SYNTOMIC REGULATORS AND p-ADIC INTEGRATION I: RIGID SYNTOMIC REGULATORS

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ABSTRACT. We construct a new version of syntomic cohomology, called rigid syntomic cohomology, for smooth schemes over the ring of integers of a p-adic field. This version is more refined than previous constructions and naturally maps to most of them. We construct regulators from K-theory into rigid syntomic cohomology. We also define a "modified" syntomic cohomology, which is better behaved in explicit computations yet is isomorphic to rigid syntomic cohomology in most cases of interest.

1. INTRODUCTION

The syntomic cohomology, more precisely the cohomology of the sheaves $s(n)$ on the syntomic site of a scheme, were introduced in [FM87] in order to prove comparison isomorphisms between crystalline and p-adic étale cohomology. It can be seen as an analogue of the Deligne-Beilinson cohomology in the p-adic world (for an excellent discussion see [Nek98]). In particular, when X is a smooth scheme over the ring of integers $\mathcal{V}$ of a finite extension K of $\mathbb{Q}_p$ there should exist higher Chern classes from algebraic K-theory into the syntomic cohomology of X. Such classes have been constructed, sometimes under certain additional assumptions, by Gros [Gro90] and by Nizioł [Niz97].

Syntomic cohomology comes in different flavors (much like Deligne-Beilinson cohomology). The versions discussed above are well behaved only for proper schemes. In particular, they do not have the homotopy property for affine spaces. This makes computations difficult because most constructions in K-theory go through non proper schemes.

In [Gro94], Gros introduced, using the rigid cohomology of Berthelot [Ber96, Ber97], **rigid syntomic cohomology** for a scheme X which is smooth over an unramified base. When the scheme X is affine he constructed rigid syntomic regulators,

$$c_{i,j}\colon K_j(X) \to H^{2i-j}(X, s(i)_{X/K,\mathrm{rig}}) ,$$

from K-theory into his rigid syntomic cohomology. Gros was able to show that the value of the syntomic regulator on certain cyclotomic elements in the higher K-theory of number fields is, when properly normalized, given by the values of p-adic polylogarithms at roots of unity.

It should be mentioned here that there is another method of "controlling" syntomic cohomology, due to Somekawa [Som99]. In this method one assumes X has a compactification where the complement is a relative normal crossings divisor. Somekawa was able to prove the result of Gros for all cyclotomic elements.

The following philosophy exists:

Philosophy 1. *There should be a p-adic Beilinson conjecture that relates special values of p-adic L-functions to syntomic regulators.*

Special cases of this are the results of [Gro90] and [KNQD98]. One should be able to derive some general conjecture from [PR95]. For results about CM elliptic curves see the discussion below.

The main result of this work is an extension of the constructions of Gros to an arbitrary smooth finite type $\mathcal{V}$-scheme X, where $\mathcal{V}$ is any complete discrete valuation ring with a perfect residue field of characteristic p. For such a scheme X we define in section 6 syntomic cohomology $H^i_{\mathrm{syn}}(X,n)$ and in section 7 we construct Chern classes from K theory to it. Our definition takes into account more growth conditions than that of Gros: we also consider log singularities. The result is that H_{syn} is always finite dimensional (proposition 6.3). Our cohomology maps when possible to the version of Gros (proposition 9.5) and to the version of Nizioł (proposition 9.9).

Another objective of this work is to begin to develop tools for computations in syntomic cohomology. Our main result here is the construction of a **modified syntomic cohomology**, denoted $H^*_{\mathrm{ms}}(X,*)$, in section 8. This cohomology is related to syntomic cohomology by a natural map (proposition 8.6.2) which is an isomorphism in most cases of interest (proposition 8.6.3). It is significantly easier to compute when the base $\mathcal{V}$ is ramified. We have also found that the original rigid syntomic cohomology of Gros (without log singularities), extended to the case of ramified base, is useful in some computations (see for example [BdJ99]). It again can come with an original or modified flavor, the latter being most useful.

Let us discuss a bit of applications. In a sequel to this paper [Bes99a] we compute the syntomic regulator $K_2(X) \to H^2_{\mathrm{syn}}(X,2)$ when X is smooth and proper of relative dimension 1 over $\mathcal{V}$. We show that there is a precise relation between this regulator and the p-adic regulator constructed by Coleman and de Shalit [CdS88]. In particular, for elliptic curves with complex multiplication their results in conjunction with ours relate the syntomic regulator with special values of a p-adic L-function of E, in line with the philosophy 1.

In [Bes99] we will build on the results of this paper and embed syntomic cohomology in some other "cohomology theory" which has Poincaré duality. This is very useful for computations involving cycles. We will show how to relate p-adic Abel-Jacobi maps to a generalization of Coleman's p-adic integration theory [Col85].

Finally, in [BdJ99] we intend to show how to compute syntomic regulators on the wedge complexes introduced in [DJ95] using p-adic polylogarithms. This is a generalization of the results of Gros on cyclotomic elements described above.

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Notation: Throughout this work $\mathcal{V}$ is a complete discrete valuation ring with maximal ideal $\mathfrak{p}$, quotient field K and residue field κ of characteristic p. When κ is perfect we let $\mathcal{V}_0 \subset \mathcal{V}$ be the Witt ring of κ and K_0 its quotient field. Much of the theory can be carried out without the assumption that $\mathcal{V}$ is discrete but since some results we need are not yet documented in the literature we prefer to make this additional assumption.

All schemes will be separated and of finite type over their respective bases. We let Compl (respectively Compl_K when K is a field) denote the category of bounded below cohomological complexes of abelian groups (respectively K-vector spaces).

2. Canonical resolutions

One of the difficulties in constructing Chern classes from higher K-theory into a cohomology theory is that one has to realize the cohomology theory as the cohomology of a complex which is functorial on the category of schemes. This means that many of the constructions one makes on the level of cohomology or of the derived category have to be refined by using some canonical resolution. As an example, in [Hub95] Huber systematically uses a Godement resolution to eventually obtain Chern classes into her absolute cohomology. In our constructions we will need to consider the cohomology of rigid analytic spaces, which unfortunately do not have enough points so that at least naively a Godement construction is impossible on them. In fact, One can replace the usual rigid spaces by spaces with enough points obtaining an equivalent sheaf theory [vdPS95]. Since not everything we need is documented in the literature at the moment, we have chosen to use instead the approach of Beilinson [Bei85, 1.6.5] where it was used to construct Zariski sheaves computing Deligne cohomology.

Suppose we are given a category $\mathcal{A}$ and a functor $\mathcal{D} \mapsto U_{\mathcal{D}}, f \mapsto f_U$, from $\mathcal{A}$ into the category of sites with continuous morphisms (we could work directly with toposes instead). Then one can consider the category $\mathcal{B}$ whose objects are pairs $(\mathcal{D}, F)$ where $\mathcal{D} \in \mathcal{A}$ and F is a sheaf on $U_{\mathcal{D}}$. A map $(\mathcal{D}, F) \to (\mathcal{D}', F')$ consists of a map $f\colon \mathcal{D} \to \mathcal{D}'$ in $\mathcal{A}$ together with a map of sheaves $F' \to f_{U*}F$. Then $\mathcal{B}$ is, in the terminology of [SD72, 1.2.2], **bifiltered by toposes over** $\mathcal{A}$ (loc. cit., 4.1.0). We can consider the section category $\underline{\Gamma}(\mathcal{B})$ of loc. cit., 1.2.8. Explicitly, an object of $F \in \underline{\Gamma}(\mathcal{B})$ is given by a collection of sheaves $F_{\mathcal{D}}$ on $U_{\mathcal{D}}$, for every $\mathcal{D} \in \mathcal{A}$, together with morphisms of sheaves $f^*\colon F_{\mathcal{D}'} \to f_{U*}F_{\mathcal{D}}$ for every morphism $f\colon \mathcal{D} \to \mathcal{D}'$ in $\mathcal{A}$ such that one has

$$(f \circ g)^* = g^* \circ f^*, \quad \mathrm{id}^* = \mathrm{id} \ . \tag{2.1}$$

By loc. cit., 1.2.12, $\underline{\Gamma}(\mathcal{B})$ is a topos. By loc. cit., 1.3.10 there is a collection $I_{\mathcal{B}}$ of abelian objects of $\underline{\Gamma}(\mathcal{B})$ such that the following two properties hold:

- Any abelian $F \in \underline{\Gamma}(\mathcal{B})$ injects into $I \in I_{\mathcal{B}}$.
- For $I \in I_{\mathcal{B}}$ and for any $\mathcal{D} \in \mathcal{A}$, the sheaf $I_{\mathcal{D}}$ on $U_{\mathcal{D}}$ is flasque (in the sense of loc. cit.).

Suppose then that we are given a complex of abelian objects $F^\bullet$ in $\underline{\Gamma}(\mathcal{B})$ and that we want to find a contravariant functor $\mathcal{A} \to \mathsf{Compl}$, $\mathcal{D} \mapsto C^\bullet_{\mathcal{D}}$, in such a way that $C^\bullet_{\mathcal{D}}$ represents $\mathbb{R}\Gamma(U_{\mathcal{D}}, F^\bullet_{\mathcal{D}})$ and that if $f\colon \mathcal{D} \to \mathcal{D}'$ is a morphism in $\mathcal{A}$, then the map $f^*\colon C^\bullet_{\mathcal{D}'} \to C^\bullet_{\mathcal{D}}$ induces the obvious map on cohomology. Since flasque sheaves are acyclic with respect to the global section functor all we have to do is find a resolution $I^\bullet \in I_{\mathcal{B}}$ (i.e., that $I^n \in I_{\mathcal{B}}$ for all n) for $F^\bullet$ and take $C^\bullet_{\mathcal{D}} = \Gamma(U_{\mathcal{D}}, I^\bullet_{\mathcal{D}})$. This is the basic construction that we need.

This idea can be extended to diagrams of sheaves: Suppose that we have a diagram of complexes in $\underline{\Gamma}(\mathcal{B})$ indexed by the small category J. In other words, for each $\mathcal{D} \in \mathcal{A}$ we are given a diagram of complexes of sheaves on $U_{\mathcal{D}}$, $j \mapsto F^\bullet_{j,\mathcal{D}}$ in such that the all diagrams,

$$\begin{array}{ccc} F^\bullet_{j,\mathcal{D}'} & \longrightarrow & F^\bullet_{j',\mathcal{D}'} \\ \downarrow & & \downarrow \\ F^\bullet_{j,\mathcal{D}} & \longrightarrow & F^\bullet_{j',\mathcal{D}} \end{array}$$

commute. We can define a functor from $J^{\mathrm{op}} \times \mathcal{A}$ to sites by composing the old functor with the projection to the second factor. Then we see that our diagram becomes an object of the corresponding section category. Resolving it we now get for each $\mathcal{D}$ and for each j a flasque resolution of $F^\bullet_{j,\mathcal{D}}$ on $U_{\mathcal{D}}$ and taking the global section functor again we obtain a functor $J \times \mathcal{A} \to \mathsf{Compl}$ whose value at $(j, \mathcal{D})$ represents $\mathbb{R}\Gamma(U_{\mathcal{D}}, F^\bullet_{j,\mathcal{D}})$.

Remark 2.1. One can try to generalize this idea to obtain completely canonical resolutions using the canonical section of the fibration $\mathcal{B} \times_{\mathcal{A}} \mathcal{B}$. Some set theoretical problems seem to arise this way. It appears however that Gabber can define completely canonical resolutions using an extension of the canonical Barr resolution.

Since the resolutions we obtained are not injective there could a-priori be a problem of the uniqueness of the theory obtained in this way. This is easily settled however: Suppose that one obtains two functorial resolutions, $I_1^\bullet$ and $I_2^\bullet$, to $F^\bullet$. Resolving now the diagram $I_1^\bullet \leftarrow F^\bullet \to I_2^\bullet$ we obtain a diagram

$$\begin{array}{ccccc} I_3^\bullet & \longleftarrow & I_4^\bullet & \longrightarrow & I_5^\bullet \\ \uparrow & & \uparrow & & \uparrow \\ I_1^\bullet & \longleftarrow & F^\bullet & \longrightarrow & I_2^\bullet \end{array}$$

and taking global sections we see that we have 5 different functorial complexes connected by functorial quasi-isomorphisms. Whichever way we continue to torture these complexes it will be clear that the 5 possible complexes one obtain preserve this connectivity property. This of course implies that the cohomology of all complexes is canonically isomorphic and also it is standard to obtain from this (see for example the proof of proposition 9.9) that the theories of Chern classes one eventually obtains are independent of choices.

