a cohomology theory, does satisfy Poincaré duality in the naive sense and this allows us to give an explicit description of Gysin maps.

The word "polynomial" in the name of H_{fp} comes from the use of polynomials in Frobenius in its definition which is a generalization of their role in Coleman's theory.

Let us explain in what sense the finite polynomial cohomology generalizes Coleman's integration. For proper X there is a fundamental short exact sequence, functorial in X,

$$0 \to H^{i-1}_{\mathrm{dR}}(X_K/K)/F^n \xrightarrow{\mathbf{i}} H^i_{\mathrm{fp}}(X,n) \xrightarrow{\mathbf{p}} F^n H^i_{\mathrm{dR}}(X_K/K) \to 0 \tag{3}$$

(see Proposition 2.5). Consider this for a moment in the case $i = n = 1$. Suppose that $\omega \in F^1 H^1_{\mathrm{dR}}(X_K/K)$ is a holomorphic 1-form on X_K. Then we may choose a lift $\tilde{\omega}$ to ω in $H^1_{\mathrm{fp}}(X,1)$. Note that this lift is unique up to an element in $H^0_{\mathrm{dR}}(X_K/K) \cong K$. There is a natural one to one correspondences between these lifts and Coleman integrals of ω. The lift $\tilde{\omega}$ yields the Coleman integral in the following way: Let $x \in X(K)$ be a point and lift it to a section $x : \mathcal{V} \to X$. We may then consider $x^*\tilde{\omega} \in H^1_{\mathrm{fp}}(\mathcal{V},1)$. The short exact sequence Eq. (3) for $X = \mathrm{Spec}\,\mathcal{V}$ shows this last group is isomorphic to $H^0_{\mathrm{dR}}(\mathrm{Spec}\,K/K) \cong K$. We can prove

Theorem 1.1. *The function $x \mapsto x^*\tilde{\omega}$ is a Coleman integral F_ω evaluated at x.*

Note that changing $\tilde{\omega}$ by a constant changes the integral by the same constant.

It is now easy to see how to use H_{fp} to integrate a cohomology class $\omega \in F^{j+1}H^{2j+1}_{\mathrm{dR}}(X_K/K)$ against a codimension i (i.e., with relative dimension j) cycle, $Z = \sum n_l(Z_l)$, which is homologically equivalent to 0, provided the components of Z are smooth over $\mathcal{V}$. To do this, one chooses a lifting of ω to $\tilde{\omega} \in H^{2j+1}_{\mathrm{fp}}(X, j+1)$. For each component Z_l one considers the pullback of $\tilde{\omega}$ to $H^{2j+1}_{\mathrm{fp}}(Z_l, j+1) \cong K$ where the last isomorphism is again a consequence of Eq. (3). Summing up these pullbacks gives the required integral

$$\int_Z \omega = \sum n_l \,\mathrm{tr}_{\,Z_l} i_l^*(\tilde{\omega}) \ . \tag{4}$$

If Z is homologically equivalent to 0 it is not too hard to show that this integral is independent of the choice of $\tilde{\omega}$. We can show the following.

Theorem 1.2. *The syntomic Abel-Jacobi map sends the homologically trivial cycle Z to the functional on $F^{j+1}H^{2j+1}_{\mathrm{dR}}(X_K/K)$ sending ω to $\int_Z \omega$.*

This theorem is in fact a corollary of the more general Theorem 5.5 showing compatibility of Gysin maps in Chow groups and in finite polynomial cohomology. In a future paper we intend to use this stronger theorem to show other applications of Coleman integration to computation of cycle classes. In particular, we will show how to compute the syntomic Abel-Jacobi image of the Heegner cycles considered by Nekovář in [Nek95] by applying Coleman's integration for sections of the standard vector bundles over a modular curve [Col94].

The structure of the paper is as follows: In Sect. 2 we define finite polynomial cohomology and state all its basic properties. We try to give the simplest definition here. For computing cup products it is more convenient to give a more sophisticated definition. This is done in Sect. 3. We next define cup products in Sect. 4. We recall the definition of the cycle class map based on K-theory in Sect. 5 and we also state the basic compatibility Theorem 5.5 and deduce Theorem 1.2 from it. The proof of Theorem 5.5 requires some technical results about pushforwards for the embedding of a smooth divisor. This is done in Sections 6 and 7, after which the proof is given in Sect. 8. The proof is almost the same as the corresponding result for crystalline cohomology given in [GM87, Part II]. We just indicate the necessary modifications. Finally, the comparison with Coleman's theory and some further remarks are given in Sect. 9.

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2. Finite polynomial cohomology

Throughout the entire paper, all schemes are of finite type over the respective base scheme.

In this section we will define finite polynomial cohomology, state and prove some of its basic properties.

We begin by recalling some of the definitions and constructions in [Bes98, Sect. 2]. We fix $\mathcal{V}$, a complete valuation ring with field of fractions K and residue field κ of characteristic p. In order for us to be able to compare the theory with Coleman's integration we will not assume in advance that $\mathcal{V}$ is discretely valued, although less can be said otherwise and it suffices to assume so for arithmetic applications. On the other hand, we always assume that κ is algebraic over the prime field. Let X be a smooth $\mathcal{V}$-scheme with generic fiber X_K and special fiber X_κ. In this situation there is a notion of a *Frobenius endomorphism* of X_κ [Bes98, 8.1], which is just a κ-linear endomorphism obtained by extension of scalars from an r-th power of the absolute Frobenius over a model of X_κ defined over a finite field for some r (this is where the assumption that κ is algebraic over the prime field enters). The number $q = p^r$ is then called the degree of φ. In loc. cit. we have associated to X_κ a complex $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ and to X_K a complex $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$ together with complexes $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$ for every non-negative integer n. These complexes are functorial with respect to morphisms of κ (respectively K) schemes. Their cohomologies give respectively the rigid cohomology (in the sense of Berthelot [Ber97b]), $H_{\mathrm{rig}}(X_\kappa/K)$, the de Rham cohomology, $H_{\mathrm{dR}}(X_K/K)$, and the pieces of its Hodge filtration. There are canonical maps $F^n\mathbb{R}\Gamma_{\mathrm{dR}} \to \mathbb{R}\Gamma_{\mathrm{dR}}$ and also a map, functorial in X, $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. This last map only exists in the derived category of K-vector spaces, but it decomposes into a composition of morphisms and inverses of quasi-isomorphisms, all of which are functorial in X, so that one may use this morphism to get canonically defined cones. We note that when X is proper the map $\mathbb{R}\Gamma_{\mathrm{dR}} \to \mathbb{R}\Gamma_{\mathrm{rig}}$ is a quasi-isomorphism. Most of our work is completely formal linear algebra so that we will not need to know the definition of these complexes at this point and they will only be recalled, as needed, in Sect. 7. The interested reader can consult [Bes98] and the references to Berthelot's work therein.

Now fix X and choose a Frobenius endomorphism φ on X_κ.

Definition 2.1. *Let $\mathfrak{P}$ be the multiplicative monoid of all polynomials $P(t) \in \mathbb{Q}[t]$ with constant coefficient 1. We write such a polynomial as $P(t) = \prod_1^{\deg P}(1-\alpha_i t)$. Let $\mathfrak{P}_{m,\varphi} \subset \mathfrak{P}$ be the submonoid of polynomials which are "pure of weight m", i.e., such that all $1/\alpha_i$ have complex absolute value $q^{m/2}$, with q the degree of φ. Most of the time φ will be fixed and we will denote the above submonoid simply by $\mathfrak{P}_m$.*

Definition 2.2. *Let $P \in \mathfrak{P}$. The* syntomic P-complex*, $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$, is defined as*

$$\mathbb{R}\Gamma_{\mathrm{f},P}(X,n) := \mathrm{MF}\left(F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \xrightarrow{P(\varphi^*)} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)\right) , \tag{5}$$

where the notation $P(\varphi^*)$ is a shorthand for the composition

$$F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \xrightarrow{P(\varphi^*)} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) .$$

The i-th cohomology of $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$ will be denoted by $H^i_{\mathrm{f},P}(X,n)$.

The notation MF stands for the mapping fiber of a map. To fix notation and signs, The mapping fiber of $f : A^\bullet \to B^\bullet$ between two cohomological complexes is given in degree i by

$$A^i \oplus B^{i-1}, \quad \text{with differential } d(a,b) = (da, f(a) - db) . \tag{6}$$

Definition 2.3. *For any P and Q in $\mathfrak{P}$ let $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n) \xrightarrow{c_Q} \mathbb{R}\Gamma_{\mathrm{f},PQ}(X,n)$ be the map induced by the commutative diagram of complexes:*

$$\begin{tikzcd}
F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \arrow[r, "P(\varphi^*)"] \arrow[d, equal] & \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \arrow[d, "Q(\varphi^*)"] \\
F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \arrow[r, "PQ(\varphi^*)"] & \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K).
\end{tikzcd} \tag{7}$$

Definition 2.4. *The* finite polynomial *complex of X of weight m twisted n times, denoted $\mathbb{R}\Gamma_{\mathrm{fp},m}(X,n)$, is defined to be*

$$\mathbb{R}\Gamma_{\mathrm{fp},m}(X,n) := \varinjlim_{P\in\mathfrak{P}_m} \mathbb{R}\Gamma_{\mathrm{f},P}(X,n) ,$$

where the limit is over all polynomials $P \in \mathfrak{P}_m$, ordered by division, and the direct limit is taken with respect to the maps of Definition 2.3. The cohomology of $\mathbb{R}\Gamma_{\mathrm{fp},m}(X,n)$ is called the finite polynomial *cohomology of X and denoted $H^*_{\mathrm{fp},m}(X,n)$. The notation $H^i_{\mathrm{fp}}(X,n)$ will stand for $H^i_{\mathrm{fp},i}(X,n)$.*

Just as in [Bes98] there is also a tilded version (Gros style) of both the finite polynomial complex and its cohomology. These are obtained by replacing the filtered part $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$ by the filtration on the complex $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ described in loc. cit. Definition 9.1. When X is proper (which is the main case that interests us here) these two complexes are canonically quasi-isomorphic. The Gros style complex is more convenient for comparison of our theory with that of Coleman so we will say more about it in Sect. 9. The main properties of the finite polynomial cohomology (stated here for the non tilded version but equally true for the tilded one) are summarized in the following proposition.

Proposition 2.5. *The finite polynomial complex $\mathbb{R}\Gamma_{\mathrm{fp},m}(X,n)$ is independent of the choice of φ up to a unique isomorphism (not just quasi-isomorphism!). It is contravariantly functorial with respect to arbitrary morphisms. The finite polynomial cohomology enjoys the following properties:*

1. *When X is proper and defined over a discrete valuation ring there exists a functorial short exact sequence,*
$$0 \to H^{i-1}_{\mathrm{dR}}(X_K/K)/F^n \xrightarrow{\mathbf{i}} H^i_{\mathrm{fp}}(X,n) \xrightarrow{\mathbf{p}} F^n H^i_{\mathrm{dR}}(X_K/K) \to 0\ . \tag{8}$$
We view this short exact sequence as defining a filtration on $H^i_{\mathrm{fp}}(X,n)$,
$$H^i_{\mathrm{fp}}(X,n) = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \mathrm{Fil}^2 = 0\ ,$$
with
$$\mathrm{Gr}\, H^i_{\mathrm{fp}}(X,n) = F^n H^i_{\mathrm{dR}}(X_K/K) \oplus H^{i-1}_{\mathrm{dR}}(X_K/K)/F^n$$
in degrees 0 *and* 1 *respectively.*
2. *There is a canonical map*
$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{fp},n}(X,n)\ , \tag{9}$$
where $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)$ is the modified syntomic complex defined in [Bes98, Definition 8.4]. If $\mathcal{V}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$ and W_0 is the Witt ring of κ with fraction field K_0, the composition of this map with the map $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \xrightarrow{\Xi} \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)$ of loc. cit. Proposition 8.6.2 *will be denoted by $\mathfrak{r}$. When $i = 2n$ and X is proper the induced map on cohomology fits into the following commutative diagram:*
$$\begin{tikzcd} H^{2n-1}_{\mathrm{dR}}(X_K/K)/F^n \arrow[r, "\mathbf{i}_{\mathrm{syn}}"] \arrow[d, equal] & H^{2n}_{\mathrm{syn}}(X,n) \arrow[r, "\mathbf{p}_{\mathrm{syn}}"] \arrow[d, "\mathfrak{r}"] & F^n H^{2n}_{\mathrm{dR}}(X_K/K) \cap (H^{2n}_{\mathrm{cr}}(X_\kappa/W_0) \otimes K_0)^{\phi=p^n} \arrow[d, hook] \\ H^{2n-1}_{\mathrm{dR}}(X_K/K)/F^n \arrow[r, "\mathbf{i}"] & H^{2n}_{\mathrm{fp}}(X,n) \arrow[r, "\mathbf{p}"] & F^n H^{2n}_{\mathrm{dR}}(X_K/K) \end{tikzcd} \tag{10}$$
where H_{cr} is crystalline cohomology and ϕ is the semilinear Frobenius on it.
3. *There is a functorial product, which is associative and anti-commutative up to homotopy,*
$$\mathbb{R}\Gamma_{\mathrm{fp},i}(X,n) \times \mathbb{R}\Gamma_{\mathrm{fp},j}(X,m) \to \mathbb{R}\Gamma_{\mathrm{fp},i+j}(X,n+m)\ .$$
It is compatible with the product on the modified syntomic complexes $\mathbb{R}\Gamma_{\mathrm{syn}}$. The induced cup product on cohomology is compatible with the filtration Fil *in part* 1 *and the induced product on the graded part is induced by the de Rham cup product.*

4. *Assume X is proper, defined over a DVR and irreducible of relative dimension d. Then the map $\mathbf{i} : H^{2d}_{\mathrm{dR}}(X_K/K)/F^{d+1} \to H^{2d+1}_{\mathrm{fp}}(X, d+1)$ is an isomorphism and there is a trace map isomorphism*

$$\mathrm{tr}_X : H^{2d+1}_{\mathrm{fp}}(X, d+1) \xrightarrow{\mathbf{i}^{-1}} H^{2d}_{\mathrm{dR}}(X_K/K)/F^{d+1} = H^{2d}_{\mathrm{dR}}(X_K/K) \xrightarrow{\mathrm{tr}_{\mathrm{dR}}} K\ .$$

The pairing

$$\langle\ ,\ \rangle_{\mathrm{fp}} : H^i_{\mathrm{fp}}(X, n) \times H^{2d+1-i}_{\mathrm{fp}}(X, d+1-n) \xrightarrow{\cup} H^{2d+1}_{\mathrm{fp}}(X, d+1) \xrightarrow{\mathrm{tr}_X} K \qquad (11)$$

is a perfect duality.

Proof. We first remark that when X is proper and defined over a DVR we may assume $\mathcal{V}$ is discretely valued. The special case of a base change for proper schemes will follow from the base change results of Proposition 2.6. Part of the proof will be done in this section while some parts which require a bit more homological algebra will be done in the following two. Let φ and φ' be two Frobenius endomorphisms. We want to show that the corresponding complexes are canonically isomorphic. As appropriate powers of any two Frobenius endomorphisms are identical [Bes98, Lemma 8.3] we may assume that $\varphi' = \varphi^r$ for some r. Let q and q^r be the degrees of φ and φ' respectively. Let $Q \in \mathfrak{P}_{m,\varphi'}$. Then $P_Q(t) := Q(t^r) \in \mathfrak{P}_{m,\varphi}$ and $P_Q(\varphi^*) = Q(\varphi'^*)$ so that $\mathbb{R}\Gamma_{\mathrm{f},P_Q}(X,n)_\varphi = \mathbb{R}\Gamma_{\mathrm{f},Q}(X,n)_{\varphi'}$ where we have added the Frobenius endomorphism to the notation to indicate which one is used. To prove the independence it is enough to note that the collection of P_Q is cofinal in $\mathfrak{P}_{m,\varphi}$ because any $P \in \mathfrak{P}_{m,\varphi}$ divides the polynomial $\prod_{\zeta^r=1} P(\zeta t) = Q(t^r)$ for some $Q \in \mathfrak{P}_{m,\varphi'}$. As we are able to modify our Frobenius endomorphism functoriality follow easily. For part 1 we notice that the long exact sequence for the cone $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$ is

$$\begin{aligned} \cdots \quad &\to \quad F^n H^{i-1}_{\mathrm{dR}}(X_K/K) \xrightarrow{P(\varphi^*)} H^{i-1}_{\mathrm{rig}}(X_\kappa/K) \to H^i_{\mathrm{f},P}(X,n) \\ &\to \quad F^n H^i_{\mathrm{dR}}(X_K/K) \xrightarrow{P(\varphi^*)} H^i_{\mathrm{rig}}(X_\kappa/K) \to \cdots\ , \end{aligned} \qquad (12)$$

from which we get a short exact sequence:

$$\begin{aligned} 0 \to H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/P(\varphi^*)(F^n H^{i-1}_{\mathrm{dR}}(X_K/K)) \to H^i_{\mathrm{f},P}(X,n) \\ \to F^n H^i_{\mathrm{dR}}(X_K/K)^{P(\varphi^*)=0} \to 0\ . \end{aligned}$$

Suppose that X is proper and that $P \in \mathfrak{P}_i$. Then we can replace H^{i-1}_{rig} by H^{i-1}_{dR} on the left. By [ClS98] $H^{i-1}_{\mathrm{rig}}(X_\kappa/K)$ and $H^i_{\mathrm{rig}}(X_\kappa/K)$ are pure of weights $i-1$ and i respectively. This means that $P(\varphi^*)$ is invertible on $H^{i-1}_{\mathrm{dR}}(X_K/K)$ and that the characteristic polynomial of φ^* acting on $H^i_{\mathrm{dR}}(X_K/K)$ belongs to $\mathfrak{P}_i$. It follows that on the left we have an isomorphism

$$H^{i-1}_{\mathrm{dR}}(X_K/K)/F^n \xrightarrow{P(\varphi^*)} H^{i-1}_{\mathrm{dR}}(X_K/K)/P(\varphi^*)F^n\ , \qquad (13)$$

whereas on the right we have $F^n H^i_{\mathrm{dR}}(X_K/K)^{P(\varphi^*)=0} = F^n H^i_{\mathrm{dR}}(X_K/K)$ on the cofinal set of polynomials P divisible by the characteristic polynomial. The short exact sequence Eq. (8) follows easily. To prove part 2 recall that $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)$ is by [Bes98, Proposition 5.6.1] isomorphic to

$$\lim_{\substack{\rightarrow \\ k}} \mathrm{MF}\left(1-\left(\frac{\varphi^*}{q^n}\right)^k : F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)\right) .$$

But for any k the polynomial $1-(t/q^n)^k$ belongs to $\mathfrak{P}_n$ which gives immediately the required map. The top line of diagram Eq. (10) can be reconstructed from [Bes98, Proposition 6.3] by a similar argument to the above and it is then easy to see that the diagram commutes. The cup product and its properties will be discussed in the following two section. To prove part 4 we notice that since X_K has dimension d it follows from Eq. (8) that $\mathbf{i}$ is an isomorphism as required. As a special case of the compatibility with the de Rham cup product in part 3, the following diagram commutes

$$\begin{tikzcd} H^i_{\mathrm{fp}}(X,n) \times H^{j-1}_{\mathrm{dR}}(X_K/K)/F^m \arrow[r, "1\times \mathbf{i}"] \arrow[d, "\mathbf{p}\times 1"'] & H^i_{\mathrm{fp}}(X,n) \times H^j_{\mathrm{fp}}(X,m) \arrow[d, "\langle\,,\,\rangle^{\mathrm{fp}}"] \\ F^nH^i_{\mathrm{dR}}(X_K/K) \times H^{j-1}_{\mathrm{dR}}(X_K/K)/F^m \arrow[r, "\langle\,,\,\rangle_{\mathrm{dR}}"] & K, \end{tikzcd} \tag{14}$$

where $i+j=2d+1$, $m+n=d+1$ and $\langle\,,\,\rangle_{\mathrm{dR}}$ is the pairing induced on de Rham cohomology by the cup product and trace map. From this it is easy to deduce that Eq. (11) is perfect. Indeed, Eq. (14) implies that we have the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H^{i-1}_{\mathrm{dR}}(X_K/K)/F^n \arrow[r, "\mathbf{i}"] \arrow[d] & H^i_{\mathrm{fp}}(X,n) \arrow[r, "\mathbf{p}"] \arrow[d] & F^nH^i_{\mathrm{dR}}(X_K/K) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (F^mH^j_{\mathrm{dR}}(X_K/K))^* \arrow[r, "\mathbf{p}^*"] & (H^j_{\mathrm{fp}}(X,m))^* \arrow[r, "\mathbf{i}^*"] & (H^{j-1}_{\mathrm{dR}}(X_K/K)/F^m)^* \arrow[r] & 0, \end{tikzcd}$$

where we use $*$ to denote dual spaces and maps over K. The vertical maps on the left and right are induced by the de Rham cup product and trace and hence are isomorphisms by Poincaré duality. We are therefore done by the five lemma.

Proposition 2.6. *Suppose that $\mathcal{V} \subset \mathcal{V}'$ is an isometric extension (possibly infinite) with fraction field K'. Then there is a canonical base change map*

$$K' \otimes H^*_{\mathrm{fp}}(X,*) \to H^*_{\mathrm{fp}}(X \otimes \mathcal{V}',*)$$

compatible with the exact sequence Eq. (8) when it exists and with the cup product. If either both $\mathcal{V}$ and $\mathcal{V}'$ are only finitely ramified or X is proper, then the base change map is a quasi-isomorphism.

Proof (sketch). Let T be a rigid analytic K-space which is quasi-separated [Ber96, 0.1.7]. Then one can define the base change of T to K', $T\otimes_K K'$ [BGR84, 9.3.6]. There is a map of topological spaces in the Grothendieck topologies, $T\otimes_K K' \xrightarrow{\pi} T$. For any sheaf $\mathcal{F}$ on T we therefore obtain a map $\mathbb{R}\Gamma(T,\mathcal{F}) \to \mathbb{R}\Gamma(T\otimes_K K', \pi^*\mathcal{F})$. Comparing with the proof of proposition 1.6 in [Ber97b] (see also [Bes98, Proposition 4.21]) one easily finds the existence of a map $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_{\kappa'}/K')$ and one can further tensor with K' on the first component. The corresponding base change on the de Rham complexes, $K'\otimes \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X_{K'}/K')$, is well known and is in fact a quasi-isomorphism. It is also clear that the two base change morphisms are compatible. It is not known in general if the rigid base change morphism is a quasi-isomorphism. It is however known in the following cases:

– The extension K'/K is finite [Ber97b, Proposition 1.6].
– K and K' are absolutely unramified [Ber97a, Final remark]
– X is the special fiber of a proper smooth $\mathcal{V}$-scheme, because then the comparison map between de Rham and rigid cohomology are quasi-isomorphisms.

The rigid base change maps are therefore quasi-isomorphisms under the assumptions of the proposition. We finish as in [Bes98, Proposition 4.21].

3. A new interpretation of the syntomic complexes

To define the product on the finite polynomial complexes and thereby complete the proof of Proposition 2.5 we will now reinterpret the syntomic P-complex $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$ in terms of the fibered product and quasi-fibered product of complexes as described in [Bes98, Sect. 3]. The reinterpretation does not involve the structure of K-vector spaces but only the map φ^*. It therefore applies equally well to the syntomic complex.

Our starting data is a complex $C^\bullet$ of $\mathbb{Q}$-vector spaces equipped with a $\mathbb{Q}$-linear endomorphism ϕ. We will think of $C^\bullet$ as a complex in the category $\mathbb{Q}[t]-\mathcal{MOD}$ of $\mathbb{Q}[t]$-modules, where of course t acts as ϕ.

Given $P \in \mathfrak{P}$, let $\mathfrak{V}_P \in \mathbb{Q}[t]-\mathcal{MOD}$ be the cyclic module whose generator, which we denote by 1, has characteristic polynomial P. In other words, it is the 0-th homology of the complex

$$V_P^\bullet : \mathbb{Q}[t] \xrightarrow{\cdot P(t)} \mathbb{Q}[t] \ ,$$

concentrated in degrees -1 and 0, which therefore serves as a projective resolution to $\mathfrak{V}_P$.

Lemma 3.1. *There is a canonical isomorphism*

$$\mathrm{Hom}\,(V_P^\bullet, C^\bullet) \cong \mathrm{MF}\,(P(\phi) : C^\bullet \to C^\bullet) \ ,$$

where Hom *refers to the Hom complex, i.e., the total complex of the double complex of Hom's of the different components.*

Proof. This is fairly clear but we take the opportunity to record our sign convention. For any two complexes $A^\bullet$ and $B^\bullet$ the Hom double complex is the one that has $\mathrm{Hom}\,(A^{-p}, B^q)$ in bidegree (p,q). Given a double complex $D^{p,q}$ with differentials

$$d^{(1,0)} : D^{p,q} \to D^{p+1,q}, \quad d^{(0,1)} : D^{p,q} \to D^{p,q+1} \ ,$$

the differential on the total complex if given by $d = d^{(1,0)}+(-1)^p d^{(0,1)}$. When we identify $\mathrm{Hom}_{\mathbb{Q}[t]-\mathcal{MOD}}(\mathbb{Q}[t], C^i) \cong C^i$ in the natural way we have pullback by multiplication by $P(t)$ transforming to multiplication by $P(\phi)$. The degree i part of $\mathrm{Hom}\,(V_P^\bullet, C^\bullet)$ is therefore

$$\mathrm{Hom}\,(V_P^0, C^i) \oplus \mathrm{Hom}\,(V_P^{-1}, C^{i-1}) = C^i \oplus C^{i-1}$$

and the differential can be checked to be as in Eq. (6).

There is a canonical map $\mathbb{Q}[t] \to V_P^\bullet$ which is given by the identity in degree 0. This map is a lifting to projective resolutions of the projection map $\mathbb{Q}[t] \to \mathfrak{V}_P$. The induced map, $\mathrm{Hom}\,(V_P^\bullet, C^\bullet) \to \mathrm{Hom}\,(\mathbb{Q}[t], C^\bullet) \cong C^\bullet$, can be identified with the canonical map $\mathrm{MF}\,(P(\phi)) \to C^\bullet$.

Recall now from [Bes98, Sect. 3] that one can define both ordinary and quasi-fibered products of complexes. Given complexes $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ with maps f and g from the first two to the third the fibered product is just the complex of componentwise fiber products while the quasi-fibered product is defined as $X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet := \mathrm{MF}\,(f-g)$. Also recall that there is a canonical map $X^\bullet \times_{Z^\bullet} Y^\bullet \to X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet$ taking (x,y) to $(x \oplus y, 0)$, which is a quasi-isomorphism if $f-g$ is surjective. The following lemma is now completely obvious.

Lemma 3.2. *Suppose there is a map of complexes $D^\bullet \xrightarrow{\alpha} C^\bullet$. Then there is a commutative diagram whose rows are isomorphisms*

$$\begin{array}{ccc}
\mathrm{MF}\,(P(\phi)\circ\alpha : D^\bullet \to C^\bullet) & \xrightarrow{\sim} & D^\bullet \times_{C^\bullet} \mathrm{Hom}\,(V_P^\bullet, C^\bullet) \\
\downarrow & & \downarrow \\
D^\bullet \tilde{\times}_{C^\bullet} \mathrm{MF}\,(P(\phi) : C^\bullet \to C^\bullet) & \xrightarrow{\sim} & D^\bullet \tilde{\times}_{C^\bullet} \mathrm{Hom}\,(V_P^\bullet, C^\bullet).
\end{array}$$

Let $P,\ Q \in \mathfrak{P}$ and consider the map $\mathfrak{V}_{PQ} \to \mathfrak{V}_P$ obtained by "reduction mod $P(t)$." This lifts to the map of projective resolutions:

$$\begin{array}{cccc}
V_{PQ}^\bullet : & \mathbb{Q}[t] & \xrightarrow{PQ(t)} & \mathbb{Q}[t] \\
\downarrow & {\scriptstyle \cdot Q(t)}\downarrow & & \| \\
V_P^\bullet : & \mathbb{Q}[t] & \xrightarrow{P(t)} & \mathbb{Q}[t]
\end{array}$$

and one easily checks the following.

Lemma 3.3. *Under the isomorphisms of Lemma* 3.1*, the map of cones induced by*

$$\begin{array}{ccc}
D^\bullet & \xrightarrow{P(\phi)\circ\alpha} & C^\bullet \\
\| & & \downarrow {\scriptstyle Q(\phi)} \\
D^\bullet & \xrightarrow{PQ(\phi)\circ\alpha} & C^\bullet
\end{array}$$

is the fiber product with $D^\bullet$ of the map $\mathrm{Hom}\,(V_P^\bullet, C^\bullet) \to \mathrm{Hom}\,(V_{PQ}^\bullet, C^\bullet)$ *induced by the map $V_{PQ}^\bullet \to V_P^\bullet$ above.*

Proposition 3.4.