With this much said we can allow ourselves a bit of freedom in choosing our complexes. As an example we will define our rigid complexes in section 4 with respect to one choice of canonical resolutions, but when we compare our rigid complexes with the de Rham complexes in section 5 these complexes will change because we will be resolving additional shaves and spaces. The above remarks guarantee that nothing is damaged by this procedure.

Finally, we can use similar ideas to define products: Suppose we have in $\underline{\Gamma}(\mathcal{B})$ a map $F^\bullet \otimes G^\bullet \to H^\bullet$. In our applications we will assume that all sheaves are of K-modules, where K is a field, so flatness problems do not arise. We can resolve $F^\bullet$ and $G^\bullet$ by $I_F^\bullet$ and $I_G^\bullet$ respectively. Now we can resolve the diagram $I_F^\bullet \otimes I_G^\bullet \leftarrow F^\bullet \otimes G^\bullet \to H^\bullet$ and get $J^\bullet_{(I_F^\bullet \otimes I_G^\bullet)} \leftarrow J^\bullet_{(F^\bullet \otimes G^\bullet)} \to J_H^\bullet$. Remembering that we have a map $I_F^\bullet \otimes I_G^\bullet \to J^\bullet_{(I_F^\bullet \otimes I_G^\bullet)}$ and taking global sections we get for each $\mathcal{D} \in \mathcal{A}$ a diagram

$$\Gamma(U_{\mathcal{D}}, I_F^\bullet) \otimes \Gamma(U_{\mathcal{D}}, I_G^\bullet) \to \Gamma(U_{\mathcal{D}}, J^\bullet_{(I_F^\bullet \otimes I_G^\bullet)}) \xleftarrow{\sim} \Gamma(U_{\mathcal{D}}, J^\bullet_{(F^\bullet \otimes G^\bullet)}) \to \Gamma(U_{\mathcal{D}}, J_H^\bullet) .$$

Here it is implicit in the notation that global sections on $U_{\mathcal{D}}$ are taken with respect to the section sheaf at $\mathcal{D}$. Thus we obtain a functorial cup product in the derives category.

3. The quasi-fibered product and cup products on cones

We describe a bit of homological algebra that we need. Nothing is really new here but we want to record thing in a way convenient for use.

Suppose we are given complexes $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ with maps $f\colon X^\bullet \to Z^\bullet$ and $g\colon Y^\bullet \to Z^\bullet$. Then one can form the naive fibered product $X^\bullet \times_{Z^\bullet} Y^\bullet$ whose n-th component is $X^n \times_{Z^n} Y^n$. It is of course equal to the kernel of $f-g\colon X^\bullet \oplus Y^\bullet \to Z^\bullet$. Therefore, one should prefer to use instead the slightly different construction, called the **quasi-fibered product**, $X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet := \mathrm{Cone}(f-g)[-1]$. We have the well known

Lemma 3.1. *In the situation above, if the map $f-g$ is surjective, then the two construction are quasi-isomorphic via the map*

$$(x, y) \to (x \oplus y, 0) . \tag{3.1}$$

It will be convenient to use both construction in what follows.

Notice that we have canonical maps

$$Z^\bullet[-1] \xrightarrow{i} X^\bullet \underset{Z^\bullet}{\tilde{\times}} Y^\bullet \xrightarrow{p} X^\bullet \oplus Y^\bullet \tag{3.2}$$

coming from the cone construction. Let us write p_A and p_B for the composition of p with the first and second projection respectively. The following construction of the cup product is a variant of one of Nizioł [Niz93], which is itself a variant of a construction of Beilinson. Alternatively, it is a special case of the construction of [Bei86, 1.11]:

Lemma 3.2. *Suppose we are given complexes $X_i^\bullet$, $Y_i^\bullet$, $Z_i^\bullet$ and maps f_i, g_i as above for $i = 1,2,3$, and that we are given maps of complexes $\cup\colon X_1^\bullet \otimes X_2^\bullet \to X_3^\bullet$, and similarly for Y and Z, which are (strictly) compatible with the maps f_i and g_i in the obvious sense. Then,*

1. *There exist a map (bottom horizontal), making the following diagram commute, where the top horizontal map is induced by the maps $\cup$.*
$$\begin{array}{ccc}
(X_1^\bullet \times_{Z_1^\bullet} Y_1^\bullet) \otimes (X_2^\bullet \times_{Z_2^\bullet} Y_2^\bullet) & \xrightarrow{\cup} & X_3^\bullet \times_{Z_3^\bullet} Y_3^\bullet \\
\downarrow & & \downarrow \\
(X_1^\bullet \tilde{\times}_{Z_1^\bullet} Y_1^\bullet) \otimes (X_2^\bullet \tilde{\times}_{Z_2^\bullet} Y_2^\bullet) & \xrightarrow{\cup} & X_3^\bullet \tilde{\times}_{Z_3^\bullet} Y_3^\bullet .
\end{array}$$
2. *On cohomology the induced cup product is compatible with the cup products on X's and Y's via the projections p_A and p_B. One has the following projection formula for $z \in H^*(Z_1^\bullet)$ and $w \in H^*(X_2^\bullet \tilde{\times}_{Z_2^\bullet} Y_2^\bullet)$:*
$$((i_1)_*(z)) \cup w = (i_3)_* \left[z \cup (g_2)_*(p_{B_2})_* w \right] .$$

Proof. (Compare with [Niz93, Prop. 3.1] or [Bei86, Lemma 1.11]) One chooses a parameter γ and defines the cup product by the formula

$$\begin{aligned}
(x_1,y_1,z_1) \cup (x_2,y_2,z_2) = \big(& x_1 \cup x_2, y_1 \cup y_2, \\
& z_1 \cup (\gamma f_2(x_2) + (1-\gamma) g_2(y_2)) \\
& + (-1)^{\deg x_1} ((1-\gamma) f_1(x_1) + \gamma g_1(y_1)) \cup z_2 \big) .
\end{aligned} \tag{3.3}$$

All of these products are known to be homotopic for different values of γ. Checking the required properties is straightforward from this formula, including the compatibility with p_A and p_B. For the projection formula one specifies $\gamma = 0$. □

The quasi-fibered product can be used to invert quasi-isomorphisms in a canonical way. This is done quite often in [Hub95] except that there the dual construction of the quasi-pushout was being used. We want to describe this in a systematic way so that we do not have to mention it explicitly any more.

The problem is as follows: Suppose we are given a map in the derived category of complexes between complexes $X^\bullet$ and $Y^\bullet$. We want to consider the cone of this map. Unfortunately it is well known that this cone is only unique up to a non-canonical quasi-isomorphism. In particular, if $X^\bullet$ and $Y^\bullet$ are instead presheaves of complexes the associated cone can not be naively made into a presheaf. If we are instead given an explicit and functorial description of the morphism in question, then we can construct a cone quite easily using the quasi-fibered product: Suppose the morphism is given by a sequence of morphisms connecting $X^\bullet$ to $Y^\bullet$ and all the morphisms that are going "in the wrong direction" are quasi-isomorphisms.

Definition 3.3. In the situation above we will say that there is a map $X^\bullet \to Y^\bullet$ in the generalized sense. If $X^\bullet$ and $Y^\bullet$ are functors $C \to \mathsf{Compl}$ we will say that the map $X^\bullet \to Y^\bullet$ in the generalized sense is functorial if all the intermediate maps are functorial as well.

We can always assume that the situation is given by the following diagram

$$X^\bullet \to Z_1^\bullet \xleftarrow{\sim} Z_2^\bullet \to Z_3^\bullet \xleftarrow{\sim} \cdots Z_n^\bullet \to Y^\bullet\ , \tag{3.4}$$

with all the left pointing arrow quasi-isomorphisms. Indeed, if two arrows point to the same direction we can simply compose them and if the leftmost or rightmost arrows were left pointing quasi-isomorphisms we could simply replace $X^\bullet$ by $Z_1^\bullet$ or $Y^\bullet$ by $Z_n^\bullet$. Thus the situation described above can always be reached. Now we simply replace this diagram by the diagram

$$(\tilde{X}^\bullet := X^\bullet \underset{Z_1^\bullet}{\tilde{\times}} Z_2^\bullet) \to Z_3^\bullet \xleftarrow{\sim} \cdots Z_n^\bullet \to Y^\bullet\ ,$$

where the first arrow is the composition of the projection on $Z_2^\bullet$ with the map $Z_2^\bullet \to Z_3^\bullet$. One checks easily that because $Z_2^\bullet \to Z_1^\bullet$ is a quasi-isomorphism so is the projection $\tilde{X}^\bullet \to X^\bullet$ and the map $\tilde{X}^\bullet \to Z_3^\bullet$ induces on cohomology the same map as $H(X^\bullet) \to H(Z_3^\bullet)$. Repeating this process we obtain a diagram

$$\begin{tikzcd}
\tilde{\tilde{X}}^\bullet \arrow[d] \arrow[dr] & \\
X^\bullet \arrow[r, dotted] & Y^\bullet
\end{tikzcd}$$

such that the vertical map is a quasi-isomorphisms and such that on cohomology, where the dotted arrow exists, the diagram is commutative. Thus, the cone of $\tilde{\tilde{X}}^\bullet \to Y^\bullet$ is a good replacement for the cone of the nonexistent map $X^\bullet \to Y^\bullet$.

Sometimes it will be possible to replace a morphism in the generalized sense with an honest morphism, as follows from the following trivial lemma.

Lemma 3.4. *Suppose that we have, in the situation of diagram* (3.4) *a map of complexes $R^\bullet \to S^\bullet$ and morphisms from $R^\bullet$ to all the $Z_i^\bullet$ making the extended diagram*

$$\begin{tikzcd}
X^\bullet \arrow[r] & Z_1^\bullet & Z_2 \arrow[l, "\sim"'] \arrow[r, dotted] & Y^\bullet \\
R^\bullet \arrow[u] \arrow[ur] \arrow[urr] \arrow[rrr] & & & S^\bullet \arrow[u]
\end{tikzcd}$$

commute. Then we get a map $R^\bullet \to \tilde{\tilde{X}}^\bullet$ *from the maps* $R^\bullet \to Z_i^\bullet$ *and this map makes the diagram*

$$\begin{array}{ccc} R^\bullet & \longrightarrow & S^\bullet \\ \downarrow & & \downarrow \\ \tilde{\tilde{X}} & \longrightarrow & Y \end{array}$$

commute

In particular, we get a map of cones $\mathrm{Cone}(R^\bullet \to S^\bullet) \to \mathrm{Cone}(\tilde{\tilde{X}}^\bullet \to Y^\bullet)$. This map will be a quasi-isomorphism if the maps $R^\bullet \to X^\bullet$ and $S^\bullet \to Y^\bullet$ are.

Finally, application of lemma 3.2 shows that if we have 3 diagrams as in (3.4), an original, primed and double primed, and if we have products $X^\bullet \otimes X'^\bullet \to X''^\bullet$ and similar products with the Z's and Y, and all are compatible, then there would also be products $\tilde{\tilde{X}}^\bullet \otimes \tilde{\tilde{X}}'^\bullet \to \tilde{\tilde{X}}''^\bullet$ compatible with the maps to the Y's.

4. Rigid complexes

Syntomic cohomology has two main components, rigid and de Rham cohomology. Rigid cohomology was defined by Berthelot (see [Ber97]) as the cohomology of an object in the derived category of vector spaces which is independent up to quasi-isomorphism of some auxiliary data. For the purpose of constructing Chern classes it is vital to replace these objects by canonically defined complexes, functorial with respect to the underlying scheme. This turns out to be a nontrivial task. We would like to thank Berthelot for pointing out a serious mistake in our original argument.

We will consider schemes which are separated and of finite type over κ. We will associate our rigid complexes with such a scheme X.

We introduce the required auxiliary data following Berthelot. Let j be an open embedding $j\colon X \hookrightarrow \overline{X}$ into a proper κ-scheme $\overline{X}$. Consider a closed immersion $\overline{X} \hookrightarrow \mathcal{P}$ into a p-adic formal $\mathcal{V}$-scheme $\mathcal{P}$ which is smooth in a neighborhood of X. We call the triple $(\overline{X}, j, \mathcal{P})$ a **rigid datum** for X and most of time, when j is clear from the situation, we will abbreviate this to $(\overline{X}, \mathcal{P})$.

For the formal scheme $\mathcal{P}$ there is an associated rigid analytic K-space, the **generic fiber** of $\mathcal{P}$, denoted $\mathcal{P}_K$. There is a canonical **specialization** map $\mathrm{sp}\colon \mathcal{P}_K \to \mathcal{P}$, which is continuous when $\mathcal{P}_K$ is given its strong Grothendieck topology and $\mathcal{P}$ its Zariski topology. Berthelot introduces the notion of a tube. If Y is a locally closed subset of the special fiber of a formal $\mathcal{V}$-scheme $\mathcal{P}$, the **tube** of Y in $\mathcal{P}$, denoted $]Y[_{\mathcal{P}}$, is a rigid analytic K-subspace of $\mathcal{P}_K$ whose underlying set is the set $\mathrm{sp}^{-1}(Y)$ of points whose specialization is in Y.

Now let $Z = \overline{X} - X$. Berthelot introduces the notion of a **strict neighborhood** of $]X[_{\mathcal{P}}$ inside $]\overline{X}[_{\mathcal{P}}$. By definition this is a subset $U \subset]\overline{X}[_{\mathcal{P}}$, open in the strong Grothendieck topology, such that $\{U,]Z[_{\mathcal{P}}\}$ is a covering of $]\overline{X}[_{\mathcal{P}}$ in the same topology. Let V be such a strict neighborhood. Berthelot defines a functor

$j^\dagger$ from the category of sheaves on V to itself by

$$j^\dagger(F) = \varinjlim_U j_{U*} F\ ,$$

where the direct limit is over all U which are strict neighborhoods of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$ contained in V and j_U is the canonical embedding.

Definition 4.1 (Berthelot)**.** In the situation described above, the **rigid complex in the derived category** of X with respect to the auxiliary data $(\overline{X}, \mathcal{P})$ is defined by

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}} := \mathbb{R}\Gamma(]\overline{X}[_{\mathcal{P}}, j^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}})\ .$$

Our task is to replace the object defined above in a derived category by a canonical complex representing it. We first explain the required functoriality for these complexes. We will call the datum $(X, \overline{X}, \mathcal{P})$ a **rigid triple**. These will form a category under a certain class of morphism which we are about to define.

Definition 4.2. Let $(X, \overline{X}, \mathcal{P})$ and $(Y, \overline{Y}, \mathcal{P}')$ be two rigid triples and let $f\colon X \to Y$ be a morphism of κ-schemes. Let V be a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$ and let $F\colon V \to \mathcal{P}'_K$ be a morphism of K-rigid spaces. We say that F is **compatible** with f if F maps $]X[_{\mathcal{P}}$ into $]Y[_{\mathcal{P}'}$ and there is a commutative diagram

$$\begin{array}{ccc}]X[_{\mathcal{P}} & \xrightarrow{F} &]Y[_{\mathcal{P}'} \\ {\scriptstyle \mathrm{sp}}\downarrow & & {\scriptstyle \mathrm{sp}}\downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

Lemma 4.3. *Suppose, in the notation of the previous definition, that U is a strict neighborhood of $]Y[_{\mathcal{P}'}$ in $]\overline{Y}[_{\mathcal{P}'}$ and that F is compatible with f. Then $F^{-1}(U)$ is a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$.*

Proof. We first prove the lemma under the assumption that $U =]\overline{Y}[_{\mathcal{P}'}$. Let $g\colon X \to X \times Y$ be defined by $g(x) = (x, f(x))$ and let $G\colon V \to \mathcal{P}_K \times \mathcal{P}'_K$ be defined by $G(x) = (x, F(x))$. Let W be the closure of the image of g in $\overline{X} \times \overline{Y}$. We claim that $G^{-1}(]W[_{\mathcal{P}\times\mathcal{P}'})$ is a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$. This will finish the proof of the lemma in this case because $G^{-1}(]W[_{\mathcal{P}\times\mathcal{P}'}) \subset F^{-1}(]\overline{Y}[_{\mathcal{P}'})$ (because $W \subset \overline{X} \times \overline{Y}$). To prove the claim, let $Z = (\overline{X} - X) \times \mathcal{P}'$. This is a closed subset of $\mathcal{P} \times \mathcal{P}'$. It is easy to see that $G(V) \subset]Z \cup W[_{\mathcal{P}\times\mathcal{P}'}$. Indeed, for $x \in V$ either $x \in]X[_{\mathcal{P}}$, in which case $F(x) \in]Y[_{\mathcal{P}'}$ and $\mathrm{sp}(F(x)) = f(\mathrm{sp}(x))$ so $\mathrm{sp}(G(x)) = g(\mathrm{sp}(x)) \in g(X) \subset W$, or $\mathrm{sp}(x) \in \overline{X} - X$ and then $\mathrm{sp}(G(x)) = (\mathrm{sp}(x), \mathrm{sp}(F(x))) \in Z$. According to [Ber96, Proposition 1.1.14.ii] the covering $]Z \cup W[_{\mathcal{P}\times\mathcal{P}'} =]Z[_{\mathcal{P}\times\mathcal{P}'} \cup]W[_{\mathcal{P}\times\mathcal{P}'}$ is admissible. Since G is continuous we find that the covering $V = G^{-1}(]Z[_{\mathcal{P}\times\mathcal{P}'}) \cup G^{-1}(]W[_{\mathcal{P}\times\mathcal{P}'})$ is also admissible. We now notice that $G^{-1}(]Z[_{\mathcal{P}\times\mathcal{P}'}) \subset V \cap]\overline{X} - X[_{\mathcal{P}}$ so finally $V = (V \cap]\overline{X} - X[_{\mathcal{P}}) \cup G^{-1}(]W[_{\mathcal{P}\times\mathcal{P}'})$ is also an admissible covering. This implies ([Ber96, Remark

1.2.3(iv)]) that $G^{-1}(]W[_{\mathcal{P}\times\mathcal{P}'})$ is a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$ and therefore the lemma in the case $U =]\overline{Y}[_{\mathcal{P}'}$.

For the general case we may therefore assume now, by shrinking V if needed, that $F(V) \subset]\overline{Y}[_{\mathcal{P}'}$. The proof now is an easy modification of [Ber96, Proposition (1.2.7)]. Since $\{U,]\overline{Y} - Y[_{\mathcal{P}'}\}$ is an admissible covering of $]\overline{Y}[_{\mathcal{P}'}$ it follows that $\{F^{-1}(U), F^{-1}(]\overline{Y} - Y[_{\mathcal{P}'})\}$ is an admissible covering of V. The assumption that F is compatible with f implies in particular that $F^{-1}(]\overline{Y} - Y[_{\mathcal{P}'}) \subset]Z[_{\mathcal{P}} \cap V$. From this the lemma follows. □

We write $\mathrm{Hom}_f(V, \mathcal{P}'_K)$ for the collection of morphisms $V \to \mathcal{P}'_K$ compatible with f.

Definition 4.4. Let $(X, \overline{X}, \mathcal{P})$ and $(Y, \overline{Y}, \mathcal{P}')$ be two rigid triples. We define $\mathrm{Hom}((X, \overline{X}, \mathcal{P}), (Y, \overline{Y}, \mathcal{P}'))$ to be the collection of all pairs (f, F) where $f\colon X \to Y$ is a κ-morphism and $F \in \varinjlim_V \mathrm{Hom}_f(V, \mathcal{P}'_K)$, where V runs over all strict neighborhoods of $]X[_{\mathcal{P}}$, i.e., F is a germ of a morphism compatible with f.

It follows immediately from lemma 4.3 that germs of compatible morphisms can be composed. Thus, the collection of rigid triples together with their morphisms form a category, which we denote by $\mathcal{RT}$.

Proposition 4.5. *There exist a functor*

$$\mathcal{RT} \to \mathsf{Compl}_K, \quad (X, \overline{X}, \mathcal{P}) \mapsto \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}} ,$$

such that for each rigid triple the corresponding complex represents its name sake in the derived category.

Proof. Consider the category $\mathcal{A}$ whose objects are fourtuples $\mathcal{D} = (X, \overline{X}, \mathcal{P}, V)$ where the first 3 objects form a rigid triple and V is a strict neighborhood of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$. A morphism in this category from $\mathcal{D}$ to $(Y, \overline{Y}, \mathcal{P}', U)$ consists of a morphism $f\colon X \to Y$ and a compatible morphism $F\colon V \to U$. Given such a morphism let j_X and j_Y be the corresponding embeddings. Using lemma 4.3 it is easy to see that there is a canonical map $F^*: j_Y^\dagger \Omega_U^\bullet \to F_* j_X^\dagger \Omega_V^\bullet$. Consider the functor from $\mathcal{A}$ to sites sending $\mathcal{D}$ to the rigid space V and let $\mathcal{B}$ and $\underline{\Gamma}(\mathcal{B})$ be as in section 2. It is straightforward to check that $\mathcal{D} \mapsto j_X^\dagger \Omega_V^\bullet$ is a complex in $\underline{\Gamma}(\mathcal{B})$. Therefore, the discussion of section 2 gives us a functor $\mathcal{A} \to \mathsf{Compl}_K$, $\mathcal{D} \mapsto C_{\mathcal{D}}^\bullet$, such that $C_{\mathcal{D}}^\bullet$ represents $\mathbb{R}\Gamma(V, j_X^\dagger \Omega_V^\bullet)$. Now set

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}} := \varinjlim_V C^\bullet_{(X,\overline{X},\mathcal{P},V)} , \tag{4.1}$$

where the limit is with respect to inclusion. It is clear that these complexes are functorial with respect to maps of rigid triples and the fact that by [Ber97, (1.2.5)] all maps in the direct limit are quasi-isomorphisms implies that $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}}$ indeed represents its name sake in the derived category as required. □

Definition 4.6. For each κ-scheme X_0 the category of **rigid data** for X_0, denoted $\mathcal{RD}(X_0)$ is the fiber of the forgetful functor $(\mathcal{RT} \to \mathsf{Sch})$ over X_0, i.e., is the collection of all triples $(X, \overline{X}, \mathcal{P})$ with $X = X_0$ with morphisms whose first component is the identity map of X_0.

Proposition 4.7. *A morphism of rigid data induces a quasi-isomorphism on rigid complexes.*

Proof. Suppose first that the morphism F between $(\overline{X}, \mathcal{P})$ and $(\overline{X}', \mathcal{P}')$ actually comes from a map of formal schemes $F\colon \mathcal{P} \to \mathcal{P}'$ sending $\overline{X}$ to $\overline{X}'$ and inducing the identity morphism on X. The proposition then follows from Berthelot's results concerning the independence of the rigid complexes of the auxiliary choices. The discussion before Theorem 1.6 in [Ber97] shows that F induces the morphism on rigid complexes of pairs $\mathbb{R}\Gamma_{\mathrm{rig}}((X, \overline{X}')/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}((X, \overline{X})/K)$ coming from the map $\overline{X} \to \overline{X}'$ induced by F. Then Theorem 1.6 in loc. cit. shows that this morphism is a quasi-isomorphism. For the general case let W be the closure of X embedded as the diagonal in $\overline{X} \times \overline{X}'$. Then $(W, \mathcal{P} \times \mathcal{P}')$ is a rigid datum for X and the two projections p_1 and p_2 from $\mathcal{P} \times \mathcal{P}'$ to the two factors induce maps of rigid data $p_1\colon (W, \mathcal{P} \times \mathcal{P}') \to (\overline{X}, \mathcal{P})$, $p_2\colon (W, \mathcal{P} \times \mathcal{P}') \to (\overline{X}', \mathcal{P}')$, which are of the type that was already proved to induce quasi-isomorphisms. Now consider $G(x) = (x, F(x))$. This gives a map of rigid data $(\overline{X}, \mathcal{P}) \to (W, \mathcal{P} \times \mathcal{P}')$. This map is a section to p_1 and therefore induces a quasi-isomorphism on rigid complexes. We can factor $F = p_2 \circ G$ hence the result. □

Remark 4.8. The proof was inspired by a discussion with Berthelot.