1. *Let $P \in \mathfrak{P}$. There are canonical maps*

$$\begin{aligned}
\mathbb{R}\Gamma_{\mathrm{f},P}(X,n) &\longrightarrow F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \times_{\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)} \mathrm{Hom}\,(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)) \\
&\longrightarrow F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \tilde{\times}_{\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)} \mathrm{Hom}\,(V_P^\bullet, \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))\,,
\end{aligned}$$

the first of which is an isomorphism while the second is a quasi-isomorphism.

2. *There are canonical maps*

$$\begin{aligned}
&\mathbb{R}\Gamma_{\mathrm{fp},m}(X,n)\\
&\quad\longrightarrow \varinjlim_{P\in\mathfrak{P}_m} F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\times_{\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)}\mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))\\
&\quad\longrightarrow \varinjlim_{P\in\mathfrak{P}_m} F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\tilde{\times}_{\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)}\mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))\ ,
\end{aligned}$$

where again the first map in an isomorphism and the second a quasi-isomorphism. The transition maps are induced from the natural maps $V_{PQ}^\bullet\to V_P^\bullet$ as described above.

3. *When X is proper the maps* **i** *and* **p** *of the short exact sequence Eq. (8) are induced by the first and second maps of the diagram*

$$\begin{aligned}
&\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)[-1]\\
&\longrightarrow \mathrm{MF}\left(F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\oplus\mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))\to\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)\right)\\
&\longrightarrow F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)
\end{aligned}\tag{15}$$

respectively.

Proof. The first two statements are just a direct application of lemmas 3.1, 3.2 and 3.3. For the last part we have to do an explicit computation. Realizing the complex $\mathrm{Hom}(V_P^\bullet,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))$ as $\mathrm{MF}(P(\varphi^*))$ we see that we can write an element of degree i in the mapping fiber of Eq. (15) as $(x\oplus(y,z),w)$ with x of degree i in $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$ and y, z and w in $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ of degrees i, $i-1$ and $i-1$ respectively. The map to $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$ simply takes this element to x. On the other hand the map to this complex from $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$ takes a pair (x,z) to $(x\oplus(x,z),0)$. The composed map therefore takes (x,z) to x, which is what it should do. For the map **i** we start with a closed element z of degree $i-1$ in $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. We first have to apply the normalization Eq. (13) to obtain $P(\varphi^*)z$. This is sent in $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$ to $(0,P(\varphi^*)z)$ and then to $(0\oplus(0,P(\varphi^*)z),0)$. On the other hand, the first map in Eq. (15) takes z to $(0\oplus(0,0),z)$. We finish by noticing that $(0\oplus(0,P(\varphi^*)z),-z)$ is the boundary of $(0\oplus(z,0),0)$.

4. Cup products

Next we define the product.

Definition 4.1. *Given 2 polynomials $P(t)=\prod_1^{\deg P}(1-\alpha_i t)$, $Q(t)=\prod_1^{\deg Q}(1-\beta_j t)$ in $\mathfrak{P}$, we defined their star product to be*

$$P*Q(t)=\prod_{i,j}(1-\alpha_i\beta_j t)\ .\tag{16}$$

The following lemma summarizes the main properties of the star product.

Lemma 4.2. *The operation $*$ defines a commutative and associative product on $\mathfrak{P}$ and is distributive over the multiplication. In addition*

1. *If P and Q are the characteristic polynomials of finite dimensional operators T and S, then $P * Q$ is the characteristic polynomial of $T \otimes S$.*
2. *There is a canonical map $\mathfrak{V}_{P*Q} \to \mathfrak{V}_P \otimes \mathfrak{V}_Q$ (here, and in what follows, the tensor product is taken over $\mathbb{Q}$ and the ϕ action is the diagonal one) sending the generator 1 to $1 \otimes 1$.*
3. *There is a map of complexes, unique up to homotopy,*

$$V^\bullet_{P*Q} \to V^\bullet_P \otimes V^\bullet_Q \ , \tag{17}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{Q}[t] & \xrightarrow{t \to t\otimes t} & \mathbb{Q}[t] \otimes \mathbb{Q}[t] \\ \downarrow & & \downarrow \\ V^\bullet_{P*Q} & \longrightarrow & V^\bullet_P \otimes V^\bullet_Q . \end{array} \tag{18}$$

4. *In the polynomial ring $\mathbb{Q}[t_1, t_2]$ one can find polynomials $a(t_1, t_2)$ and $b(t_1, t_2)$ satisfying the equation*

$$(P * Q)(t_1 \cdot t_2) = a(t_1, t_2)P(t_1) + b(t_1, t_2)Q(t_2) \ .$$

If $[a_0(t_1, t_2), b_0(t_1, t_2)]$ is one such solution, then the general one is given by

$$[a_0(t_1, t_2), b_0(t_1, t_2)] + R(t_1, t_2) \cdot [Q(t_2), -P(t_1)] \ .$$

5. *We have $\mathfrak{P}_m * \mathfrak{P}_n \subset \mathfrak{P}_{m+n}$.*

Proof. The commutativity, associativity and the distributive law are clear. The fact that $\mathfrak{P}$ is closed under $*$ follows immediately from part 1 which is itself clear and implies part 2 because P and Q are the characteristic polynomials of ϕ on $\mathfrak{V}_P$ and $\mathfrak{V}_Q$ respectively and therefore the minimal polynomial of a generator of $\mathfrak{V}_P \otimes \mathfrak{V}_Q$ is divisible by $P * Q$. Part 3 follows from part 2: Since $V^\bullet_P \otimes V^\bullet_Q$ is a projective resolution of $\mathfrak{V}_P \otimes \mathfrak{V}_Q$ there is clearly a map $V^\bullet_{P*Q} \to V^\bullet_P \otimes V^\bullet_Q$ which is a resolution of $\mathfrak{V}_{P*Q} \to \mathfrak{V}_P \otimes \mathfrak{V}_Q$. As the map $\mathbb{Q}[t] \to V_P$ is simply the inclusion of the degree 0 part, the commutativity of diagram Eq. (18) is equivalent to the fact that we may fix the map of resolutions to be the first line of the diagram in degree 0, which in turn follows from the commutativity of

$$\begin{array}{ccc} \mathbb{Q}[t] & \xrightarrow{t \to t\otimes t} & \mathbb{Q}[t] \otimes \mathbb{Q}[t] \\ \downarrow & & \downarrow \\ \mathfrak{V}_{P*Q} & \longrightarrow & \mathfrak{V}_P \otimes \mathfrak{V}_Q . \end{array}$$

To prove part 4 we identify $\mathbb{Q}[t] \otimes \mathbb{Q}[t]$ with $\mathbb{Q}[t_1, t_2]$ in the obvious way. In particular $\mathbb{Q}[t_1, t_2]$ is given the structure of a $\mathbb{Q}[t]$-module where t acts via multiplication by $t_1 \cdot t_2$. Now the proof is simply to write down explicitly the statement of part 3 in these terms: It says that there is a map

$$\begin{tikzcd}
 & -2 & -1 & 0 \\
V^\bullet_{P*Q}: & 0 \arrow[r] \arrow[d] & \mathbb{Q}[t] \arrow[r, "\cdot P*Q(t)"] \arrow[d] & \mathbb{Q}[t] \arrow[d, "f(t)\to f(t_1t_2)"'] \\
V^\bullet_P \otimes V^\bullet_Q: & \mathbb{Q}[t_1,t_2] \arrow[r, "\cdot(P(t_1),-Q(t_2))"] & \mathbb{Q}[t_1,t_2]\oplus\mathbb{Q}[t_1,t_2] \arrow[r] & \mathbb{Q}[t_1,t_2],
\end{tikzcd}$$

where the bottom right map is $(f,g) \mapsto (f\cdot Q(t_2)+g\cdot P(t_1))$. Let $(b(t_1,t_2), a(t_1,t_2))$ be the image of 1 under the degree -1 map. Then it is immediate to check that $a(t_1,t_2)$ and $b(t_1,t_2)$ satisfy the required relation. The degree of freedom corresponds to a homotopy between degrees -1 and -2. Finally, Part 5 is obvious.

We now complete the proof of Proposition 2.5. The product on $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ together with the map Eq. (17) induce a map

$$\operatorname{Hom}(V^\bullet_P, \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)) \otimes \operatorname{Hom}(V^\bullet_Q, \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)) \to \operatorname{Hom}(V^\bullet_{F*Q}, \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)) .$$

These are compatible with the product on $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ via the maps

$$\operatorname{Hom}(V^\bullet_?, \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$$

with ? = P, Q or $P*Q$ because of the commutativity of diagram Eq. (18). Since the products on $\mathbb{R}\Gamma_{\mathrm{rig}}$ and $\mathbb{R}\Gamma_{\mathrm{dR}}$ are also compatible we can use the construction of cup products on cones (see for example [Bes98, Lemma 3.2]) to obtain a product, for fixed P and Q,

$$\mathbb{R}\Gamma_{\mathrm{f},P}(X,n) \otimes \mathbb{R}\Gamma_{\mathrm{f},Q}(X,m) \to \mathbb{R}\Gamma_{\mathrm{f},PQ}(X,m+n) .$$

To construct the product on H_{fp} it is enough to show, in view of part 5 of the lemma, that these products are compatible up to homotopy under the transition maps. This follows from the commutativity up to homotopy of the diagram

$$\begin{tikzcd}
V^\bullet_{(P\cdot P')*Q} \arrow[r, equal] \arrow[d] & V^\bullet_{(P*Q)\cdot(P'*Q)} \arrow[d] \\
V^\bullet_{(P\cdot P')} \otimes V^\bullet_Q \arrow[r] & V^\bullet_P \otimes V^\bullet_Q,
\end{tikzcd}$$

which in turn follows because this diagram lifts a similar commutative diagram with $\mathfrak{V}$ in place of $V^\bullet$. To check this commutativity it is enough to note that $\mathfrak{V}_{(P\cdot P')*Q}$ is cyclic and to see that its standard generator 1 goes to $1\otimes 1 \in \mathfrak{V}_P \otimes \mathfrak{V}_Q$ in both possible ways. The final statement to be checked, regarding the induced product on the graded object associated with the filtration Fil defined in part 1 of Proposition 2.5 is an immediate consequence of the description of the maps **i** and **p** in Proposition 3.4 and of [Bes98, Lemma 3.2]

Remark 4.3. One can give the following explicit description of the cup product: Recall that we are guaranteed the existence of polynomials $a(t_1,t_2)$ and $b(t_1,t_2)$ such that $(P*Q)(t_1\cdot t_2) = a(t_1,t_2)P(t_1) + b(t_1,t_2)Q(t_2)$. Let such a polynomial act on $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)\otimes\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ by letting t_1 act as $\varphi^*\otimes\mathrm{id}$ and t_2 as $\mathrm{id}\otimes\varphi^*$. As in Eq. (6) represent an element of $\mathbb{R}\Gamma_{\mathrm{f},P}(X,n)$ by a pair (x,y) with $x\in F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$

and $y \in \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ of degrees i and $i-1$ respectively. Then the cup product can be given by the formula

$$(x_1, y_1) \cup (x_2, y_2) = \left(x_1 \cup x_2, (-1)^{\deg x_1} \bigcup (b(t_1, t_2)(x_1 \otimes y_2) + a(t_1, t_2)(y_1 \otimes x_2)) \right) \ .$$

One can reconstruct from this description the cup product on the syntomic and modified syntomic complexes of [Bes98]. In particular, one obtains a conceptual proof of the compatibility of the cup products in the proof of part 5 of Proposition 8.6 in loc. cit.

5. Pushforwards and cycle classes

Let X and Y be smooth connected and proper over $\mathcal{V}$ and let $f : X \to Y$ be an arbitrary morphism. Using Poincaré duality we obtain a pushforward map

$$f_* : H^i_{\mathrm{fp}}(X, n) \to H^{i+2d}_{\mathrm{fp}}(Y, n+d) \ ,$$

with $d = \mathrm{reldim}\, Y - \mathrm{reldim}\, X$. For completely formal reasons we have the projection formula

$$f_*(x \cdot f^* y) = f_*(x) \cdot y \tag{19}$$

For a morphism f as above.

Lemma 5.1. *The pushforward map is compatible with pushforward in de Rham cohomology via the map* $\mathbf{p}$*. In other words, there exists a commutative diagram*

$$\begin{array}{ccc} H^i_{\mathrm{fp}}(X, n) & \xrightarrow{f_*} & H^{i+2d}_{\mathrm{fp}}(Y, n+d) \\ \downarrow{\scriptstyle \mathbf{p}} & & \downarrow{\scriptstyle \mathbf{p}} \\ F^n H^i_{\mathrm{dR}}(X_K/K) & \xrightarrow{f_*^{\mathrm{dR}}} & F^{n+d} H^{i+2d}_{\mathrm{dR}}(Y_K/K). \end{array}$$

Proof. Let $x \in H^i_{\mathrm{fp}}(X, n)$ and $y \in H^{2\mathrm{reldim}\, X - i}_{\mathrm{dR}}(Y_K/K)$. Then we have

$$\begin{aligned} \langle \mathbf{p} f_* x, y \rangle_{\mathrm{dR}} &\overset{Eq.\ (14)}{=} \langle f_* x, \mathbf{i} y \rangle_{\mathrm{fp}} = \langle x, f^* \mathbf{i} y \rangle_{\mathrm{fp}} \\ &= \langle x, \mathbf{i} f^*_{\mathrm{dR}} y \rangle_{\mathrm{fp}} && \text{by functoriality of Eq. (8)} \\ &\overset{Eq.\ (14)}{=} \langle \mathbf{p} x, f^*_{\mathrm{dR}} y \rangle_{\mathrm{dR}} = \langle f_*^{\mathrm{dR}} \mathbf{p} x, y \rangle_{\mathrm{dR}} \ , \end{aligned}$$

which shows that $\mathbf{p} f_*(x) = f_*^{\mathrm{dR}} \mathbf{p}(x)$.

When $i = 0$ and f is the embedding into Y of the closed smooth scheme X of codimension d we may apply f_* to the canonical element

$$1_X \in H^0_{\mathrm{dR}}(X_K/K) \cong H^0_{\mathrm{fp}}(X, 0)$$

and obtain a cycle class $c\ell_{\mathrm{fp}}(X) \in H^{2d}_{\mathrm{fp}}(Y, d)$. By definition, this cycle class is characterized by the property

$$\langle c\ell_{\mathrm{fp}}(X), y \rangle_{\mathrm{fp}} = \mathrm{tr}_X(f^* y) \tag{20}$$

for all $y \in H^{2\mathrm{reldim}\, X+1}_{\mathrm{fp}}(Y, \mathrm{reldim}\, X + 1)$.