We are now ready to construct canonical rigid complexes by removing the dependency of the auxiliary data. For reasons to be discussed later, this is a bit more complicated then it may seem at first sight. The result that we get is the following.

Proposition 4.9. *There is a way to construct the following functors:*

1. *A big rigid realization functor*

$$\mathcal{RT} \to \mathsf{Compl}_K, \quad (X, \overline{X}, \mathcal{P}) \mapsto \widetilde{\mathbb{R}\Gamma}_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}} .$$

2. *A rigid realization functor*

$$\mathsf{Sch}/\kappa \to \mathsf{Compl}_K, \quad X \mapsto \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) .$$

In addition, there are quasi-isomorphisms

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \leftarrow \widetilde{\mathbb{R}\Gamma}_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}}$$

which are functorial with respect to maps of rigid triples (on the left hand side we remember only the scheme component of a morphism).

To prove this proposition, the naive idea would be to produce the rigid complex of X by taking a direct limit over all possible rigid data for X. The problem is that the category of these data is not filtered (one can not equalize morphisms of formal schemes) and so the direct limit may fail to be quasi-isomorphic to the terms of the limit. The solution to this problem is to look at the way Berthelot proves that all possible rigid complexes are quasi-isomorphic and to consider the limit only over those morphisms that show up in the proofs. This implies we should take limits over sets of a finite number of objects of $\mathcal{RD}(X)$. To handle functoriality, however, one needs to consider more objects.

Definition 4.10. For a κ-scheme X, the set $\mathcal{PT}_X$ is the set of all pairs $(f,(\overline{Y},\mathcal{P}))$, where $f\colon X\to Y$ is a morphism of κ-scheme and $(\overline{Y},\mathcal{P})\in\mathcal{RD}(Y)$. The subset $\mathcal{PT}_X^0$ contains all pairs where the first component is the identity map of X.

As usual we ignore all set theoretic problems which can be solved by using a particular universe. If we assume that our schemes are quasi-projective we can consider only projective compactifications and formal schemes inside a particular choice of a projective space avoiding all these problems.

Definition 4.11. The category $\mathcal{SET}_X$ is the category whose objects are all finite subsets of $\mathcal{PT}_X$ and whose morphisms are inclusions. We denote by $\mathcal{SET}_X^0$ the full subcategory whose objects are all subsets with a non-empty intersection with $\mathcal{PT}_X^0$.

Let $g\colon Z\to X$ be a morphism of schemes. There is then a map of sets $g^\circ\colon\mathcal{PT}_X\to\mathcal{PT}_Z$ given by $g^\circ(f,(\overline{Y},\mathcal{P}))=(f\circ g,(\overline{Y},\mathcal{P}))$. Clearly $(gh)^\circ=h^\circ\circ g^\circ$ for any two composable morphisms g and h. The map g° induces a functor, still denoted by g°, from $\mathcal{SET}_X$ to $\mathcal{SET}_Z$, by sending $A\subset\mathcal{PT}_X$ to $g^\circ(A)\subset\mathcal{PT}_Z$. Notice that this functor does not send $\mathcal{SET}_X^0$ to $\mathcal{SET}_Z^0$.

From now on it will be more convenient to denote elements of $\mathcal{PT}_X$ by letters like a and to denote the associated auxiliary datum by $(f_a,(\overline{Y}_a,\mathcal{P}_a))$. This also allows to have multiplicities by associating the same data to different elements.

Lemma 4.12. *For $A\in\mathcal{SET}_X^0$ consider the formal scheme $\mathcal{P}_A:=\prod_{a\in A}\mathcal{P}_a$. Let $\overline{X}_A$ be the closure of the image of X under the map $X\to\prod_{a\in A}\overline{Y}_a$ given by $j_A(x)=(f_a(x)\in\overline{Y}_a)_a$. Then $j_A\colon X\to\overline{X}_A$ is an open immersion and $\mathcal{P}_A$ is smooth in a neighborhood of X. Thus $(\overline{X}_A,\mathcal{P}_A)\in\mathcal{RD}(X)$. When $A\subset B$, the natural projections $\overline{X}_B\to\overline{X}_A$ and $\mathcal{P}_B\to\mathcal{P}_A$ induce a map or rigid data $(\overline{X}_B,\mathcal{P}_B)\to(\overline{X}_A,\mathcal{P}_A)$ and the induced morphism $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_A,\mathcal{P}_A}\to\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_B,\mathcal{P}_B}$ is a quasi-isomorphism by proposition* 4.7.

Proof. The only thing one needs to remark is that the assumption that A includes at least one pair $(\mathrm{id},(?,?))$ guarantees that the map $X\to\overline{X}_A$ is an open embedding. ☐

To simplify notation, we will now define

$$\mathcal{F}_X(A):=\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X}_A,\mathcal{P}_A}\ .$$

This is then a covariant functor $\mathcal{SET}_X^0\to\mathsf{Compl}_K$ and all morphisms of complexes one obtains are quasi-isomorphisms. We will also need to consider the following more complicated situation. Suppose there are sets $\mathcal{PT}'_X$ and ${\mathcal{PT}^0_X}'$ with a projection map $\Pi\colon \mathcal{PT}'_X \to \mathcal{PT}_X$ sending ${\mathcal{PT}^0_X}'$ to $\mathcal{PT}^0_X$. Then for $a \in \mathcal{PT}'_X$ we can define $\overline{Y}_a := \overline{Y}_{\Pi(a)}$, $\mathcal{P}_a := \mathcal{P}_{\Pi(a)}$ and define $\overline{X}_A$ and $\mathcal{P}_A$ for $A \subset \mathcal{PT}'_X$ in the same way as before. We obtain a functor $\mathcal{F}_X : {\mathcal{SET}^0}'_X \to \mathsf{Compl}_K$ as before, where ${\mathcal{SET}^0}'_X$ denotes the obvious construction. For $A \in {\mathcal{SET}^0}'_X$ the canonical projection $\Pi\colon A \to \Pi(A)$ induces canonical maps, "diagonals on identical elements", $\Delta_A\colon (X_{\Pi(A)}, \mathcal{P}_{\Pi(A)}) \to (X_A, \mathcal{P}_A)$ which by proposition 4.7 induce quasi-isomorphisms on rigid complexes:

$$\mathbb{R}\Gamma_{\mathrm{rig}}(\Delta_A)\colon \mathcal{F}_X(A) \to \mathcal{F}_X(\Pi(A)) . \tag{4.2}$$

Going to the limit we obtain a map, which is again a quasi-isomorphism as in both limits all maps are,

$$\boldsymbol{\Delta}\colon \varinjlim_{A\in {\mathcal{SET}^0}'_X} \mathcal{F}_X(A) \to \varinjlim_{A\in \mathcal{SET}^0_X} \mathcal{F}_X(A) . \tag{4.3}$$

Definition 4.13. We define the rigid complex

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) := \varinjlim_{A\in \mathcal{SET}^0_X} \mathcal{F}_X(A) .$$

Since $\mathcal{SET}^0_X$ is clearly filtered this complex is quasi isomorphic to each of the complexes in the limit.

When given a map $g\colon Z \to X$ and sets $B \in \mathcal{SET}^0_Z$ and $C \in \mathcal{SET}^0_X$, the map $g^\circ\colon C \to B \cup g^\circ(C)$ induces a projection $\mathcal{P}_{B\cup g^\circ(C)} \to \mathcal{P}_C$ which is compatible with the map g in the obvious way. The induced map $\mathcal{F}_X(C) \to \mathcal{F}_Z(B \cup g^\circ(C))$ we denote by g°_*.

Proposition 4.14. *for any map of κ-schemes $g\colon Z \to X$ there is a unique map $g^*\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(Z/K)$ making the following diagram commute*

$$\begin{tikzcd}
\varinjlim_C \mathcal{F}_X(C) \arrow[r, "g^*"] & \varinjlim_B \mathcal{F}_Z(B) \\
\varinjlim_{(B,C)} \mathcal{F}_X(C) \arrow[u, "(B,C)\mapsto C"] \arrow[r, "g^\circ_*"] & \varinjlim_{(B,C)} \mathcal{F}_Z(B\cup g^\circ(C)) \arrow[u, "(B,C)\mapsto B\cup g^\circ(C)"']
\end{tikzcd}$$

where in this notation one takes the limits over $B \in \mathcal{SET}^0_Z$ and $C \in \mathcal{SET}^0_X$. For any two composable morphisms $W \xrightarrow{h} Z \xrightarrow{g} X$ we have $(gh)^* = h^* \circ g^*$.

Proof. The existence and uniqueness of g^* follow immediately if we show that the vertical maps in the diagram are isomorphisms (not just quasi-isomorphisms). In fact, it is enough to show that for the left vertical map. Both maps are induced from the indicated functors between the index categories and both functors are easily seen to be final hence the result. To show the compatibility with composable maps consider the following diagram whose commutativity is obvious

$$\begin{tikzcd}
\varinjlim\limits_{(A,B,C)} \mathcal{F}_X(C) \arrow[r, "g^\circ_*"] \arrow[rr, bend right, "(gh)^\circ_*"'] & \varinjlim\limits_{(A,B,C)} \mathcal{F}_Z(B\cup g^\circ(C)) \arrow[r, "h^\circ_*"] & \varinjlim\limits_{(A,B,C)} \mathcal{F}_W(A\cup h^\circ(B)\cup (gh)^\circ(C)).
\end{tikzcd}$$

The result now follows by checking that each of the arrows in the diagram is isomorphic to the corresponding arrow between the rigid complexes, i.e., by checking the commutativity and the fact that the vertical arrows are isomorphisms in the following diagrams:

$$\begin{tikzcd}
\varinjlim\limits_{(B,C)} \mathcal{F}_X(C) \arrow[r, "g^\circ_*"] & \varinjlim\limits_{(B,C)} \mathcal{F}_Z(B\cup g^\circ(C)) \\
\varinjlim\limits_{(A,B,C)} \mathcal{F}_X(C) \arrow[r, "g^\circ_*"] \arrow[u, "(A,B,C)\mapsto(B,C)"] & \varinjlim\limits_{(A,B,C)} \mathcal{F}_Z(B\cup g^\circ(C)) \arrow[u, "(A,B,C)\mapsto(B,C)"]
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim\limits_{(A,B)} \mathcal{F}_Z(B) \arrow[r, "h^\circ_*"] & \varinjlim\limits_{(A,B)} \mathcal{F}_W(A\cup h^\circ(B)) \\
\varinjlim\limits_{(A,B,C)} \mathcal{F}_Z(B\cup g^\circ(C)) \arrow[r, "h^\circ_*"] \arrow[u, "(A,B,C)\mapsto(A,B\cup g^\circ(C))"] & \varinjlim\limits_{(A,B,C)} \mathcal{F}_W(A\cup h^\circ(B)\cup (gh)^\circ(C)) \arrow[u, "(A,B,C)\mapsto(A,B\cup g^\circ(C))"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\varinjlim\limits_{(A,C)} \mathcal{F}_X(C) \arrow[r, "(gh)^\circ_*"] & \varinjlim\limits_{(A,C)} \mathcal{F}_W(A\cup g^\circ(C)) \\
\varinjlim\limits_{(A,B,C)} \mathcal{F}_X(C) \arrow[r, "(gh)^\circ_*"] \arrow[u, "(A,B,C)\mapsto(A,C)"] & \varinjlim\limits_{(A,B,C)} \mathcal{F}_W(A\cup h^\circ(B)\cup g^\circ(C)).\arrow[u, "(A,B,C)\mapsto(A,C)"]
\end{tikzcd}$$

□

This completes the construction of the functor $\mathbb{R}\Gamma_{\mathrm{rig}}$. The following lemma is obvious.

Lemma 4.15. *In the situation described before* (4.2) *suppose we have a map* $g'_\circ\colon \mathcal{PT}'_X \to \mathcal{PT}'_Z$ *compatible with the map* g° *and preserving the* 0 *subsets. Then we obtain a map* g'^* *between the obvious complexes compatible with the maps* $\mathbb{R}\Gamma_{\mathrm{rig}}(\Delta_A)$. *For composable maps* g' *and* h' *we have* $(g'h')^* = g'^*\circ h'^*$.

Suppose now that we have a fixed rigid triple $(X,\overline{X},\mathcal{P}_X)$. Consider then the set $\mathcal{PT}_{X,\overline{X},\mathcal{P}_X}$ of all morphisms of rigid triples from $(X,\overline{X},\mathcal{P}_X)$ to another rigid triple and the subset $\mathcal{PT}^0_{X,\overline{X},\mathcal{P}_X}$ consisting of the identity morphism. There is then a canonical forgetful projection $\Pi\colon \mathcal{PT}_{X,\overline{X},\mathcal{P}_X} \to \mathcal{PT}_X$ and of course $\Pi(\mathcal{PT}^0_{X,\overline{X},\mathcal{P}_X}) \subset \mathcal{PT}^0_X$. Therefore we can use the considerations preceding (4.2). In particular we can define a category $\mathcal{SET}^0_{X,\overline{X},\mathcal{P}_X}$ in the obvious manner.

Definition 4.16. We define

$$\widetilde{\mathbb{R}\Gamma}_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}_X} := \varinjlim_{A\in\mathcal{SET}^0_{X,\overline{X},\mathcal{P}_X}} \mathcal{F}_X(A)\ .$$

We can now check that with this definition proposition 4.9 holds. The functoriality and the natural transformation to the complexes $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$ are an immediate application of (4.3) and lemma 4.15. Suppose $A \in \mathcal{SET}^0_{X,\overline{X},\mathcal{P}_X}$ and consider the rigid datum $(\overline{X}_A, \mathcal{P}_A)$ defined in lemma 4.12. We define map of rigid data $(\overline{X}, \mathcal{P}) \to (\overline{X}_A, \mathcal{P}_A)$ as follows: By definition there is a strict neighborhood U of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$ and, for each $a \in A$, a map of rigid spaces $F_a\colon U \to (\mathcal{P}_a)_K$ which maps $]X[_{\mathcal{P}}$ to $]Y_a[_{\mathcal{P}_a}$ and is compatible under the specialization map with $f_a\colon X \to Y_a$. The product of these maps gives a map $F\colon U \to (\mathcal{P}_A)_K$ whose restriction to $]X[_{\mathcal{P}}$ lands in $]j_A(X)[_{\mathcal{P}_A}$ and is compatible under the specialization map with j_A hence is a morphism of rigid data. It induces a quasi-isomorphism $\mathcal{F}_X(A) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}_X}$. These maps are compatible with the maps in $\mathcal{SET}^0_{X,\overline{X},\mathcal{P}_X}$. Going to the limit we obtain the desired map $\widetilde{\mathbb{R}\Gamma}_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}}$ and it is easy to check that it has the required naturality. This completes the proof of proposition 4.9.

Remark 4.17. One might hope that maps induced by morphisms of rigid triples on the complexes $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}}$ will be compatible with the maps between the $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$. Unfortunately we do not know how to prove this and at the moment we do not believe this to be true. The heuristic reason is that the complexes $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X},\mathcal{P}}$ have no reason to be very "big" in general. On the other hand, a compatibility like we suggested would mean that all pullbacks of an element by maps of rigid triples have to map to the same element in $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$, which seems implausible. We will be satisfied instead with the situation described in proposition 4.9.

Definition 4.18. Let $X_\bullet$ be a simplicial κ-scheme. By applying in each component of $X_\bullet$ the functor $\mathbb{R}\Gamma_{\mathrm{rig}}(?/K)$ we obtain a cosimplicial object in the category of complexes of K-vector spaces. We define $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\bullet/K)$ to be the total complex of the associated double complex.

This construction is functorial on the category of simplicial κ-schemes. We have the usual spectral sequence:

Proposition 4.19. *Let $X_\bullet = (X_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a simplicial κ-scheme. Then there exists a spectral sequence*

$$E_2^{i,j} = H^i_{\mathrm{rig}}(X_j/K) \Rightarrow H^{i+j}_{\mathrm{rig}}(X_\bullet/K)\ .$$

Proposition 4.20. *Let X be a κ-scheme and let $\mathcal{U}_\bullet \to X$ be the covering associated to a finite Čech covering of X (we view X as a simplicial scheme which is X in each degree). Then the canonical map $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{U}_\bullet/K)$ is a quasi-isomorphism.*

Proof. Let the Čech covering be $\{U_1, \dots, U_n\}$. Then

$$\mathcal{U}_n = \coprod_{|I|=n+1} U_I, \qquad U_I := \bigcap_{i\in I} U_i \ .$$

We choose a compactification $j\colon X \to \overline{X}$ and an embedding $\overline{X} \hookrightarrow \mathcal{P}$. This then defines a compactification $U_I \to X \to \overline{X}$, denoted j_I, for each U_I and we thus get a rigid datum $(\overline{X}, j_I, \mathcal{P})$ for each U_I. The identity maps on $\mathcal{P}$ defines, for every morphisms $U_I \to U_J$ that appear in the definition of $\mathcal{U}_\bullet$, a map compatible with this morphism. It follows that $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{U}_\bullet/K)$ is quasi-isomorphic to the total complex of the double complex

$$\bigoplus_{|I|=n+1} \mathbb{R}\Gamma(]\overline{X}[_{\mathcal{P}}, j_I^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}}) \ .$$

It follows from [Ber96, Prop. 2.1.8] or [Ber97, 1.2.ii] that this last complex is quasi-isomorphic to $\mathbb{R}\Gamma(]\overline{X}[_{\mathcal{P}}, j^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}})$ and hence to $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$. □

We state 4.21, 4.22 and 4.23 below for schemes but they immediately extend to simplicial schemes as well.

Proposition 4.21. *Let $\mathcal{V} \to \mathcal{V}'$ be a finite map of discrete valuation rings where $\mathcal{V}'$ has residue field κ' and fraction field K' and let X be a κ-scheme. Then there is a canonical base change map*

$$K' \otimes_K \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X \otimes \kappa'/K') \ ,$$

which is a quasi-isomorphism. The base change map is functorial in the obvious sense with respect to diagrams $\mathcal{V} \to \mathcal{V}' \to \mathcal{V}''$ and commutes with the maps induced by morphisms of κ-schemes.

Proof. Let $(\overline{X}, \mathcal{P})$ be a rigid datum for X over $\mathcal{V}$. One obtains a rigid datum $(\overline{X}', \mathcal{P}') = (\overline{X} \otimes \kappa', \mathcal{P} \otimes_{\mathcal{V}} \mathcal{V}')$ for $X \otimes \kappa'$ over $\mathcal{V}'$. In this situation the proof of [Ber97, Proposition 1.8] shows the existence of a map $K' \otimes_K \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\overline{X}', \mathcal{P}'} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X \otimes \kappa'/K')_{\overline{X}', \mathcal{P}'}$ which is a quasi-isomorphism. One checks that this is functorial in rigid triples so gives rise to the indicated map which is then forced to be a quasi-isomorphism. The functoriality statements are straightforward. □

Corollary 4.22. *Suppose κ is perfect and recall that $\mathcal{V}_0$ is the Witt ring of κ. Let $\sigma\colon \mathcal{V}_0 \to \mathcal{V}_0$ be the map induced by the p-power map on κ. Then there exists a canonical and natural σ-semilinear map $\phi\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0)$.*

Proof. Let π be the projection $X \to \kappa$ and let $X \xrightarrow{\mathrm{Fr}_X \times \pi} X \otimes_{\kappa, \mathrm{Fr}_\kappa} \kappa$ be the relative Frobenius map. Here the map $\kappa \to \kappa$ in the last tensor product is the Frobenius map of κ, i.e., the p-power map. The map ϕ is obtained as the composition

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0) &\xrightarrow{1\otimes \mathrm{id}} K_0 \otimes_\sigma \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0)\\ &\xrightarrow{\text{base change}} \mathbb{R}\Gamma_{\mathrm{rig}}(X \otimes_{\kappa,\mathrm{Fr}_\kappa} \kappa/K_0) \xrightarrow{(\mathrm{Fr}_X \times \pi)^*} \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0),\end{aligned}$$

where the base change map is with respect to the map σ. Naturality is easily verified. □

The following lemma will be needed for the comparison between syntomic cohomology and modified syntomic cohomology. Its truth is obtained by a careful application of the functoriality properties of the base change.

Lemma 4.23. *Suppose, under the assumptions of corollary 4.22 that κ is a finite field with $q = p^r$ elements, which implies that $\mathrm{Fr}^r \colon X \to X$ is κ-linear. Then $\phi^r = (\mathrm{Fr}^r)^*$ as endomorphisms of $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$.*

5. de Rham complexes and comparison

The next step is to define a de Rham complex. This was already done by Huber [Hub95, Chapter 7] so we do not go into all the details. We need to know not only a complex computing de Rham cohomology, but also complexes computing all the filtered parts. Here K can be any field of characteristic 0. Let X be a smooth K-scheme. A de Rham datum for X is an injection $i\colon X \hookrightarrow Y$ where Y is a smooth and proper K-scheme and $D := Y - X$ is a divisor with normal crossings.

Definition 5.1. To a de Rham datum (Y) and to every $k \in \mathbb{Z}_{\geq 0}$ we associate a complex, called the k-th filtered part of the **de Rham complex** of X with respect to the datum (Y), defined by

$$\mathrm{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y := \mathbb{R}\Gamma(Y, \Omega^{\geqslant k}_{Y/K}\langle \log D\rangle) .$$

The k-th filtered part of the **de Rham complex** of X is defined by

$$\mathrm{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) := \varinjlim_{Y} \mathrm{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_Y ,$$

where the limit is over all de Rham data.

We will write $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ for $\mathrm{Fil}^0 \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$. Note that the Fil^k, in spite of their name, are not subcomplexes of $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ but there are natural maps

$$\mathrm{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) . \tag{5.1}$$

The final ingredient needed for the construction of syntomic cohomology is a comparison between de Rham and rigid cohomology. Let X be a smooth $\mathcal{V}$-scheme with generic fiber X_K and closed fiber X_κ. We will define a functorial map in the generalized sense, $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. We stress that this map is not a quasi-isomorphism in general. First we take the limit over all de Rham data (Y) of the map

$$\begin{aligned}(5.2)\qquad &\mathbb{R}\Gamma(Y,\Omega^\bullet_{Y/K}\langle\log(Y-X_K)\rangle)\to\mathbb{R}\Gamma(Y,i_*\Omega^\bullet_{X_K/K})\\ &\to\mathbb{R}\Gamma(X_K,\Omega^\bullet_{X_K/K})\to\mathbb{R}\Gamma(X_K^{\mathrm{an}},\Omega^\bullet_{X_K^{\mathrm{an}}})\;,\end{aligned}$$

where X_K^{an} is the rigid analytic K-space associated with X_K [Ber96, 0.3.3], to obtain a map $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\to\mathbb{R}\Gamma(X_K^{\mathrm{an}},\Omega^\bullet_{X_K^{\mathrm{an}}})$. Now consider a compactification $X\xrightarrow{j}\overline{X}$. Let $\hat{\overline{X}}$ be the p-adic completion of $\overline{X}$. It is not so hard to see that X_K^{an} is a strict neighborhood of $]X_\kappa[_{\hat{\overline{X}}}$ inside $]\overline{X}_\kappa[_{\hat{\overline{X}}}=\overline{X}_K^{\mathrm{an}}$. By (4.1) we have a functorial quasi-isomorphism

$$\mathbb{R}\Gamma(X_K^{\mathrm{an}},j_\kappa^\dagger\Omega^\bullet_{X_K^{\mathrm{an}}})\to\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X}_\kappa,\hat{\overline{X}}}\;.$$

We have a map

$$\mathbb{R}\Gamma(X_K^{\mathrm{an}},\Omega^\bullet_{X_K^{\mathrm{an}}})\to\mathbb{R}\Gamma(X_K^{\mathrm{an}},j_\kappa^\dagger\Omega^\bullet_{X_K^{\mathrm{an}}})\;,$$

which is not a quasi-isomorphism in general. Composing we get a map

$$\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\to\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X}_\kappa,\hat{\overline{X}}}\;.$$

By taking the limit over all $\overline{X}$ we obtain the required diagram

$$\begin{aligned}(5.3)\qquad \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)&\to\varinjlim_{\overline{X}}\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X}_\kappa,\hat{\overline{X}}}\\ &\xleftarrow{\sim}\varinjlim_{\overline{X}}\widetilde{\mathbb{R}\Gamma}_{\mathrm{rig}}(X_\kappa/K)_{\overline{X}_\kappa,\hat{\overline{X}}}\xrightarrow{\sim}\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K).\end{aligned}$$

One checks readily that the entire diagram is functorial with respect to X.