We want to compare this cycle class with a cycle class obtained via syntomic regulators in K-theory. These have been constructed by Gros [Gro90] and Niziol [Niz97] but we will use the version of [Bes98]. We first recall the essential information.

Let X be a noetherian and separated scheme. Let $K_0(X)$ (resp. $K_0'(X)$) be the Grothendieck group of the category of locally free finite rank (resp. coherent) sheaves on X. The group $K_0'(X)$ has a canonical filtration by codimension of support which is called the *topological filtration* and denoted $F_{\mathrm{top}}^{\bullet}$.

Suppose now that X is regular. Then it is well known that $K_0(X)$ and $K_0'(X)$ are canonically isomorphic, which implies that there is also a canonical topological filtration on $K_0(X)$. We may describe this filtration directly using K-theory with support (see [Sou92, I3.2]): Denote by $K_0^Z(X)$, for $Z \subset X$ a closed subscheme, the 0-th K-group of X with support in Z (see loc. cit.). Then

$$F_{\mathrm{top}}^j = \bigcup_{Z \subset X,\ \mathrm{codim}\, Z \geq j} \mathrm{Im}\, \left(K_0^Z(X) \to K_0^X(X) = K_0(X)\right) \ .$$

The connection with Chow groups is given by the following theorem:

Theorem 5.2. *(*[GS87, Theorem 8.2] or [Sou92, I, Thm 3]) *If X is regular and of finite dimension, then there is a canonical isomorphism*

$$CH^j(X) \otimes \mathbb{Q} \cong \mathrm{Gr}_{\mathrm{top}}^j K_0(X) \otimes \mathbb{Q}$$

given by $Z \to i_*^Z \mathcal{O}_Z$ with $i^Z : Z \to X$ the canonical injection.

When X is regular, $K_0(X)$ has the structure of a λ-ring (see [Sou92, I Theorem 4]). Then $K_0(X)$ acquires a filtration, called the γ-filtration. It follows from [GS87, Proposition 5.3] that the graded pieces of the topological and γ-filtrations agree up to torsion.

Suppose now that X is smooth over a DVR $\mathcal{V}$. By [Bes98, Sect. 7] there exists a theory of Chern classes $c_j^{\mathrm{syn}} : K_0(X) \to H_{\mathrm{syn}}^{2j}(X, j)$. Similarly, when κ is algebraic over the prime field there is a theory of modified syntomic Chern classes which is compatible with the syntomic ones under the map Ξ. We can define fp Chern classes by composing the modified syntomic ones with the map Eq. (9),

$$c_j^{\mathrm{fp}} : K_0(X) \to H_{\mathrm{fp}}^{2j}(X, j) \ . \tag{21}$$

As usual we have a projective bundle theorem.

Lemma 5.3. *Let X be a proper smooth $\mathcal{V}$-scheme, E/X a locally free sheaf of rank $d+1$ over X, $Y = \mathbb{P}(E) \xrightarrow{\pi} X$ the projective bundle over X. Then the map*

$$\begin{aligned} \bigoplus_{i=0}^{d} H_{\mathrm{fp}}^{2r-2i}(X, r-i) \quad &\to \quad H_{\mathrm{fp}}^{2r}(Y, r) \\ x \in H_{\mathrm{fp}}^{2r-2i}(X, r-i) \quad &\mapsto \quad \pi^* x \cdot c_1^{\mathrm{fp}}(\mathcal{O}_Y(1))^i \end{aligned}$$

is an isomorphism.

Proof. Let $j = r - i$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \bigoplus_{i=0}^{d} H_{\mathrm{dR}}^{2j-1}(X_K/K)/F^j & \to & \bigoplus_{i=0}^{d} H_{\mathrm{fp}}^{2j}(X,j) & \to & \bigoplus_{i=0}^{d} F^j H_{\mathrm{dR}}^{2j}(X_K/K) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & H_{\mathrm{dR}}^{2r-1}(Y_K/K)/F^r & \to & H_{\mathrm{fp}}^{2r}(Y,r) & \to & F^r H_{\mathrm{dR}}^{2r}(Y_K/K) & \to 0
\end{array}$$

By Proposition 2.5.3 the two maps on the side are given by cup product with the appropriate power of $c_1^{\mathrm{dR}}(\mathcal{O}_Y(1))$. These maps are isomorphisms by the projective bundle theorem in de Rham cohomology so we are done by the five lemma.

From the argument of [GM87, Theorem 4.3.1] we obtain the following:

Proposition 5.4. *If X is smooth over a DVR $\mathcal{V}$, then there exist cycle class maps:*

$$\eta_X^{\mathrm{syn}} : CH^*(X) \to H_{\mathrm{syn}}^{2*}(X,*) ,$$

and if further the residue field κ of $\mathcal{V}$ is algebraic over the prime field then there exist modified syntomic and finite polynomial cycle class maps

$$\eta_X^{\mathrm{ms}} : CH^*(X) \to H_{\mathrm{ms}}^{2*}(X,*), \qquad \eta_X^{\mathrm{fp}} : CH^*(X) \to H_{\mathrm{fp}}^{2*}(X,*) ,$$

with the obvious compatibilities satisfied, defined by

$$\eta_X^{?}[Y] := \frac{(-1)^{j-1}}{(j-1)!} c_j^{?}(\mathcal{O}_Y) ,$$

with $? = \mathrm{syn}$ *(resp.* $= \mathrm{ms}$, $= \mathrm{fp}$*) and Y irreducible of codimension j.*

Following closely [GM87], we are able to prove the following theorem

Theorem 5.5. *Let $\mathcal{V}$ be a DVR with a residue field algebraic over the prime field and let X and Y be smooth, connected and proper schemes over $\mathcal{V}$. Let $f : X \to Y$ be a projective morphism. Then the following diagram commutes:*

$$\begin{array}{ccc}
CH^*(X) & \xrightarrow{\eta_X} & H_{\mathrm{fp}}^{2*}(X,*) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
CH^*(Y) & \xrightarrow{\eta_Y} & H_{\mathrm{fp}}^{2*}(Y,*).
\end{array}$$

The proof of this theorem will be given in Sect. 8. Here we collect a number of corollaries:

Corollary 5.6. *Suppose under the same assumptions on $\mathcal{V}$ that X is smooth and proper over $\mathcal{V}$ and that $Y \subset X$ is a closed irreducible subscheme which is also smooth over $\mathcal{V}$. Then $c\ell_{\mathrm{fp}}(Y) = \eta^{\mathrm{fp}}(Y)$.*

Proof. This is clear from the definition of $c\ell_{\mathrm{fp}}$ and from the obvious fact that $\eta_Y^{\mathrm{fp}}[Y] = 1_Y$.

Proof of Theorem 1.2. Let $Z = \sum n_l(Z_l)$ be a cycle of relative dimension j with smooth components, let $\omega \in F^{j+1}H^{2j+1}_{\mathrm{dR}}(X_K/K)$ and choose $\tilde{\omega} \in H^{2j+1}_{\mathrm{fp}}(X, j+1)$ such that $\mathbf{p}(\tilde{\omega}) = \omega$ as in the statement of the theorem. Let i_l be the inclusion of Z_l into X. We have

$$\begin{aligned}
\int_Z \omega &= \sum n_l \operatorname{tr}_{Z_l} i_l^*(\tilde{\omega}) \text{ by Eq. (4)} \\
&= \sum n_l \langle c\ell_{\mathrm{fp}}(Z_l), \tilde{\omega}\rangle_{\mathrm{fp}} \text{ by Eq. (20)} \\
&= \langle c\ell_{\mathrm{fp}}(Z), \tilde{\omega}\rangle_{\mathrm{fp}} \\
&= \langle \eta_X^{\mathrm{fp}}(Z), \tilde{\omega}\rangle_{\mathrm{fp}} \text{ by Corollary 5.6.}
\end{aligned}$$

Consider now the commutative diagram in Proposition 2.5.2. Since Z is de Rham homologically equivalent to 0 we have $\mathbf{p}_{\mathrm{syn}}(\eta_X^{\mathrm{syn}}(Z)) = 0$. By definition, the syntomic Abel-Jacobi map sends Z to $\mathbf{i}_{\mathrm{syn}}^{-1}(\eta_X^{\mathrm{syn}}(Z)) = \mathbf{i}_{\mathrm{fp}}^{-1}(\eta_X^{\mathrm{fp}}(Z)) \in H^{2i-1}_{\mathrm{dR}}(X_K/K)/F^i$, with $i + j = \operatorname{reldim} X$ as usual. It now follows from Eq. (14) that

$$\int_Z \omega = \langle \mathbf{i}_{\mathrm{syn}}^{-1}(\eta_X^{\mathrm{syn}}(Z)), \omega\rangle_{\mathrm{dR}} \ .$$

6. Localized cup products in cohomology

As we will see in Sect. 8, the proof of [GM87] transforms quite well to the fp setup. The one additional ingredient required is provided by a localized cup product, which may be of interest in itself, and which we describe in the following two sections.

We begin by discussing the formal properties we expect of a localized cup product and the consequences that these have. We assume we are given a cohomology theory for schemes, $H^i(X, n)$, taking values in K-vector spaces, for a fixed field K, and that for a closed embedding $i : Y \hookrightarrow X$ we are also able to define a cohomology theory with supports $H^i_Y(X, n)$ such that

(S1) There exists a canonical map $H^i_Y(X, n) \xrightarrow{\mathfrak{g}} H^i(X, n)$.
(S2) The groups $H^i_Y(X, n)$ are contravariant with respect to cartesian maps of pairs,

$$\begin{tikzcd}
Y' \arrow[r, "i'"] \arrow[d, "f_Y"'] & X' \arrow[d, "f"] \\
Y \arrow[r, "i"] & X,
\end{tikzcd} \tag{22}$$

as is the map $\mathfrak{g}$.

Suppose that our cohomology theory has cup products. By a *localized cup product* for an embedding $i : Y \hookrightarrow X$ we mean a product $H^i(Y, n) \times H^j_Y(X, m) \to H^{i+j}_Y(X, n+m)$ satisfying the following properties:

(L1) The diagram

$$\begin{array}{ccc} H^i(X,n)\times H^j_Y(X,m) & \xrightarrow{i^*\times 1} & H^i(Y,n)\times H^j_Y(X,m) \\ \Big\downarrow{\scriptstyle 1\times\mathfrak{g}} & & \Big\downarrow{\scriptstyle\cup} \\ H^i(X,n)\times H^j(X,m) & & H^{i+j}_Y(X,n+m) \\ \Big\downarrow{\scriptstyle\cup} & & \Big\downarrow{\scriptstyle\mathfrak{g}} \\ H^{i+j}(X,n+m) & = & H^{i+j}(X,n+m) \end{array} \tag{23}$$

commutes.

(L2) The product $H^*(Y,*)\times H^*(Y,*)\times H^*_Y(X,*)\to H^*_Y(X,*)$ is independent of the order of multiplication.

(L3) The cup product is functorial with respect to cartesian maps of pairs of the form Eq. (22), i.e, in the situation described there one has $f^*(x\cup y)=f^*_Y(x)\cup f^*(y)$ for $x\in H^*(Y,*)$ and $y\in H^*_Y(X,*)$.

Suppose now that we have in addition a Poincaré duality in our cohomology theory. This would mean that when X is connected there is a trace map $\mathrm{tr}_X : H^{i_X}(X,n_X)\to K$, where i_X and n_X denotes certain indices depending on the dimension of X, and that the composition of the trace map and the cup product gives a perfect pairing between cohomology theories in complimentary dimensions.

If Poincaré duality exists, a (strong) fundamental class is a rule which associates to $i : Y\hookrightarrow X$ a cohomology class $c\ell_Y\in H^{i_X-i_Y}_Y(X,n_X-n_Y)$ such that the following properties are satisfied:

(P1) The diagram

$$\begin{array}{ccc} H^{i_Y}(Y,n_Y) & \xrightarrow{\cup c\ell_Y} & H^{i_X}_Y(X,n_X) \\ \Big\downarrow{\scriptstyle\mathrm{tr}_Y} & & \Big\downarrow{\scriptstyle\mathfrak{g}} \\ K & \xleftarrow{\mathrm{tr}_X} & H^{i_X}(X,n_X) \end{array}$$

commutes.

(P2) The formation of $c\ell_Y$ is functorial with respect to the cartesian diagram Eq. (22).

Sometimes we will be in a situation where we have cup products and Poincaré duality only for certain schemes, but the formation of the fundamental class exists for more general schemes and continues to satisfy property (P2). In these cases we will call $\mathfrak{g}(c\ell_Y)$ a *weak* fundamental class for Y in X

An easy consequence of the properties described above is the following proposition.

Proposition 6.1. *If the cohomology theory H has a localized cup product, Poincaré duality and admits fundamental classes, then*

1. *Given $i : Y\hookrightarrow X$, The maps $i_* : H^j(Y,m)\to H^{j+i_X-i_Y}(X,m+n_X-n_Y)$ adjoint to the pullback maps $i^* : H^i(X,n)\to H^i(Y,n)$ (Gysin map) are given by the composition*

$$H^j(Y,m)\xrightarrow{\cup c\ell_Y} H^{j+i_X-i_Y}_Y(X,m+n_X-n_Y)\xrightarrow{\mathfrak{g}} H^{j+i_X-i_Y}(X,m+n_X-n_Y)\,.$$

2. *In the situation of diagram Eq. (22) the following diagram commutes*

$$\begin{tikzcd} H^j(Y,m) \arrow[r, "i_*"] \arrow[d, "f_Y^*"'] & H^{j+i_X-i_Y}(X, j+n_X-n_Y) \arrow[d, "f^*"] \\ H^j(Y',m) \arrow[r, "i'_*"] & H^{j+i_X-i_Y}(X', j+n_X-n_Y). \end{tikzcd}$$

Remark 6.2. In fact, the result of 1. requires only the Poincaré duality for X and Y, the other properties of the cohomology theory only for the injection $Y \hookrightarrow X$ and does not require functoriality.