Remark 5.2. Note that the morphisms in the diagrams above are not as innocent as they may seem, and most of them are in fact only morphisms in the generalized sense. For example, let us see how one gets a morphism $\mathbb{R}\Gamma(Z,\Omega^\bullet_{Z/K})\to\mathbb{R}\Gamma(Z^{\mathrm{an}},\Omega^\bullet_{Z^{\mathrm{an}}})$ for a K-variety Z functorial with respect to maps of K-varieties. First one resolves $\Omega^\bullet_{Z^{\mathrm{an}}}$ in a functorial way with respect to all maps of analytic spaces (this will be required later for other comparisons) to get a complex $I_Z^\bullet$. Then one gets maps $\Omega^\bullet_{Z/K}\to u_*^Z I_Z^\bullet$, where u_Z is the map between the analytic and the algebraic sites of Z. Now we can use these maps to find resolutions $J_Z^\bullet$ of $\Omega^\bullet_{Z/K}$ and $L_Z^\bullet$ of $I_Z^\bullet$ together with maps $J_Z^\bullet\to u_*^Z L_Z^\bullet$. The required map can now be written as $\Gamma(Z,J_Z^\bullet)\to\Gamma(Z^{\mathrm{an}},L_Z^\bullet)\xleftarrow{\sim}\Gamma(Z^{\mathrm{an}},I_Z^\bullet)$. If however Z is affine it is easily seen using the discussion in section 3 and in particular lemma 3.4 that this generalized map can be replaced by the map $\Gamma(Z,\Omega^\bullet_{Z/K})\to\Gamma(Z^{\mathrm{an}},\Omega^\bullet_{Z^{\mathrm{an}}})$.

6. Syntomic cohomology

We are now ready to define syntomic cohomology. We assume that κ is perfect. Let X be a smooth $\mathcal{V}$-scheme. By combining (5.1) and diagram (5.3) we have a (generalized) map $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. In addition, we have a semi-linear Frobenius map $\phi\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0)$. We deduce a map

$$\operatorname{Cone}\left(1-\frac{\phi}{p^n}\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0)\right)[-1] \\ \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)\ .$$

Definition 6.1. The **syntomic complex** of X twisted by n is defined to be

$$\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) := \operatorname{Cone}\left(1-\frac{\phi}{p^n}\right)[-1] \underset{\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)}{\tilde{\times}} \operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\ .$$

The i-th cohomology of $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n)$ will be denoted $H^i_{\mathrm{syn}}(X,n)$.

The above construction is evidently functorial in X. We can therefore define $\mathbb{R}\Gamma_{\mathrm{syn}}$ for simplicial schemes as in definition 4.18. We have the analogue of proposition 4.20:

Proposition 6.2. *Let X be a smooth $\mathcal{V}$-scheme and let $\mathcal{U}_\bullet \to X$ be the covering associated to a finite Čech covering of X. Then the canonical map $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{syn}}(\mathcal{U}_\bullet,n)$ is a quasi-isomorphism for any $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Because $\mathbb{R}\Gamma_{\mathrm{syn}}$ is defined as an iterated cone, it is enough to check the statement of the proposition on each of the components of the cone. But for the de Rham components it is well known and for the rigid components it was proved in proposition 4.20. ☐

We proceed to show some of the fundamental properties of syntomic cohomology.

Proposition 6.3. *There is a long exact sequence,*

(6.1)

$$\cdots \to H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \oplus \operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \xrightarrow{①} H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \oplus H^{i-1}_{\mathrm{rig}}(X_\kappa/K) \\ \to H^i_{\mathrm{syn}}(X,n) \\ \to H^i_{\mathrm{rig}}(X_\kappa/K_0) \oplus \operatorname{Fil}^n H^i_{\mathrm{dR}}(X_K/K) \xrightarrow{②} H^i_{\mathrm{rig}}(X_\kappa/K_0) \oplus H^i_{\mathrm{rig}}(X_\kappa/K) \to \cdots$$

where the maps ① and ② are given in the appropriate degrees by

$$(x,y) \mapsto \left(\left(1-\frac{\phi}{p^n}\right)x, x-y\right)\ . \tag{6.2}$$

Here, for the second component we have identified both x and y with their images in $H^i_{\mathrm{rig}}(X_\kappa/K)$. In particular, if K is finite over K_0, then $H^i_{\mathrm{syn}}(X,n)$ is a finite dimensional K_0-vector space for every i and n.

Proof. By writing explicitly the quasi-fibered product in term of cones, one finds

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \cong \mathrm{Cone}(&\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\\ &\to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))[-1],\end{aligned}$$

where the map defining the cone is given by (6.2) (the reader should compare at this point the construction of [Niz97, 2.1]). This immediately gives the result. □

Remark 6.4. Let us consider the special case where X is a smooth K-scheme considered as a $\mathcal{V}$-scheme. In this case we have $X_\kappa = \emptyset$, so $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/?) = 0$ with $? = K$ or K_0. The long exact sequence (6.1) shows that

$$H^i_{\mathrm{syn}}(X,n) \cong \mathrm{Fil}^n H^i_{\mathrm{dR}}(X/K)\ .$$

This is perhaps to be expected since this is the "absolute" cohomology for varieties over a field.

Definition 6.5. The cup product map on syntomic cohomology,

$$\cup\colon H^i_{\mathrm{syn}}(X,n) \times H^j_{\mathrm{syn}}(X,m) \to H^{i+j}_{\mathrm{syn}}(X,n+m)\ ,$$

is constructed as follows: By lemma 3.2 it is enough to construct a product $\mathrm{Cone}(1-\phi_n) \times \mathrm{Cone}(1-\phi_m) \to \mathrm{Cone}(1-\phi_{n+m})$, with $\phi_n = \phi/p^n$. This is achieved by the formula, similar to (3.3),

$$\begin{aligned}(x_1,z_1)\cup(x_2,z_2) = \big(&x_1\cup x_2,\\ &z_1\cup(\gamma x_2 + (1-\gamma)\phi_m(x_2))\\ &+(-1)^{\deg x_1}((1-\gamma)x_1+\gamma\phi_n(x_1))\cup z_2\big)\ .\end{aligned} \tag{6.3}$$

This definition is compatible with the definitions given by Nizioł, Kato, Gros and many others.

7. Construction of syntomic regulators

In this section we construct syntomic Chern classes,

$$c_j^p\colon K_p(X) \to H^{2j-p}_{\mathrm{syn}}(X,j)\ .$$

The method follows mostly Huber [Hub95, Chapter 18] with some input from Gros [Gro90] and Deligne [Del74].

The main step in the construction is to repeat the computation of the de Rham cohomology of $\mathbf{B}_\bullet\mathrm{GL_n}$ by Deligne [Gro90, Chapter II] for rigid cohomology. We briefly recall the setup from loc. cit., but using the notation of Deligne in [Del74, 6].

We will work simultaneously over any of the bases κ, $\mathcal{V}$, $\mathcal{V}_0$ K or K_0, making the needed adjustments. If G is an algebraic group (over any of the bases above) acting on a scheme X we let $[X/G]_\bullet$ be the simplicial scheme such that $[X/G]_n = (G^{\Delta_n}\times X)/G$ where G acts by $g\cdot(g_0,\dots,g_n,x) = (g_0g^{-1},\dots,g_ng^{-1},gx)$ and the face and degeneracy maps are the obvious ones [Del74, 6.1.2]. Note that the quotients are well defined and in fact there is an isomorphism $G^n\times X \xrightarrow{\sim} [X/G]_n$ given by (for example) $(g_1,\dots g_n,x)\mapsto(1,g_1,\dots g_n,x)$.

Lemma 7.1. *Let X be a principal G-bundle over $S = X/G$. Then the map $[X/G]_\bullet \to S$ induces an isomorphism on rigid cohomology.*

Proof. (sketch). If we knew how to write rigid cohomology as a sheaf cohomology this would follow from [Del74, 6.1.2.2]. We need to check that what we know about rigid cohomology is sufficient for a proof. Let $X_\bullet = \operatorname{cosq}(X \to S)$. There is a canonical isomorphism of simplicial schemes over S,

$$X_\bullet \cong [X/G]_\bullet$$

[Del74, 6.1.2.a]. If there is a section $S \to X$, then it extends to a section $s\colon S \to X_\bullet$ to the canonical map $\pi\colon X_\bullet \to S$. It is well known that the map $s \circ \pi$ is homotopic to the identity map of $X_\bullet$ and this homotopy induces a homotopy on the rigid complexes showing the result in this case. In the general case we have a finite covering, $U = \coprod U_i$, of S such that the restriction of X to each U_i has a section. Let $\mathcal{U}_\bullet = \operatorname{cosq}(U \to S)$. An application of the spectral sequence 4.19, proposition 4.20, and the special case of a map with a section discussed above now shows that the cohomology of the bisimplicial set

$$(\operatorname{cosq}(X \times_S \mathcal{U}_n \to \mathcal{U}_n))_{?,n} = (\operatorname{cosq}(X_m \times_S U \to X_m))_{m,?}$$

is isomorphic to the cohomology of $\mathcal{U}_\bullet$, hence of S, on the one hand, and to the cohomology of $X_\bullet$ on the other hand. □

Fix $N \geq n$ in $\mathbb{Z}_{\geq 0}$. Let $E = \mathbb{G}_a^{\oplus n}$ and $F = \mathbb{G}_a^{\oplus N}$ be two vector group schemes and let $\underline{\mathrm{Hom}}(E,F)$ be the corresponding scheme of homomorphisms. There is a filtration of $\underline{\mathrm{Hom}}(E,F)$ by open subschemes

$$\underline{\mathrm{Hom}}(E,F) = U_n \supset U_{n-1} \supset \cdots \supset U_0 \ ,$$

where U_l is defined by the invertability of at least one $n-l$ minor.

Lemma 7.2. *The scheme $U_l - U_{l-1}$ is a smooth subscheme of $\underline{\mathrm{Hom}}(E,F)$ of codimension $l(l-n+N)$.*

Proof. This is proved in [Gro90, II.2.4] for schemes over $\mathcal{V}_0$ but the proof is the same in any of the other cases. □

The group $G = \mathrm{GL}_n$ acts on $\underline{\mathrm{Hom}}(E,F)$ in the obvious manner, preserving the filtration by the U_i. The scheme U_0 is the so called Stiefel variety of n-frames on F and is denoted by $\underline{\mathrm{Stief}}(E,F)$. We have

$$\underline{\mathrm{Stief}}(E,F)/G \cong \underline{\mathrm{Grass}}_n(F) \ , \tag{7.1}$$

where $\underline{\mathrm{Grass}}_n(F)$ is the grassmanian of n-dimensional subspaces of F.

Proposition 7.3. *The canonical map*

$$H^*_{\mathrm{rig}}([\underline{\mathrm{Hom}}(E,F)/G]_\bullet/K) \to H^*_{\mathrm{rig}}([\underline{\mathrm{Stief}}(E,F)/G]_\bullet/K)$$

is an isomorphism in degrees $\leq 2(N-n)$.