Proof. The second part follows from the first using the functoriality of the fundamental class (P2) and of the localized cup product (L3). To prove the first part we have to show that for any two classes $y \in H^j(Y,m)$ and $x \in H^{i_Y-j}(X, n_Y-m)$ we have the formula

$$\operatorname{tr}_Y(i^*x \cup y) = \operatorname{tr}_X(x \cup \mathfrak{g}(y \cup c\ell_Y)) .$$

This is derived as follows.

$$\begin{aligned} \operatorname{tr}_X(x \cup \mathfrak{g}(y \cup c\ell_Y)) &= \operatorname{tr}_X(\mathfrak{g}(i^*x \cup (y \cup c\ell_Y))) && \text{by (L1)} \\ &= \operatorname{tr}_X(\mathfrak{g}((i^*x \cup y) \cup c\ell_Y)) && \text{by (L2)} \\ &= \operatorname{tr}_Y(i^*x \cup y) \qquad \text{by (P1).} \end{aligned}$$

7. Localized cup products for Rigid, de Rham and syntomic cohomologies

In this section we show that syntomic and finite polynomial cohomologies satisfy the formal properties described in the previous section, at least in some situations.

Proposition 7.1. *In syntomic, modified syntomic and finite polynomial cohomologies there is a theory of cohomology with supports, $H_Y^*(X,*)$. In all theories there exists a construction of weak fundamental classes, $c\ell_Y \in H_Y^{2d}(X,d)$, with Y of codimension d in X.*

Proof. As usual, one can define, for $Y \subset X$ closed, the cohomology of X with support in Y as the cohomology of X relative to $X-Y$, i.e.,

$$H^i_{Y,?}(X,n) = H^i(\mathrm{MF}\,(\mathbb{R}\Gamma_?(X,n) \to \mathbb{R}\Gamma_?(X-Y,n))) , \tag{24}$$

with ? denoting syn, ms or fp. The formal properties (S1) and (S2) are clear from this construction (one should maybe remark that in fp-cohomology the map $\mathfrak{g}$ does not fit into a long exact because the weight of the polynomials used changes with the degree of the cohomology). One can check that also rigid and de Rham cohomologies with support can be defined in a similar fashion. We do the construction of the fundamental class in the fp situation, which is the one we need. The syntomic and ms cases are handled similarly. It is formal to check that for each $P \in \mathfrak{P}$ we have the long exact sequence, analogous to Eq. (12),

$$\begin{aligned}\cdots \quad &\to \quad F^n H^{i-1}_{\mathrm{dR},Y_K}(X_K/K) \xrightarrow{P(\varphi^*)} H^{i-1}_{\mathrm{rig},Y_\kappa}(X_\kappa/K) \to H^i_{\mathrm{f},P,Y}(X,n)\\ &\to \quad F^n H^{i}_{\mathrm{dR},Y_K}(X_K/K) \xrightarrow{P(\varphi^*)} H^{i}_{\mathrm{rig},Y_\kappa}(X_\kappa/K) \to \cdots \ .\end{aligned}$$

Specializing now to $n = d$, $i = 2d$ with d the codimension of Y, we can use purity results. In particular, by [Ber97b, Corollaire 5.7] we have $H^{2d-1}_{\mathrm{rig},Y_\kappa}(X_\kappa/K) = 0$. Purity for de Rham cohomology tells us that $H^0_{\mathrm{dR}}(Y_K/K) \cong H^i_{\mathrm{dR},Y_K}(X_K/K)$ and the image of 1 under this isomorphism, which is the de Rham fundamental class, belongs to F^d. This image will give us our desired class if we can show that it is killed by $P(\varphi^*)$ for sufficiently divisible P because this class will be compatible with cartesian diagrams by the corresponding compatibility of the de Rham fundamental class. Thus, the following lemma finishes the proof.

Lemma 7.2. *Let $Y \subset X$ be a closed immersion of codimension d of schemes over κ. Assume that the Frobenius endomorphism φ of degree q stabilizes Y. Then φ^* acts on $H^{2d}_{\mathrm{rig},Y}(X/K)$ by q^d.*

Proof. The cup product map,

$$H^{2d}_{\mathrm{rig},Y}(X/K) \otimes H^{2\dim Y}_{c,\mathrm{rig}}(Y/K) \to H^{2\dim X}_{c,\mathrm{rig}}(X/K)$$

[Ber97a], is equivariant with respect to φ^* and is an isomorphism of 1-dimensional K-vector spaces. Thus, the lemma follows from the fact that φ^* acts on $H^{2\dim Y}_{c,\mathrm{rig}}(Y/K)$ by $q^{\dim Y}$ and on $H^{2\dim X}_{c,\mathrm{rig}}(X/K)$ by $q^{\dim X}$ as follows from [ClS98, 2.5].

There are also cup products for cohomology with supports

Proposition 7.3. *For $Y_1, Y_2 \subset X$ closed we have a cup product*

$$H^i_{?,Y_1}(X,n) \times H^j_{?,Y_2}(X,m) \to H^{i+j}_{?,Y_1\cap Y_2}(X,n+m)\ ,$$

where ? is any of syntomic, modified syntomic or finite polynomial, satisfying the relation $\mathfrak{g}(x \cup y) = \mathfrak{g}(x) \cup \mathfrak{g}(y)$.

Proof. From the description Eq. (24) of cohomology with support we can deduce this from the following purely abstract lemma.

Lemma 7.4. *Let X, U_1 U_2 and U_{12} be 4 complexes fitting into a commutative diagram*

$$\begin{tikzcd} & U_1 \arrow[dr, "|_{U_{12}}"] & \\ X \arrow[ur, "|_{U_1}"] \arrow[dr, "|_{U_2}"'] & & U_{12} \\ & U_2 \arrow[ur, "|_{U_{12}}"'] & \end{tikzcd}$$

Consider the map $U_1 \oplus U_2 \to U_{12}$ given by $(x,y) \mapsto (x-y)|_{U_{12}}$, and the map $X \to MF(U_1 \oplus U_2 \to U_{12})$ given by $x \mapsto (x,x,0)$. Suppose there are compatible cup products on all 4 complexes. Then there is a cup product

$$MF(X \to U_1) \times MF(X \to U_2) \to MF(X \to MF(U_1 \oplus U_2 \to U_{12}))$$

which is compatible with the product on X.

Proof. The target of the cup product is given in degree i by $X^i \oplus U_1^{i-1} \oplus U_2^{i-1} \oplus U_{12}^{i-2}$ with differential $d(x, u_1, u_2, u_{12}) = (dx, x|_{U_1} - du_1, x|_{U_2} - du_2, du_{12} - u_1|_{U_{12}} + u_2|_{U_{12}}$. The cup product is now given by the formula

$$(x_1, u_1) \cup (x_2, u_2) = (x_1 \cup x_2, u_1 \cup x_2|_{U_1}, (-1)^{\deg x_1} x_1|_{U_2} \cup u_2, -(-1)^{\deg x_1} u_1 \cup u_2) .$$

It is immediate that this product is compatible with the one on X. The relevant formula for checking that it is indeed a product, $d((x_1, u_1) \cup (x_2, u_2)) = d(x_1, u_1) \cup (x_2, u_2) + (-1)^{\deg x_1}(x_1, u_1) \cup d(x_2, u_2)$, is easily checked.

The lemma directly applies to our situation because for each choice of a cohomology theory for ? we have, setting $U_i = X - Y_i$, quasi-isomorphism $\mathbb{R}\Gamma_?(U_1 \cup U_2, *) \xrightarrow{\sim} MF(\mathbb{R}\Gamma_?(U_1, *) \oplus \mathbb{R}\Gamma_?(U_2, *) \to \mathbb{R}\Gamma_?(U_1 \cap U_2, *)$. This was checked in [Bes98, Proposition 6.2] for syntomic complexes, but the same proof works in all situations: Every complex is an iterated cone with rigid and de Rham component and the map on each component is a quasi-isomorphism.

The following is the main result of this section.

Proposition 7.5. *Suppose that in the situation of the previous section Y and X are smooth and proper schemes over $\mathcal{V}$ and that i embeds Y as a divisor in X. Then there exist localized cup product satisfying properties* (L1)–(L3) *for both syntomic and finite polynomial cohomology. Furthermore, this cup product is compatible with localized cup products in de Rham cohomology. For the finite polynomial cohomology the weak fundamental class is also a strong fundamental class in this situation, i.e, it satisfies property* (P1).

Using the previous section we obtain immediately

Corollary 7.6. *Suppose we are given a cartesian diagram of the form Eq. (22), with X, Y, X' and Y' are all smooth and proper over $\mathcal{V}$ and where Y and Y' are of codimension* 1 *in X and X' respectively. Then the diagram*

$$\begin{tikzcd} H^j_{\mathrm{fp}}(Y, m) \arrow[r, "i_*"] \arrow[d, "f^*_{Y'}"'] & H^{j+2}_{\mathrm{fp}}(X, j+1) \arrow[d, "f^*"] \\ H^j_{\mathrm{fp}}(Y', m) \arrow[r, "i'_*"] & H^{j+2}_{\mathrm{fp}}(X', j+1) \end{tikzcd}$$

commutes.

Both syntomic cohomology and finite polynomial cohomology are glued from rigid and de Rham cohomologies. What we therefore need to do is first define the localized cup products on these cohomologies. To glue we will have to get them on the level of complexes of sheaves

One can define the localized cup product in rigid cohomology without codimension assumptions. To simplify matters we will still assume that we are dealing with proper schemes.

At this point we will finally need to recall the construction of the rigid complexes, in a somewhat simplified version as we assume our schemes are proper. Let $i : Y \to X$ be a closed embedding of proper κ-schemes of finite type. We choose a closed embedding $X \to \hat{\mathcal{P}}$ into a smooth $\mathcal{V}$-formal scheme $\hat{\mathcal{P}}$. This embedding then also defines an embedding for Y. Of course, there are more embeddings to Y but we will restrict to those given above. For every $Z \subset X$, the tube $]Z[_{\hat{\mathcal{P}}}$ is the rigid analytic space of points in the analytic space associated to $\hat{\mathcal{P}}$ reducing to a point of Z. Let $W = X - Y$ and let $j : W \to X$ be the canonical embedding. Then Berthelot defines a functor $j^\dagger$ from the category of sheaves on $]X[_{\hat{\mathcal{P}}}$ to itself by

$$j^\dagger(\mathfrak{F}) = \lim_{\substack{\to \\ V}} j_{V*}\mathfrak{F} \ ,$$

where the direct limit is over all V which are strict neighborhoods of $]W[_{\hat{\mathcal{P}}}$ in $]X[_{\hat{\mathcal{P}}}$ and j_V is the canonical embedding in of V in $]X[_{\hat{\mathcal{P}}}$. Recall that $V \supset]W[_{\hat{\mathcal{P}}}$ is a strict neighborhood of $]W[_{\hat{\mathcal{P}}}$ if it is open and $\{V,]Y[_{\hat{\mathcal{P}}}\}$ is an admissible covering of $]X[_{\hat{\mathcal{P}}}$. Berthelot shows that the canonical map $\mathfrak{F} \to j^\dagger(\mathfrak{F})$ is surjective and defines $\underline{\Gamma}^\dagger_{]Y[}(\mathfrak{F})$, called the sheaf of sections of $\mathfrak{F}$ with overconvergent support on Y, as its kernel.

Lemma 7.7. *The functors $j^\dagger$ and $\underline{\Gamma}^\dagger_{]Y[}$ satisfy the following properties.*

1. (Berthelot [Ber96, 2.1.3(iii) and 2.1.6] or [Ber97b, 1.2]) *They are exact.*
2. *They send flasque (in particular injective) sheaves to $\Gamma(]X[_{\hat{\mathcal{P}}}, ?)$-acyclics.*

Proof. A presheaf $\mathfrak{F}$ is a flasque sheaf if for every covering $\mathcal{U} = \{U_i\}$ of every open U the Čech cohomology $\check{H}^i(\mathcal{U}, \mathfrak{F})$ vanishes for $i > 0$ and equals $\mathfrak{F}(U)$ for $i = 0$. Consider now the presheaf $\mathfrak{G}(U) = \lim_{\to V} \mathfrak{F}(U \cap V)$ where the limit is over all strict neighborhoods V of $]W[_{\hat{\mathcal{P}}}$ in $]X[_{\hat{\mathcal{P}}}$. By the exactness of filtered direct limits it follows that $\mathfrak{G}$ is a flasque sheaf as well. Since it is a sheaf it is clearly equal to $j^\dagger\mathfrak{F}$. In particular, $j^\dagger\mathfrak{F}$ is $\Gamma(]X[_{\hat{\mathcal{P}}}, ?)$-acyclic. Also, by flasqueness again it follows that $\mathfrak{F}(]X[_{\hat{\mathcal{P}}}) \to \mathfrak{G}(]X[_{\hat{\mathcal{P}}})$ is surjective. Therefore, the long exact cohomology sequence for

$$0 \to \underline{\Gamma}^\dagger_{]Y[}\mathfrak{F} \to \mathfrak{F} \to j^\dagger\mathfrak{F} \to 0$$

shows that $\underline{\Gamma}^\dagger_{]Y[}\mathfrak{F}$ is acyclic as well.

Definition 7.8. *The rigid complex of X over K with respect to $\hat{\mathcal{P}}$ is the complex*

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\hat{\mathcal{P}}} := \mathbb{R}\Gamma(]X[_{\hat{\mathcal{P}}}, \Omega^\bullet_{]X[_{\hat{\mathcal{P}}}}) \ .$$

The rigid complex of X with support in Y is the complex

$$\mathbb{R}\Gamma_{\mathrm{rig},Y}(X/K)_{\hat{\mathcal{P}}} := \mathbb{R}\Gamma(]X[_{\hat{\mathcal{P}}}, \underline{\Gamma}^\dagger_{]Y[}\Omega^\bullet_{]X[_{\hat{\mathcal{P}}}}) \ .$$

Theorems of Berthelot show that this is independent of the choice of $\hat{\mathcal{P}}$ and that the theory is functorial. It is clear that rigid cohomology, defined as the cohomology of the rigid complexes, satisfies properties (S1) and (S2). From the description of rigid complexes in general [Bes98, Sect. 4] it follows that $\mathbb{R}\Gamma_{\mathrm{rig},Y}(X/K)_{\hat{\mathcal{P}}} = \mathrm{MF}(\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\hat{\mathcal{P}}} \to \mathbb{R}\Gamma_{\mathrm{rig}}(W/K)_{\hat{\mathcal{P}}})$.

We do not get the product directly on the rigid complexes but rather on quasi-isomorphic ones. This is irrelevant for the cohomology but we need to embed this product afterwards in a product of cones so we need to be careful to work on the level of complexes.

Since $]Y[_{\hat{\mathcal{P}}}$ is open in $]X[_{\hat{\mathcal{P}}}$ we have a quasi-isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig}}(Y/K)_{\hat{\mathcal{P}}} = \mathbb{R}\Gamma(]Y[_{\hat{\mathcal{P}}}, \Omega^\bullet_{]Y[_{\hat{\mathcal{P}}}}) \xleftarrow{\sim} \mathbb{R}\Gamma(]Y[_{\hat{\mathcal{P}}}, \Omega^\bullet_{]X[_{\hat{\mathcal{P}}}}) \ .$$

By Lemma 7.7 we have a quasi-isomorphism

$$\mathbb{R}\Gamma_{\mathrm{rig},Y}(X/K)_{\hat{\mathcal{P}}} \xleftarrow{\sim} \mathbb{R}\Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \Omega^{\bullet}_{]X[_{\hat{\mathcal{P}}}}) ,$$

where $\mathbb{R}\Gamma_{]Y[}$ is the derived functor of global sections with support in $]Y[_{\hat{\mathcal{P}}}$.

Lemma 7.9. *Let $i_{]Y[} :]Y[_{\hat{\mathcal{P}}} \to]X[_{\hat{\mathcal{P}}}$ be the canonical injection. For any two flasque sheaves $\mathfrak{F}$ and $\mathfrak{G}$ on $]X[_{\hat{\mathcal{P}}}$ there exist canonical maps,*

$$i_{]Y[_*} i^*_{]Y[}(\mathfrak{G}) \otimes \underline{\Gamma}^{\dagger}_{]Y[}(\mathfrak{F}) \to \underline{\Gamma}^{\dagger}_{]Y[}(\mathfrak{G} \otimes \mathfrak{F})$$

and

$$\Gamma(]Y[_{\hat{\mathcal{P}}}, \mathfrak{G}) \otimes \Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \mathfrak{F}) \to \Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \mathfrak{G} \otimes \mathfrak{F}) ,$$

such that the diagram

$$\begin{array}{ccc} \Gamma(]X[_{\hat{\mathcal{P}}}, \mathfrak{G}) \otimes \Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \mathfrak{F}) & \longrightarrow & \Gamma(]Y[_{\hat{\mathcal{P}}}, \mathfrak{G}) \otimes \Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \mathfrak{F}) \\ \downarrow & & \downarrow \\ \Gamma(]X[_{\hat{\mathcal{P}}}, \mathfrak{G}) \otimes \Gamma(]X[_{\hat{\mathcal{P}}}, \mathfrak{F}) & & \Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \mathfrak{G} \otimes \mathfrak{F}) \\ \downarrow & & \downarrow \\ \Gamma(]X[_{\hat{\mathcal{P}}}, \mathfrak{G} \otimes \mathfrak{F}) & = & \Gamma(]X[_{\hat{\mathcal{P}}}, \mathfrak{G} \otimes \mathfrak{F}) \end{array} \tag{25}$$

commutes.

Proof. Let $U \subset]X[_{\hat{\mathcal{P}}}$ be an admissible open. A section of $\underline{\Gamma}^{\dagger}_{]Y[}(\mathfrak{F})$ on U is by definition $f \in \mathfrak{F}(U)$ such that $\mathrm{res}_{U \cap V}(f) = 0$ for some V which is a strict neighborhood of $]W[_{\hat{\mathcal{P}}}$, i.e., that $(V,]Y[_{\hat{\mathcal{P}}})$ is an admissible covering of $]X[_{\hat{\mathcal{P}}}$. A section g of $i_{]Y[_*} i^*_{]Y[}\mathfrak{G}$ on U is just an element $g \in \mathfrak{G}(U \cap]Y[_{\hat{\mathcal{P}}})$. Given f and g we can define an element $g \otimes f \in \underline{\Gamma}^{\dagger}_{]Y[}(\mathfrak{G} \otimes \mathfrak{F})(U)$ as follows: Since $(U \cap]Y[_{\hat{\mathcal{P}}}, U \cap V)$ is an admissible cover of U, it is enough to define the section on these two subsets and make sure they match. We define $\mathrm{res}_{(U \cap]Y[_{\hat{\mathcal{P}}})}(g \otimes f)$ to be $g \otimes \mathrm{res}_{(U \cap]Y[_{\hat{\mathcal{P}}})}(f)$ and $\mathrm{res}_{(U \cap V)}(g \otimes f)$ to be 0. The two sections match because we know that $\mathrm{res}_{(U \cap V)}(f) = 0$ and so $\mathrm{res}_{(U \cap V \cap]Y[_{\hat{\mathcal{P}}})}(g \otimes f) = 0$. This gives the first map. The second is obtained by taking global sections. The commutativity of diagram Eq. (25) is obvious.

Proposition 7.10. *There exists a canonical product*

$$\mathbb{R}\Gamma(]Y[_{\hat{\mathcal{P}}}, \Omega^{\bullet}_{]X[_{\hat{\mathcal{P}}}}) \otimes \mathbb{R}\Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \Omega^{\bullet}_{]X[_{\hat{\mathcal{P}}}}) \to \mathbb{R}\Gamma^{\dagger}_{]Y[}(]X[_{\hat{\mathcal{P}}}, \Omega^{\bullet}_{]X[_{\hat{\mathcal{P}}}}) .$$

The induced cup product on rigid cohomology satisfies properties (L1)–(L3).

Proof. The existence of the product is an immediate consequence of the lemma and the properties of the induced product on cohomology are clear with the exception of property (L1) which follows from diagram Eq. (25).

Now we turn to the de Rham side. The construction of the de Rham complex is described in [Bes98, Sect. 5]: To construct a de Rham complex for a smooth K-variety U we must choose a "de Rham datum" comprising of a compactification $U \subset X$ such that the complement $D = X - U$ is a divisor with normal crossings. The de Rham complex $\mathbb{R}\Gamma_{\mathrm{dR}}(U/K)_X$ relative to the datum X is then given by $\mathbb{R}\Gamma(X, \Omega_X^\bullet\langle\log D\rangle)$. The complexes for varying data are quasi-isomorphic. To define the cup product it is possible to give a purely algebraic treatment using Hartshorne's definition of de Rham cohomology. However, to compare with the rigid side it is best to work in a rigid setting to begin with. The setting will be as follows: We are given proper smooth $\mathcal{V}$-schemes X and Y together with an embedding $Y \subset X$ of Y as a divisor in X.

To obtain de Rham complexes we need to compactify $X_K - Y_K$ to a normal crossings complement situation. Because of our assumptions we can simply choose the embedding into X_K as our compactification.

The de Rham complex with supports in Y_K (relative to our chosen compactification) is the cone

$$\mathbb{R}\Gamma_{\mathrm{dR},Y_K}(X_K/K)_{X_K} := \mathrm{MF}(\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)_{X_K} \to \mathbb{R}\Gamma_{\mathrm{dR}}(X_K - Y_K/K)_{X_K})\ .$$

We will define the product not on this cone, though, but on a quasi-isomorphic one which we now derive. We have

$$\mathbb{R}\Gamma_{\mathrm{dR},Y_K}(X_K/K)_{X_K} = \mathbb{R}\Gamma(X_K, \mathrm{MF}(\Omega_{X_K}^\bullet \to \Omega_{X_K}^\bullet\langle\log Y_K\rangle))\ .$$

Using GAGA we can switch to the analytic category to get quasi-isomorphisms

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{dR},Y_K}(X_K/K)_{X_K} &\xrightarrow{\sim} \mathbb{R}\Gamma(X_K^{\mathrm{an}}, \mathrm{MF}(\Omega_{X_K^{\mathrm{an}}}^\bullet \to \Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle))\\ &\xrightarrow{\sim} \mathbb{R}\Gamma(X_K^{\mathrm{an}}, (\Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle/\Omega_{X_K^{\mathrm{an}}}^\bullet)[-1])\ .\end{aligned}$$

To tie this further with the rigid situation we can take $\hat{\mathcal{P}} = \hat{X}$ to be the p-adic completion of X. Then $]X_\kappa[_{\hat{X}} \cong X_K^{\mathrm{an}}$. We have $Y_K^{\mathrm{an}} \subset\ ]Y_\kappa[_{\hat{X}}$ and in fact $X_K - Y_K$ is a strict neighborhood of $]X_\kappa - Y_\kappa[_{\hat{X}}$. Since the sheaf $\Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle/\Omega_{X_K^{\mathrm{an}}}^\bullet$ is supported on Y_K, there is no harm in replacing global sections by global sections with overconvergent support on $]Y_\kappa[_{\hat{X}}$. Indeed, by Lemma 7.7 we have for a complex $\mathfrak{F}^\bullet$ of sheaves supported on Y_K^{an} that

$$\mathbb{R}\Gamma_{]Y_\kappa[}(X_K^{\mathrm{an}}, \mathfrak{F}^\bullet) = \mathbb{R}\Gamma(X_K^{\mathrm{an}}, \underline{\Gamma}_{]Y[}^\dagger(X_K^{\mathrm{an}}, \mathfrak{F}^\bullet)) = \mathbb{R}\Gamma(X_K^{\mathrm{an}}, \mathfrak{F}^\bullet)\ .$$

Thus we obtain the following diagram.

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{dR},Y_K}(X_K/K)_{X_K} &\xrightarrow{\sim} \mathbb{R}\Gamma(X_K^{\mathrm{an}}, (\Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle/\Omega_{X_K^{\mathrm{an}}}^\bullet)[-1])\\ &\xleftarrow{\sim} \mathbb{R}\Gamma_{]Y_\kappa[}^\dagger(X_K^{\mathrm{an}}, (\Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle/\Omega_{X_K^{\mathrm{an}}}^\bullet)[-1])\ .\end{aligned}$$

The same considerations apply if we replace $\Omega^\bullet$ with $\Omega^{\geq k}$ in all of the above. On the left of the diagram above we then get by definition the k-th filtered part of the de Rham complex, so we have the diagram

$$\begin{aligned}&F^k\mathbb{R}\Gamma_{\mathrm{dR},Y_K}(X_K/K)_{X_K}\\ &\xrightarrow{\sim} \mathbb{R}\Gamma(X_K^{\mathrm{an}}, (F^k\Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle/F^k\Omega_{X_K^{\mathrm{an}}}^\bullet)[-1])\\ &\xleftarrow{\sim} \mathbb{R}\Gamma_{]Y_\kappa[}^\dagger(X_K^{\mathrm{an}}, (F^k\Omega_{X_K^{\mathrm{an}}}^\bullet\langle\log Y_K^{\mathrm{an}}\rangle/F^k\Omega_{X_K^{\mathrm{an}}}^\bullet)[-1])\ .\end{aligned}$$

We next give a similar rigid interpretation of the de Rham cohomology of Y_K. This is given in the following diagram.

$$\mathbb{R}\Gamma(]Y_\kappa[_{\hat{X}}, \Omega^\bullet_{X_K^{\mathrm{an}}}) \xrightarrow{\sim} \mathbb{R}\Gamma(Y_K^{\mathrm{an}}, \Omega^\bullet_{Y_K^{\mathrm{an}}}) \xleftarrow{\sim} \mathbb{R}\Gamma(Y_K, \Omega^\bullet_{Y_K}) \ .$$

The first map is induced by the map of tubes induced by the map or formal schemes $\hat{Y} \to \hat{X}$ and is a quasi-isomorphism by [Ber97b, Theorem 1.4]. It is further known from the work of Berthelot that to carry this quasi-isomorphism to the filtered parts one has to filter $\Omega^\bullet_{X_K^{\mathrm{an}}}$ as follows: Let J be the sheaf of ideals on X_K^{an} defining Y_K^{an}. Then one takes

$$F_J^k \Omega^\bullet_{X_K^{\mathrm{an}}} = (J^k \Omega^0_{X_K^{\mathrm{an}}} \to J^{k-1} \Omega^1_{X_K^{\mathrm{an}}} \to \cdots) \ , \tag{26}$$

where by definition $J^i = \mathcal{O}_{X_K^{\mathrm{an}}}$ if $i \le 0$.

The key observation that makes everything work is the following

Lemma 7.11. *There are products*

$$F_J^k \Omega^\bullet_{X_K^{\mathrm{an}}} \otimes (F^j \Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / F^j \Omega^\bullet_{X_K^{\mathrm{an}}}) \to F^{k+j} \Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / F^{k+j} \Omega^\bullet_{X_K^{\mathrm{an}}} \ ,$$

for all non negative integers k and j, compatible with the obvious product

$$\Omega^\bullet_{X_K^{\mathrm{an}}} \otimes (\Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / \Omega^\bullet_{X_K^{\mathrm{an}}}) \to \Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / \Omega^\bullet_{X_K^{\mathrm{an}}} \ .$$

Proof. This would clearly follow if we show that $J \cdot (\Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / \Omega^\bullet_{X_K^{\mathrm{an}}}) = 0$. This last equality is also clear: Locally, J is generated by the single function t such that $t = 0$ defines Y_K in X_K. At the same time, $\Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / \Omega^\bullet_{X_K^{\mathrm{an}}}$ is generated by forms $\mathrm{dlog}\, t \wedge \omega$ for some $\omega \in \Omega^\bullet_{X_K^{\mathrm{an}}}$. As $t \cdot \mathrm{dlog}\, t \wedge \omega \in \Omega^\bullet_{X_K^{\mathrm{an}}}$ we are done.

Corollary 7.12. *There exist compatible products*

$$\begin{aligned} &\mathbb{R}\Gamma(]Y_\kappa[_{\hat{X}}, F_J^k \Omega^\bullet_{X_K^{\mathrm{an}}}) \otimes \mathbb{R}\Gamma^\dagger_{]Y_\kappa[}(X_K^{\mathrm{an}}, (F^j \Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / F^j \Omega^\bullet_{X_K^{\mathrm{an}}})) \\ &\quad \to \mathbb{R}\Gamma^\dagger_{]Y_\kappa[}(X_K^{\mathrm{an}}, (F^{k+j} \Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / F^{k+j} \Omega^\bullet_{X_K^{\mathrm{an}}})) \ . \end{aligned}$$

The induced cup product on the cohomology theory $H^i(X,n) = H^i_{\mathrm{dR}}(X_K/K)$ satisfy (L1)–(L3).