Proof. (Compare [Gro90, Corollaire II.2.8]). It is enough to show the same for the map induced on rigid cohomology by each of the inclusions $[U_{l-1}/G]_\bullet \to [U_l/G]_\bullet$. By lemma 7.2 we see that on the n-th component, $[U_l/G]_n - [U_{l-1}/G]_n$ is a closed subscheme of $[U_l/G]_n$ of codimension $l(l-n+N) \geq N-n+1$. By purity for rigid cohomology [Ber97, Corollaire 5.7] the map $H^i_{\mathrm{rig}}([U_l/G]_n/K) \to H^i_{\mathrm{rig}}([U_{l-1}/G]_n/K)$ is an isomorphism if $i \leq 2(N-n)$. The result now follows from the spectral sequence 4.19. $\square$

Proposition 7.4. *There are canonical classes $x_i \in \mathrm{Fil}^i H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K)$ such that we have isomorphisms*

$$K[x_1,\dots,x_n] \xrightarrow{\sim} H^*_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \xrightarrow{\sim} H^*_{\mathrm{rig}}(\mathbf{B}_\bullet \mathrm{GL_n}/K)\ . \tag{7.2}$$

If $K = K_0$ and we identify the classes x_i with their images in H_{rig}, then we have $\phi(x_i) = p^i x_i$.

Proof. Let $*$ be the one point space. Then $\mathbf{B}_\bullet \mathrm{GL_n} = [*/G]_\bullet$ [Del74, 6.1.3]. We have a G-equivariant diagram,

$$\begin{tikzcd}
\underline{\mathrm{Stief}}(E,F) \arrow[rr] \arrow[dr] & & \underline{\mathrm{Hom}}(E,F) \arrow[dl, "\pi"'] \\
& * \arrow[ur, dotted, bend right, "0"'] &
\end{tikzcd}$$

where 0 denotes the 0 section to π. It induces a corresponding diagram of cohomologies,

$$\begin{tikzcd}
H^i_{\mathrm{rig}}([\underline{\mathrm{Stief}}(E,F)/G]_\bullet/K) & & H^i_{\mathrm{rig}}([\underline{\mathrm{Hom}}(E,F)/G]_\bullet/K) \arrow[ll] \arrow[dl, dotted, bend left] \\
& H^i_{\mathrm{rig}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \arrow[ul] \arrow[ur, "\pi^*"] &
\end{tikzcd}$$

An easy diagram chase using proposition 7.3 shows that the left diagonal map is injective for $i \leq 2(N-n)$. A similar argument shows the same for de Rham cohomology. By lemma 7.1 and (7.1) it now follows that the two horizontal maps in the commutative diagram

$$\begin{tikzcd}
H^i_{\mathrm{rig}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \arrow[r] & H^i_{\mathrm{rig}}(\underline{\mathrm{Grass}_n}(F)/K) \\
H^i_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \arrow[r, "\alpha"] \arrow[u] & H^i_{\mathrm{dR}}(\underline{\mathrm{Grass}_n}(F)/K) \arrow[u, "\sim"']
\end{tikzcd} \tag{7.3}$$

are injective for $i \leq 2(N-n)$. The map on the right is an isomorphism since $\underline{\mathrm{Grass}_n}(F)$ is proper.

In de Rham cohomology we have a good theory of characteristic classes. Let $x_i \in \mathrm{Fil}^i H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K)$ be the i-th Chern class of the universal bundle. Then $\alpha(x_i)$ are the Chern classes of the universal vector bundle over $\underline{\mathrm{Grass}_n}(F)$ and it is known that these generate the cohomology ring of $\underline{\mathrm{Grass}_n}(F)$. It follows that α is surjective, hence that if $i \leq 2(N-n)$ all maps in diagram (7.3) are isomorphisms. Varying N we find the isomorphisms (7.2). It now follows that the properties of the classes x_i can be tested in the cohomology of $\underline{\mathrm{Grass}}_n(F)$, where they are well known: As $\underline{\mathrm{Grass}}_n(F)$ is proper we have an isomorphism

$$H^{2i}_{\mathrm{dR}}(\underline{\mathrm{Grass}}_n(F)/K_0) \cong H^{2i}_{\mathrm{cr}}(\underline{\mathrm{Grass}}_n(F)/\mathcal{V}_0) \otimes K_0 \ ,$$

under which x_i correspond to the crystalline Chern classes of the universal bundle and therefore have the right behavior under Frobenius. □

We can now define Chern classes in syntomic cohomology. From proposition 7.4 it follows that $\mathbf{B}_\bullet\mathrm{GL}_\mathrm{n}$ has cohomology only in even dimensions. Using the long exact sequence (6.1) we easily obtain an isomorphism

$$H^{2i}_{\mathrm{syn}}(\mathbf{B}_\bullet\mathrm{GL}_\mathrm{n} \otimes \mathcal{V}, i) \cong \{x \in \mathrm{Fil}^i\, H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet\mathrm{GL}_\mathrm{n}/K_0) \colon \phi(x) = p^i x\} \ . \tag{7.4}$$

In particular, we see that the classes x_i of proposition 7.4 define classes, denoted C_i^n, in $H^{2i}_{\mathrm{syn}}(\mathbf{B}_\bullet\mathrm{GL}_\mathrm{n} \otimes \mathcal{V}, i)$. Considering the usual inductive system of $\mathbf{B}_\bullet\mathrm{GL}_\mathrm{n}$-s, obtained by the inclusions "in the upper left corner" $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$, we see that the C_i^n are compatible under the induced maps on cohomology because the de Rham universal classes are known to do so. We thus obtained cohomology classes C_i in the cohomology of the ind-scheme $\mathbf{B}_\bullet\mathrm{GL}$ which we call the **universal syntomic Chern classes**.

Theorem 7.5. *Let X be a smooth $\mathcal{V}$-scheme. There exist functorial Chern classes*

$$c_j^p \colon K_p(X) \to H^{2j-p}_{\mathrm{syn}}(X, j) \ ,$$

such that their composition with the map $H^{2j-p}_{\mathrm{syn}}(X,j) \to \mathrm{Fil}^j\, H^{2j-p}_{\mathrm{dR}}(X_K/K)$ obtained from the sequence(6.1) *gives the usual Chern classes in de Rham cohomology.*

Proof. We follow Huber's treatment in [Hub95, Chapter 18]. By [Hub95, Proposition 18.1.5] (whose proof is also valid in our case) we have an isomorphism

$$\varinjlim_{U_\bullet} \pi_p \,\mathrm{Tot}(\mathbb{Z} \times \mathbb{Z}_\infty \mathbf{B}_\bullet\mathrm{GL}(U_\bullet)) \xrightarrow{\sim} K_p(X) \ ,$$

where the direct limit is over all finite affine Čech coverings $U_\bullet$ of X. By [Hub95, Proposition 18.1.7 b] there are induced maps

$$K_p(X) \to \varinjlim_{U_\bullet} \pi_p \,\mathrm{Tot}(\mathbb{Z}_\infty \mathbf{B}_\bullet\mathrm{GL}(U_\bullet)), \qquad K_0(X) \to \mathbb{Z} \ . \tag{7.5}$$

For simplicial schemes $U_\bullet$ and $Y_\bullet$, let $B(U_\bullet, Y_\bullet)$ be the simplicial cosimplicial abelian group which is the $\mathbb{Q}$-vector space generated by $\mathrm{Hom}(U_n, Y_m)$ in degree (m,n) and let $A(U_\bullet, Y_\bullet)$ be the associated complex [Hub95, Definition 18.2.1]. Summation of pullback maps give a map of simplicial cosimplicial groups,

$$B(U_\bullet, Y_\bullet) \to [\mathrm{Hom}(\mathbb{R}\Gamma_{\mathrm{syn}}(Y_m, j), \mathbb{R}\Gamma_{\mathrm{syn}}(U_n, j))]_{m,n} \ ,$$

where Hom here means in the category of complexes (this is why we insisted on defining the syntomic cohomology on the level of complexes). By taking the associated complexes and then the total complexes we obtain a map

$$A(U_\bullet, Y_\bullet) \to \mathbb{R}\mathrm{Hom}(\mathbb{R}\Gamma_{\mathrm{syn}}(Y_\bullet, j), \mathbb{R}\Gamma_{\mathrm{syn}}(U_\bullet, j)) \; . \tag{7.6}$$

In the special case that $Y_\bullet = \mathbf{B}_\bullet\mathrm{GL}$, we have by [Hub95, Lemma 18.2.4] a map

$$\pi_p \operatorname{Tot}(\mathbb{Z}_\infty \mathbf{B}_\bullet\mathrm{GL}(U_\bullet)) \to H^{-p}(A(U_\bullet, \mathbf{B}_\bullet\mathrm{GL})) \; .$$

Composing this with the map induced on cohomology by (7.6) and applying to the universal class $C_j \in H^{2j}(\mathbb{R}\Gamma_{\mathrm{syn}}(\mathbf{B}_\bullet\mathrm{GL}, j))$ we get a map

$$\pi_p \operatorname{Tot}(\mathbb{Z}_\infty \mathbf{B}_\bullet\mathrm{GL}(U_\bullet)) \to H^{2j-p}_{\mathrm{syn}}(U_\bullet, j) \; .$$

If $U_\bullet$ is as in (7.5), then $H^{2j-p}_{\mathrm{syn}}(U_\bullet, j) \cong H^{2j-p}_{\mathrm{syn}}(X, j)$ by proposition 6.2. This completes the construction. The result about the composition with the projection to de Rham cohomology follows from the universal case and functoriality. □

Definition 7.6. The Chern character $ch\colon K_i(X) \to \bigoplus_j H^{2j-i}_{\mathrm{syn}}(X, j)$ is given by

$$ch = \sum_{j\geq 1} -\frac{(-1)^{j-1}}{(j-1)!} c^i_j \quad (+\,\mathrm{Rank} \quad \text{if} \quad i = j = 0) \; .$$

Proposition 7.7. *The Chern character is multiplicative.*

Proof. Everything is reduced to the properties of the universal Chern classes (see for example [Hub95, 18.3]). These properties are deduced in the following way. By (7.4), the natural map $H^{2i}_{\mathrm{syn}}(\mathbf{B}_\bullet\mathrm{GL_n} \otimes \mathcal{V}, i) \to H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet\mathrm{GL_n}/K_0)$ is an injection. Since this natural map is compatible with cup products by lemma 3.2 all properties of Chern classes for syntomic cohomology can be read off from the corresponding results for de Rham Chern classes, which are well known. □

8. Modified syntomic cohomology

In this section we define a certain modification of the rigid syntomic cohomology of section 6. The difference is that we replace the semi-linear Frobenius by a linear Frobenius. This makes the theory easier to compute. For the purpose of computing regulators in higher K-theory the modified theory is as good as the original one.

In this section we need the additional assumption that $\kappa \subset \bar{F}_p$. The following notion is due to Coleman.

Definition 8.1. Let X be a κ-scheme. A **Frobenius endomorphism**, $\varphi\colon X \to X$, of degree $q = p^r$ is any κ-endomorphism of X obtained in the following way: Let X' be an F_q-scheme and let $\alpha\colon X \xrightarrow{\sim} X' \otimes_{F_q} \kappa$ be a κ-isomorphism. Then $\varphi = \alpha^{-1} \circ (\mathrm{Fr}^r \otimes \mathrm{id}_\kappa) \circ \alpha$.

It is clear that if φ is a Frobenius endomorphism of degree q then φ^k is a Frobenius endomorphism of degree q^k.

Definition 8.2. The category of Frobenius endomorphisms of X is the category whose objects are Frobenius endomorphisms $\varphi\colon X \to X$. There is a unique morphism between φ and φ^k for any $k \geq 1$.