Proof. Apply Lemma 7.9 to the product of Lemma 7.11

The next step is to compare the cup product on the de Rham side to the cup product on the rigid side. Consider the diagram of complexes

$$\begin{aligned} (\Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / \Omega^\bullet_{X_K^{\mathrm{an}}})[-1] \quad &\xleftarrow{\sim} \quad \mathrm{MF}(\Omega^\bullet_{X_K^{\mathrm{an}}} \to \Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle) \\ &\to \quad \mathrm{MF}(\Omega^\bullet_{X_K^{\mathrm{an}}} \to j^\dagger \Omega^\bullet_{X_K^{\mathrm{an}}}) \to \Omega^\bullet_{X_K^{\mathrm{an}}} \ . \end{aligned} \tag{27}$$

Applying the functor $\underline{\Gamma}^\dagger_{]Y_\kappa[}$ gives the required map

$$\mathbb{R}\Gamma^\dagger_{]Y_\kappa[}(X_K^{\mathrm{an}}, (\Omega^\bullet_{X_K^{\mathrm{an}}} \langle \log Y_K^{\mathrm{an}} \rangle / \Omega^\bullet_{X_K^{\mathrm{an}}})[-1]) \to \mathbb{R}\Gamma^\dagger_{]Y_\kappa[}(X_K^{\mathrm{an}}, \Omega^\bullet_{X_K^{\mathrm{an}}}) \ .$$

There are natural products $\Omega^\bullet_{X_K^{\mathrm{an}}} \otimes C^\bullet \to C^\bullet$ where $C^\bullet$ denotes any one of the complexes in Eq. (27). These products are clearly compatible, which shows the compatibility of the products in de Rham and rigid cohomologies.

Proof of Proposition 7.5. Using the products defined above, their compatibilities and the cup product on cones [Bes98, Lemma 3.2] one can define the required products. The details are left to the reader. The resulting product on cohomology satisfies properties (L1) – (L3). It remain to prove property (P1), i.e., the commutativity of the diagram

$$\begin{tikzcd} H_{\mathrm{fp}}^{2d_Y+1}(Y, d_Y+1) \arrow[r, "\cup c\ell_Y"] \arrow[d, "\mathrm{tr}_Y"'] & H_{\mathrm{fp},Y}^{2d_X+1}(X, d_X+1) \arrow[d, "\mathfrak{g}"] \\ K & H_{\mathrm{fp}}^{2d_X+1}(X, d_X+1), \arrow[l, "\mathrm{tr}_X"'] \end{tikzcd}$$

when Y is of codimension d in X and d_Y and d_X denote the relative dimensions of Y and X over $\mathcal{V}$ (here $d = 1$ but the argument would be valid for all d if we only knew how to construct the cup product). We have the canonical isomorphism, through which tr_X factors,

$$H_{\mathrm{dR}}^{2d_Y}(Y_K/K) \xrightarrow{\mathbf{i}} H_{\mathrm{fp}}^{2d_Y+1}(Y, d_Y+1) .$$

Therefore, in view of the projection formula Eq. (14), which by its purely formal derivation also holds for the localized cup product, the commutativity of the diagram above follows from that of

$$\begin{tikzcd} H_{\mathrm{dR}}^{2d_Y}(Y_K/K) \arrow[r, "\cup c\ell_{\mathrm{dR},Y}"] \arrow[d, "\mathrm{tr}_{Y_K}"'] & H_{\mathrm{dR},Y_K}^{2d_X}(X_K/K) \arrow[d, "\mathfrak{g}"] \\ K & H_{\mathrm{dR}}^{2d_X}(X_K/K), \arrow[l, "\mathrm{tr}_{X_K}"'] \end{tikzcd}$$

which is just the corresponding property (P1) for the de Rham fundamental class.

8. Proof of Theorem 5.5

In this section we prove Theorem 5.5. We will in fact rely heavily on the proof of the corresponding theorem for crystalline regulators in [GM87, 1.1.3]. To make it easier to both see only the necessary changes but also to be able to follow the argument independently we repeat at least part of the argument of loc. cit. Any place where a serious modifications occurs we mark with a lemma. Also, to make this section more compatible with [GM87], we make the following notational changes: We write $H^{2i}(X)$ for $H_{\mathrm{fp}}^{2i}(X, i)$ and we also write the Chern and cycle classes without an fp subscript. When reference to another cohomology theory is made this will be marked explicitly. We write the Chow ring of X as $A^*(X)$ and we use a dot to mark cup product in cohomology.

As in loc. cit. 1.1.4 one only needs to prove the theorem in the following cases:

1. E is a vector bundle on Y, X is the projective bundle $\mathbb{P}(E)$ and f is the natural projection.
2. N is a locally free sheaf on X, Y is $\mathbb{P}(N \oplus \mathcal{O}_X)$ and f is the inclusion of X as the zero section.
3. f is an arbitrary closed immersion.

This is because by [SGA71, II 2.2.7] and [GD67, II 3.6.2], any projective morphism of proper smooth $\mathcal{V}$-schemes factors as a product of morphisms of the above types.

In the first case, suppose E has rank $d+1$. Let $\zeta = c_1(\mathcal{O}_X(1))$. An element $x \in A^r(X)$ can be uniquely written as $\sum_{i=0}^{d} \alpha_{r-i}\zeta^i$ with $\alpha_{r-i} \in A^{r-i}(Y)$. By dimension considerations $f_*(\zeta^i) = 0$ for $i < d$ so by the projection formula we get $f_*(x) = \alpha_{r-d} f_*(\zeta^d)$. If U is an open of Y on which E trivializes we get a commutative diagram

$$\begin{tikzcd} A^d(U \times \mathbb{P}^d) \arrow[d, "f|_{U*}"'] & A^d(X) \arrow[l] \arrow[d, "f_*"] \\ A^0(U) & A^0(X) \arrow[l, "\sim"'] \end{tikzcd}$$

yielding

$$f_*\zeta^d = f|_{U*}((\zeta|_U))^d = f|_{U*}(H_1 \cdots H_d) = 1_U \ ,$$

where the H_i are the coordinate hyperplanes, and therefore $f_*(x) = \alpha_{r-d}$. Since η is compatible with inverse images we find $\eta(x) = \sum_{i=0}^{d} \eta(\alpha_{r-i})\eta(\zeta)^i$. For dimension reasons again we have $f_*(\eta(\zeta)^i) = 0$ for $i < d$ and so the first case will be proved if we show the following lemma.

Lemma 8.1. *We have* $f_*(\eta(\zeta)^d) = 1_{H^*(Y)}$.

Proof. In [GM87] the proof of the corresponding assertion in crystalline cohomology requires a long proof. Fortunately for us this is an equality in $H^0_{\mathrm{fp}}(Y, 0) = H^0_{\mathrm{dR}}(Y_K/K)$, and therefore follows from the corresponding result in de Rham cohomology and from Lemma 5.1.

The proof in the second case is exactly the same as the proof in [GM87] on pages 517–519. The use of cohomology is completely formal. The two properties of cohomology used are the projective bundle theorem, which we proved (Lemma 5.3), and Lemma 8.1.

It is the third case that poses the most difficulties. This is because the proof uses Berthelot's result on the compatibility of Gysin maps with pullbacks. We will observe that this result is only used for an embedding of a smooth divisor and this particular case we know by Corollary 7.6. We now check this in detail.

Let $i : X \hookrightarrow Y$ be a closed immersion of codimension d. Let W be the blow up of $Y \times \mathbb{P}^1$ along $X \times \infty$. The properties 1–4 listed in 1.3.1 of [GM87] still hold as well as diagram (1.3.2) there: W is flat over $\mathbb{P}^1$ and its fibers W_t are isomorphic to Y for $t \neq \infty$ while $W_\infty = \hat{N} \cup \tilde{Y}$ where $\hat{N}$ and $\tilde{Y}$ are smooth schemes meeting transversely: $\hat{N}$ is $\mathbb{P}(N \oplus \mathcal{O}_Y)$ where N is the conormal sheaf to X while $\tilde{Y}$ is the blow up of Y along X. The divisor W_0 is linearly equivalent to $\hat{N} + \tilde{Y}$ and $W' := W - \tilde{Y}$ is smooth over $\mathbb{P}^1$. We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "s"] \arrow[d, hook, "l_\infty"'] & \hat{N} \arrow[d, hook, "k"] & \\
X \times \mathbb{P}^1 \arrow[r, hook, "f"] \arrow[d, equal] & W \arrow[d] \arrow[dr, "r"] & \\
X \times \mathbb{P}^1 \arrow[r, hook, "j"] \arrow[d, "g"'] & Y \times \mathbb{P}^1 \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[ul, bend right, "\lambda_0"'] \\
X & \mathbb{P}^1 &
\end{tikzcd}$$

In the diagram λ_0 identifies Y with W_0 so $r \circ \lambda_0 = \mathrm{id}_Y$, and s embeds X in $\hat{N}$ as the zero section. The upper square is cartesian.

Given $\xi \in F_\gamma^p K^0(X)$ we want to compute $c_{p+d}(i_*(\xi))$. We have $i_*(\xi) = \lambda_0^* f_* g^*(\xi)$, which is 1.3.3 in loc. cit., since this does not involve cohomology and the proof is the same. Then $c_{p+d}(i_*(\xi)) = \lambda_0^* c_{p+d}(f_* g^*(\xi))$ since c_{p+d} is contravariant. The projection formula in fp-cohomology gives $\lambda_0^*(t) = r_* \lambda_{0*} \lambda_0^*(t) = r_*[t \cdot \lambda_{0*}(1_Y)]$ for any cohomology class t and in particular

$$c_{p+d}(i_*(\xi)) = r_*[c_{p+d}(f_* g^*(\xi)) \cdot \lambda_{0*}(1_Y)] \ .$$

Lemma 8.2. *Consider the diagram*

$$\begin{tikzcd}
Y \arrow[r, "\sim"] \arrow[rr, bend left, "\lambda_0"] & W_0 \arrow[r, hook] \arrow[d] & W \arrow[d, "w"] \\
& \{0\} \arrow[r, "i"] & \mathbb{P}^1
\end{tikzcd}$$

We have

$$\lambda_{0*}(1_Y) = w^*(c\ell(\{0\}))$$

and

$$c\ell(\{0\}) = i_*(1_0) = c_1(\mathcal{O}_{\{0\}}) \ .$$

Proof. This corresponds to 1.3.6 in loc. cit. The first equality follows from Corollary 7.6. The second assertion is a special case of the second case, which we already checked.

Since W_0 is linearly equivalent to $\hat{N} + \tilde{Y}$ we have $c_1(\mathcal{O}_{W_0}) = c_1(\mathcal{O}_{\hat{N}}) + c_1(\mathcal{O}_{\tilde{Y}})$.

Lemma 8.3. *We have $c_{p+d}(f_* g^*(\xi)) \cdot c_1(\mathcal{O}_{\tilde{Y}}) = 0$.*

Proof. This is 1.3.8 in loc. cit. The proof given there translates to our situation using the cup product constructed in Proposition 7.3: Since $f_* g^*(\xi)$ dies in $K^0(W - X \times \mathbb{P}^1)$ we have $c_{p+d}(f_* g^*(\xi)) \in \mathrm{Im}\,[H^{2p+2d}_{X \times \mathbb{P}^1}(W) \to H^{2p+2d}(W)]$. Similarly we have $c_1(\mathcal{O}_{\tilde{Y}}) \in \mathrm{Im}\,[H^2_{\tilde{Y}}(W) \to H^2(W)]$ and the cup product of the two classes is therefore 0 because it is supported on $Y \cap (X \times \mathbb{P}^1) = \emptyset$.

We then have $c_{p+d}(i_*(\xi)) = r_*[c_{p+d}(f_* g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}})]$.

Lemma 8.4. *We have* $c_{p+d}(f_*g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}}) = k_*k^*c_{p+d}(f_*g^*(\xi))$.

Proof. This is 1.3.10 of loc. cit. In translating the argument of 1.3.10 one has to be more careful about the difference between weak and strong fundamental classes. When $i : Z' \hookrightarrow Z$ is a closed embedding we can write $i_*(1_{Z'})$, even when Z and Z' are not proper, for the image of the weak fundamental class of Z' under the corresponding map $\mathfrak{g} : H^*_{Z'}(Z) \to H^*(Z)$. This coincides with the usual definition if Z and Z' are proper but otherwise it does not so we can not use a projection formula for example.

We already saw that $c_{p+d}(f_*g^*(\xi))$ is the image of $a \in H^{2p+2d}_{X\times\mathbb{P}^1}(W)$. It follows that $c_{p+d}(f_*g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}})$ is the image of $a \cdot c_1(\mathcal{O}_{\hat{N}}) \in H^{2p+2d+2}_{X\times\mathbb{P}^1}(W)$ (recall that cup products in cohomology with supports were defined in Proposition 7.3). Since W' is an open subset of W containing $X \times \mathbb{P}^1$ we have an excision isomorphism (in fp cohomology it is deduced as usual from the corresponding isomorphisms in rigid and de Rham cohomologies) $H_{X\times\mathbb{P}^1}(W) \to H_{X\times\mathbb{P}^1}(W')$. The image under this isomorphism of $a \cdot c_1(\mathcal{O}_{\hat{N}})$ is $a \cdot c_1(\mathcal{O}_{\mathbb{V}(N)})$. One considers the following diagram

$$\begin{tikzcd}
\hat{N} \arrow[r, hook, "k"] & W \\
\mathbb{V}(N) \arrow[u, hook] \arrow[r, hook, "k'"] \arrow[d] & W' \arrow[u] \arrow[d] \\
Y \times \{\infty\} \arrow[r, hook, "\mu_\infty"] & Y \times \mathbb{P}^1,
\end{tikzcd}$$

where the two squares are cartesian. Here, $\hat{N}$ and W are proper but $\mathbb{V}(N)$ and W' are not. Thus, the pushforward $k'_*(1_{\mathbb{V}(N)})$ has to be understood as discussed at the beginning of the proof. Note that this remark is also true for the proof of Gillet and Messing (we would like to thank Gillet for clarifying this point). As in the proof of Lemma 8.2 we have $\mu_{\infty *}(1_{Y\times\{\infty\}}) = c_1(\mathcal{O}_{Y\times\{\infty\}})$. Since the weak fundamental class is functorial with respect to arbitrary cartesian diagrams we find $k'_*(1_{\mathbb{V}(N)}) = c_1(\mathcal{O}_{\mathbb{V}(N)})$. For the same reason $k_*(1_{\hat{N}})$ restricts to $k'_*(1_{\mathbb{V}(N)})$ and it follows that $a \cdot c_1(\mathcal{O}_{\hat{N}})$ and $a \cdot k_*(1_{\hat{N}})$ have the same restriction to $H^{2p+2d+2}_{X\times\mathbb{P}^1}(W')$ and are therefore equal. We thus obtain

$$c_{p+d}(f_*g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}}) = c_{p+d}(f_*g^*(\xi)) \cdot k_*(1_{\hat{N}}) \ .$$

Here, $k_*(1_{\hat{N}})$ is the weak fundamental class of $\hat{N}$ but as $\hat{N}$ is a proper smooth divisor it is also the strong fundamental class. Therefore, one can apply the projection formula to complete the argument and obtain

$$c_{p+d}(f_*g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}}) = k_*k^*(c_{p+d}(f_*g^*(\xi))) \ .$$

The rest of the argument continues exactly as in [GM87]: In $K^0(\hat{N})$ we have, by transversality of f and k, that $s_*l^*_\infty = k^*f_*$. Thus $s_*(\xi) = k^*f_*g^*(\xi)$ and since k^* commute with Chern classes it follows that $c_{p+d}(i_*(\xi)) = r_*k_*c_{p+d}(s_*(\xi))$. Since s is a closed immersion of the same codimension as i and is a morphism of the type for which the theorem has already been proved (second case), and as $r \circ k \circ s = i$, the third case follows immediately from the second case.

9. Comparison with Coleman's theory

We now prove Theorem 1.1. We first need to recall briefly Coleman's integration theory. The basic reference is [Col85].

We allow $\mathcal{V}$ to be non-discrete here (Coleman takes $\mathcal{V} = \mathcal{O}_{\mathbb{C}_p}$) but in this case we require that our schemes be base changes from discrete valuation rings. Let X be a proper smooth $\mathcal{V}$-scheme and let $\omega \in F^1 H^1_{\mathrm{dR}}(X_K/K)$ be a holomorphic form on X_K. For $x \in X_\kappa$ let $U_x =]x[_{\hat{X}}$ be the "residue disc" of x in X_K, consisting of all points of X_K reducing to x. This is a rigid analytic space over K. We will abuse the notation in such a way that when $y \in X(K)$ extends to a section, which we continue to call y, from $\mathcal{V}$ to X we also denote by U_y the residue disc of the intersection of this section with X_κ. The Coleman integral F_ω is a locally analytic primitive for ω. This means that the function $F_\omega : X(K) \to K$ is analytic on each residue disc and satisfies $dF_\omega = \omega$. Its construction, which will be recalled soon, is such that it is unique up to an additive constant. Note that defining F_ω as a function really only makes sense if K is algebraically closed. Otherwise one should more precisely think of it as a collection of rigid analytic functions on the various U_x, i.e., power series with coefficients in K converging on a unit polydisc, and think of F_ω as their evaluation on K-valued points in the obvious sense.

To construct F_ω Coleman first covers X_K by "Zariski affinoids" U_i (loc. cit. Sect. 1). With the notation of Berthelot we have $U_i =]Z_i[_{\hat{X}}$ with Z_i belonging to a covering of X_κ. We may choose a Frobenius endomorphism φ of X_κ and assume that each Z_i is stable under φ. One can then choose for each i a lifting ϕ_i of φ to an endomorphism of U_i.

Let $Q(t)$ be the characteristic polynomial of φ^* acting on $H^1_{\mathrm{rig}}(X_\kappa/K)$ (which is isomorphic to $H^1_{\mathrm{dR}}(X_K/K)$). Then Coleman shows [Col85, Theorem 2.1] that on each $U = U_i$ there is a unique up to constant locally analytic primitive F_U to ω with the property that $Q(\phi_i^*)F_U = g \in A(U)$, i.e., is a globally analytic function on U. The last step in the construction is then to show that by adjusting the constants for the F_{U_i} they can be glued together to obtain F_ω.

To prove Theorem 1.1 it is convenient to use the Gros style fp-cohomology. To define this for X one needs to embed it into a proper $\mathcal{V}$-scheme $\bar{X}$ and then embed $\bar{X}$ into a smooth $\mathcal{V}$-scheme P. Now in the definition of $\mathbb{R}\Gamma_{\mathrm{fp}}$ one simply replaces the complexes $F^n\mathbb{R}\Gamma_{\mathrm{dR}}$ by complexes $F^n\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\hat{P}}$ where the filtration is now induced by the ideal of $\bar{X}_K$ inside $\hat{P}_K$ as in Eq. (26). If $P = \bar{X}$, then this ideal is 0 and we get the usual Hodge filtration. The other case of interest is that where $X = \operatorname{Spec}\mathcal{V}$ embedded in P via some section. If the generic fiber of this section is the point $x \in P(K)$ then we have $]X_\kappa[_{\hat{P}} = U_x$ and the filtration is by order of vanishing at x. We will need to know the following simple result

Lemma 9.1. *In the above situation the map $\Omega^\bullet(U_x) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\hat{P}}$ is a quasi-isomorphism.*

Proof. Since P is smooth U_x is isomorphic to a polydisc. But a polydisc is a quasi-Stein space. Hence by [Kie67, Satz 2.4] one has $H^i(U_x, \Omega^j(U_x)) = 0$ for $i > 0$.

Proof of Theorem 1.1. As finite polynomial cohomology for proper schemes is the same for the tilded (Gros style) and non-tilded versions, we can use $\tilde{H}_{\mathrm{fp}}$ instead of H_{fp} in the proof. We consider the point $x \in X(K)$ extended to a section $x : \operatorname{Spec}\mathcal{V} \to X$.

Since we may vary our Frobenius endomorphism without changing our pullbacks and the Coleman integral, the latter by [Col85, Corollary 2.1d], we may assume that κ is finite and that φ fixes the intersection of x with the special fiber. We choose an affine open $Y \subset X$ such that x factors as $\operatorname{Spec}\mathcal{V} \to Y \xrightarrow{i} X$. We lift the form $\omega \in F^1 H^1_{\mathrm{dR}}(X_K/K)$ to $\widetilde{\omega} \in \tilde{H}^1_{\mathrm{fp}}(X,1)$. We want to compute $x^*\widetilde{\omega} \in \tilde{H}^1_{\mathrm{fp}}(\operatorname{Spec}\mathcal{V},1)$. We can break this into the composition $x^*i^*\widetilde{\omega}$ and so we begin by computing $i^*\widetilde{\omega} \in \tilde{H}^1_{\mathrm{fp}}(Y,1)$. For this last cohomology group we can write an explicit representation as follows (compare [Bes98, Proposition 10.1]) : We compactify Y via its embedding into X. Let $Y = \operatorname{Spec} B$ and denote by $B^\dagger$ the weak completion of B corresponding to the embedding $Y \hookrightarrow X$. We can also choose a Frobenius lift $\phi : B^\dagger \to B^\dagger$. The rigid cohomology of Y_κ is the same as the Monsky-Washnitzer cohomology, which is the cohomology of the complex $\Omega^\bullet_\dagger(Y) := B^\dagger \otimes \Omega^\bullet_{B_K/K}$. It then follows that

$$\tilde{H}^1_{\mathrm{fp}}(Y,1) \cong \{(\eta,h) :\ \eta \in \Omega^1_\dagger(Y),\ h \in B^\dagger,\ Q(\phi)\eta = dh\}\ .$$

Because of the compatibility with maps to de Rham cohomology it is obvious that $i^*(\widetilde{\omega}) = (\omega|_{\Omega^1_\dagger(Y)}, g)$, for some $g \in B^\dagger$, in this representation. We can consider now the restriction to the Zariski affinoid $U :=]Y_\kappa[_{\hat{X}}$. We have on U the equality $Q(\phi)\omega = dg$, which is exactly the relation required by Coleman to carry out the integration on U (compare [Col85, Theorem 2.1]). Now we take the pullback by x. We use the embedding of $\operatorname{Spec}\mathcal{V}$ into X via x for the computation of rigid cohomology. The tube $]\operatorname{Spec}(\kappa)[_{\hat{X}}$ is exactly the residue disc U_x. By Lemma 9.1 the complex computing rigid cohomology is just $\Omega^\bullet(U_x)$, the Frobenius endomorphism can be taken to be the restriction of ϕ to U_x and the filtration is given by the order of vanishing of a function or form at x. Thus we obtain the following representation for fp-cohomology:

$$\tilde{H}^1_{\mathrm{fp}}(\operatorname{Spec}(\mathcal{V}),1) \cong \frac{\{(\eta,h) :\ \eta \in \Omega^1(U_x),\ h \in A(U_x),\ Q(\phi)\eta = dh\}}{\{(ds, Q(\phi)s) :\ s \in A(U_x),\ s(x) = 0\}}$$

(recall that Q is the characteristic polynomial of Frobenius on $H_{\mathrm{dR}}(X_K/K)$). According to our convention this is isomorphic to K via the map $\alpha \mapsto (0, Q(\phi)\alpha)$ and we are looking for the value α of the inverse to this isomorphism applied to $x^*(\widetilde{\omega})$. With this α we obtain the existence of $s \in A(U_x)$ with $s(x) = 0$ such that

$$(\omega, g)|_{U_x} = (ds, Q(\phi)s) + (0, Q(\phi)\alpha) = (ds, Q(\phi)(s+\alpha))\ .$$

Thus if we set $f = s + \alpha \in A(U_x)$ we see that $df = \omega|_{U_x}$, $Q(\phi)f = g|_{U_x}$ and $f(x) = \alpha$. The first two conditions on f show that if we construct similar functions on each residue disc we obtain a function F_U on U satisfying $dF_U = \omega|_U$, $Q(\phi)F_U = g|_U \in A(U)$ and therefore F_U is a Coleman integral of $\omega|_U$. In addition we have $F_U(x) = \alpha = \operatorname{tr} x^*(\widetilde{\omega})$. This shows that the various F_U glue together to give a global Coleman integral, whose value at x is what we wanted.

In the present situation, i.e., for zero-cycles, one can find a more general connection between syntomic regulators and Coleman integration, which is valid also in the bad reduction case and which gives another, very simple albeit much less direct, proof of Theorem 1.1.

We first recall that one can define the Coleman integral of one-forms without assumptions of good reduction. This is already evident in [Col85] and has recently been developed into a full blown theory including forms of the second and third kind by Zarhin [Zar96] and by Colmez [Col98].

Since there are no assumptions on the reduction now, the theory needs only the generic fiber and we therefore switch notation and X will now denote a proper smooth variety over K and we let ω be a holomorphic one-form on X as before. We want to construct a canonical primitive F_ω. Suppose first that X is an abelian variety. Then ω is an invariant differential. By changing the constant of integration we may assume that $F_\omega(0) = 0$. One then finds that F_ω should be an additive function. This and the fact that $dF_\omega = \omega$ is enough to determine it uniquely. Indeed, for each $x \in X(K)$ there is some $0 \neq N \in \mathbb{Z}$ such that $Nx \in U_0$. By additivity we get $F_\omega(x) = N^{-1}F_\omega(Nx)$. Since on U_0 we know F_ω by the assumption that $F_\omega(0) = 0$ and by knowing its derivative this determines F_ω completely. This shows that a good candidate for the integral is given by setting $\int_0^x \omega = \mathrm{Log}\,(x)(\omega)$ where $\mathrm{Log} : X(K) \to \mathrm{Lie}\, X$ is the logarithm map associated with $X(K)$ ([Bou72]), i.e, the unique additive map which near 0 is the inverse of the exponential map, and where we evaluate an element of $\mathrm{Lie}\, X$ on an invariant differential at the origin. This is indeed equal to the Coleman integral in the case of good reduction by [Col85, Theorem 2.8].

For an arbitrary variety X let A be the Albanese variety of X and let $h : X \to A$ be the canonical map (after fixing a point of X). We know that $\omega = h^*\omega'$ for some ω' holomorphic on A. Then functoriality gives $F_\omega = h^* F_{\omega'}$.

On the other hand, we can give an indirect definition of the syntomic Abel-Jacobi map by extending the relation with the étale Abel-Jacobi map mentioned in the introduction. There is a p-adic étale Abel-Jacobi map that associates to a codimension i cycle Z on X which is homologically equivalent to 0 a class $\alpha_p^i(Z) \in H^1(K, V)$, where V is the $\mathrm{Gal}\,(\bar{K}/K)$-representation $H^{2i-1}(X_{\bar{K}}, \mathbb{Q}_p(i))$. By the de Rham conjecture of Fontaine, proved by Faltings [Fal89], We know that $D_{\mathrm{dR}}(V) = H_{\mathrm{dR}}^{2i-1}(X/K)(i)$ where D_{dR} is one of the functors defined by Fontaine and the notation (i) means shifting the filtration. By the work of Bloch and Kato there therefore exist an exponential map,

$$\exp : H_{\mathrm{dR}}^{2i-1}(X/K)/F^i \to H^1(K, V) \ ,$$

whose image is the so called space of finite cohomology classes. One conjectures that the image of α_p^i is contained in the image of exp. This is known to be true in the case of good reduction by [Niz97, Theorem 3.2]. It is also true for 0-cycles (see below). Assuming this conjecture (for X) and assuming that the exponential map is injective one can compose α with the inverse of exp and using duality as in the introduction get a map $\alpha_{\mathrm{syn}} : A^i(X)_0 \to (F^{j+1} H_{\mathrm{dR}}^{2j+1}(X/K))^*$ where $i + j = d = \dim X$ and $*$ denotes duality over K. This map extends to the case of bad reduction our syntomic Abel-Jacobi map from the introduction. We can prove the following result.

Proposition 9.2. *When $j = 0$ and X is projective we have* $\mathrm{Im}\,(\alpha) \subset \mathrm{Im}\,(\exp)$ *and Theorem* 1.2 *remains true with the extension of the Coleman integration and the syntomic Abel-Jacobi maps described above.*

Proof. Let A and $h : X \to A$ be the Albanese variety of X and an Albanese map as before. According to a theorem of Raskind [Ras95, Appendix] the p-adic (or in this case the ℓ-adic for any ℓ) map of X is the composition

$$A^d(X)_0 \to A\,(K) \xrightarrow{\text{Kummer}} H^1(K, V_p(A_{\bar{K}})) \cong H^1(K, H_{\text{ét}}^{2d-1}(X_{\bar{K}}, \mathbb{Q}_p(d))) \ .$$

In other words, the Abel-Jacobi map for X factors through that of A. By the functoriality of the Coleman integral it is not too hard to see that it is enough to prove the proposition for A, i.e., show that for any holomorphic one-form ω on A and for any $x \in A(K)$, $\exp^{-1}(\mathrm{Kummer}(x))(\omega) = \int_0^x \omega$. Since both sides are additive in x we may, by multiplying x by an appropriate integer $\neq 0$, assume that $x \in U_0$. We may identify $H^1_{\mathrm{dR}}(A/K)/F^0$ with the Lie algebra of the p-adic Lie group $A(K)$. Under this identification [BK90, Example 3.11] shows that the exponential map becomes the composition of the usual exponential map Lie $(A(K)) \to A(K)$ with the Kummer map. This easily gives the result.

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