Lemma 8.3. *The category of Frobenius endomorphisms of X is filtered.*

Proof. It is not hard to see that sufficiently high powers of any two Frobenius endomorphisms become identical. □

Fix an integer n. We associate to each Frobenius endomorphism a certain complex, in such a way that we get a functor on the category of all Frobenius endomorphisms. To φ of degree q we associate the complex

$$\operatorname{Cone}\left(1 - \frac{\varphi^*}{q^n}\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)\right)[-1]\,.$$

To the morphism $\varphi \to \varphi^m$ we associate the map of cones induced by the commutative diagram

$$\begin{array}{ccc} \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) & \xrightarrow{1-(\varphi^*/q^n)} & \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \\ = \big\downarrow & & \big\downarrow \sum_{s=0}^{m-1}(\varphi^*/q^n)^s \\ \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) & \xrightarrow{1-(\varphi^*/q^n)^n} & \mathbb{R}\Gamma_{\mathrm{rig}}(X/K). \end{array} \tag{8.1}$$

Definition 8.4. The **modified syntomic complex** associated with a Frobenius endomorphism φ is the complex

$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi := \operatorname{Cone}\left(1 - \frac{\varphi^*}{q^n}\right)[-1] \underset{\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)}{\tilde{\times}} \operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\,,$$

where q is the degree of φ and the cone is the one discussed above. The **modified syntomic complex** of X is

$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) = \varinjlim_{\varphi} \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi\,,$$

where the direct limit is over the category of all Frobenius endomorphisms and the connecting maps are the ones defined above. The cohomology of the modified syntomic complex is called **modified syntomic cohomology** and denoted by $H^i_{\mathrm{ms}}(X,n)$.

Lemma 8.5. *The modified syntomic complexes, and hence the modified syntomic cohomologies, are functorial.*

Proof. One need only observe that any morphism of varieties over κ is already defined over some finite field, which implies that for any morphism $f\colon X \to Y$ and for a cofinal collection of Frobenius endomorphisms $\varphi\colon Y_\kappa \to Y_\kappa$ there is a Frobenius endomorphism $\varphi'\colon X_\kappa \to X_\kappa$ making the obvious diagram commute. □

Most of the basic properties of the modified syntomic cohomology are concentrated in the following proposition.

Proposition 8.6. 1. *There is a canonical quasi-isomorphism,*

(8.2)
$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \cong \varinjlim_{\varphi} \mathrm{Cone}\left(1-\frac{\varphi^*}{q^n}\colon\ \mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)\right)[-1]\,,$$

where the limit is over all Frobenius endomorphisms φ, the notation $1-\varphi^*/q^n$ stands for this map composed with the map $\mathbb{R}\Gamma_{\mathrm{dR}} \to \mathbb{R}\Gamma_{\mathrm{rig}}$ and the transition maps are constructed using a diagram analogous to (8.1). *Furthermore, if φ is any fixed Frobenius endomorphism of degree q, then we also have the quasi-isomorphism*

(8.3)
$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \cong \varinjlim_{k} \mathrm{Cone}\left(1-\left(\frac{\varphi^*}{q^n}\right)^k\colon\ \mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)\right)[-1]\,.$$

2. *If κ is a finite field, then there is a canonical and functorial map*
$$\Xi\colon \mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)\,.$$
3. *There are canonical and functorial maps*

(8.4)
$$\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)[-1] \to \mathbb{R}\Gamma_{\mathrm{syn}}(X,n), \qquad \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)[-1] \to \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)\,.$$

When κ is a finite field these maps are compatible with the map Ξ. These maps induce isomorphisms,

(8.5)
$$H^i_{\mathrm{syn}}(X,n) \cong H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\,\mathrm{Fil}^n\,H^{i-1}_{\mathrm{dR}}(X_K/K)\,,$$

and, if κ is finite,

(8.6)
$$H^i_{\mathrm{ms}}(X,n) \cong H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\,\mathrm{Fil}^n\,H^{i-1}_{\mathrm{dR}}(X_K/K)\,,$$

(at least) in the following two cases:
- *X is proper over $\mathcal{V}$ and $2n \neq i, i-1, i-2$,*
- *X is affine and $n \geq i > \mathrm{reldim}\, X$.*

In particular, if in either of these cases κ is a finite field, then Ξ induces an isomorphism on degree i cohomology.

4. *Suppose $\mathcal{V}'$ is a finite extension of $\mathcal{V}$ with field of fractions K' and let $X' = X \otimes_{\mathcal{V}} \mathcal{V}'$. Then there exists a canonical base change quasi-isomorphism $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \otimes_K K' \to \mathbb{R}\Gamma_{\mathrm{ms}}(X',n)$.*
5. *There are cup products in modified syntomic cohomology compatible with the products in syntomic cohomology under the map Ξ and also compatible with base change.*

Proof. 1. Let φ be a Frobenius endomorphism of X_κ. In the definition 8.4 of $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi$ we may replace the quasi-fibered product $\tilde{\times}$ by an ordinary product because the cone on the left hand side surjects on $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. The resulting complex is easily seen to be isomorphic to the level φ complex of (8.2). The second part of the assertion follows because for a fixed φ the collection of powers

φ^k is cofinal in the category of Frobenius endomorphisms.
2. We first construct a map $\mathrm{Cone}(1-\phi/p^n\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \circlearrowleft) \to \mathrm{Cone}(1-\varphi^*/q^n\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \circlearrowleft)$ for some Frobenius endomorphism φ (the notation $\circlearrowleft$ means that the map is from the complex to itself). Suppose κ is a finite field with $q=p^r$ elements. Then $\varphi = \mathrm{Fr}^r$ is a Frobenius endomorphism of X_κ and by lemma 4.23 we have $\phi^r = \varphi^*$ on $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0)$. It follows that we can define the required map by using a diagram similar to (8.1). This map can then be composed with the extension of scalars map to give a map

$$\mathrm{Cone}(1-\phi/p^n\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \circlearrowleft) \to \mathrm{Cone}(1-\varphi^*/q^n\colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \circlearrowleft)\ .$$

By the construction of the (modified) syntomic complexes we now obtain a map $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi$ by taking the identity maps on the other components of the quasi-fibered product. This map we may compose with the map to the limit on all Frobenius endomorphisms to complete the construction. For schemes over κ our particular φ commutes with all maps and this easily gives functoriality.
3. The maps (8.4) are instances of the map (3.2) (or a limit of a map of this kind for modified syntomic cohomology) with $X^\bullet = \mathrm{Cone}\,(1-\varphi^*/q^n)\,[-1]$ for the left hand map, $X^\bullet = \mathrm{Cone}\,(1-\phi/p^n)\,[-1]$ for the right hand map and $Y^\bullet = \mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)$, $Z^\bullet = \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. The compatibility with the map Ξ is clear since it is induced by a map between the components which is the identity on the $Z^\bullet$ component. We show that the maps (8.4) induce isomorphisms on cohomology in the stated cases for syntomic cohomology, the proof for modified cohomology being essentially the same. We abbreviate Cone for $\mathrm{Cone}(1-\phi/p^n)[-1]$. From the construction of syntomic cohomology as a quasi-fibered product, which is again a cone, we get the following long exact sequence.

$$\begin{aligned}\cdots &\to H^{i-1}(\mathrm{Cone}) \oplus \mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \to H^{i-1}_{\mathrm{rig}}(X_\kappa/K) \to H^i_{\mathrm{syn}}(X,n)\\ &\to H^i(\mathrm{Cone}) \oplus \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) \to H^i_{\mathrm{rig}}(X_\kappa/K) \to \cdots\ .\end{aligned}$$

It follows that the map $H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\,\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \to H^i_{\mathrm{syn}}(X,n)$ is an isomorphism if $H^{i-1}(\mathrm{Cone}) = H^i(\mathrm{Cone}) = 0$ and the map $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) \to H^i_{\mathrm{rig}}(X_\kappa/K)$ is an injection. This last requirement holds in the cases considered because in the proper case $H^i_{\mathrm{dR}}(X_K/K) \cong H^i_{\mathrm{rig}}(X_\kappa/K)$ and when $n > \mathrm{reldim}\, X$ we have $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) = 0$. The long exact sequence for the cohomology of Cone,

$$\begin{aligned}\cdots &\to H^{i-2}_{\mathrm{rig}}(X_\kappa/K_0) \xrightarrow{1-\phi/p^n} H^{i-2}_{\mathrm{rig}}(X_\kappa/K_0) \to H^{i-1}(\mathrm{Cone})\\ &\to H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \xrightarrow{1-\phi/p^n} H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \to H^{i-1}(\mathrm{Cone})\\ &\to H^{i}_{\mathrm{rig}}(X_\kappa/K_0) \xrightarrow{1-\phi/p^n} H^{i}_{\mathrm{rig}}(X_\kappa/K_0) \to \cdots\ ,\end{aligned}$$

shows that the i-th and $i-1$-th cohomologies of Cone vanish when $1-\phi/p^n$ is an isomorphism on $H_{\mathrm{rig}}(X_\kappa/K_0)$ in degrees i, $i-1$ and $i-2$. This now follows from

the theory of weights: By [ClS98] and [Chi98] the K_0-linear Frobenius, which is a power of ϕ, has weight j when acting on $H^j_{\mathrm{rig}}(X_\kappa/K_0)$ when X is proper and has mixed weights between j and $2j$ in general. In the proper case it follows that if $2n \neq j$ for $j = i-2$, $i-1$ and i, then the operator ϕ/p^n has no fixed vector on $H^j_{\mathrm{rig}}(X_\kappa/K_0)$ because some power of it does not. It follows that $1-\phi/p^n$ is injective, hence bijective, on the degree $i-2$, $i-1$ and i cohomologies. In the second case this is no longer true a-priori for $j = i$ but $H^i_{\mathrm{rig}} = 0$ because X is affine and $i > \operatorname{reldim} X$.
4. Apply base change (proposition 4.21 for rigid cohomology) in each component.
5. The construction of the cup product is almost identical to the one we did for syntomic cohomology. The cup product on $\operatorname{Cone}(1-\varphi^*/q^n \colon \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \circlearrowleft)$ is given by the formula (6.3) with $\phi_m = \varphi^*/q^m$. One then needs to check that these products are compatible up to homotopy under the transition maps. This can be done by a direct laborious computation. A much more conceptual and general way of understanding this is given in [Bes99]. This type of compatibility also implies that the product is compatible with the map Ξ. Compatibility with base change is clear. $\square$

Remark 8.7.
1. We expect the base change isomorphism of proposition 8.6.4 to exist for infinite extensions as well, at least on the level of cohomology.
2. Using the model (8.3) for modified syntomic cohomology it is easy to see that the cup product is given in level k by the formula (6.3) with ϕ_m being $(\varphi^*/q^n)^k$ composed with $\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$.
3. Suppose $K = K_0$. An argument similar to the proof of proposition 8.6.1 shows that
$$\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \cong \operatorname{Cone}(\operatorname{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \xrightarrow{1-\phi/p^n} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))[-1]\,.$$
This gives rise to a long exact sequence
$$\begin{aligned}\cdots \to \operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) &\xrightarrow{1-\phi/p^n} H^{i-1}_{\mathrm{rig}}(X_\kappa/K) \to H^i_{\mathrm{syn}}(X,n)\\ \to \operatorname{Fil}^n H^i_{\mathrm{dR}}(X_K/K) &\xrightarrow{1-\phi/p^n} H^i_{\mathrm{rig}}(X_\kappa/K) \to \cdots\,.\end{aligned}$$
In the cases discussed in proposition 8.6.1 this reduces to a short exact sequence
$$0 \to \operatorname{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \xrightarrow{1-\phi/p^n} H^{i-1}_{\mathrm{rig}}(X_\kappa/K) \to H^i_{\mathrm{syn}}(X,n) \to 0\,.$$
The isomorphism (8.6) is induced by the map sending $x \in H^{i-1}_{\mathrm{rig}}(X_\kappa/K)$ to the image in $H^i_{\mathrm{syn}}(X,n)$ of $(1-\phi/p^n)x$. A similar analysis applies to modified syntomic cohomology. See proposition 10.1.3 for a special case.

When we compose the syntomic Chern classes with the canonical map of cohomology theories $\Xi \colon H_{\mathrm{syn}} \to H_{\mathrm{ms}}$ we obtain modified syntomic Chern classes and Chern characters. Alternatively, one can construct these directly using the same techniques as before and universal Chern classes which are the images of the syntomic ones under the map Ξ in the cohomology of $\mathbf{B}_\bullet \mathrm{GL_n}$. This makes the following lemma evident

Proposition 8.8. *The modified syntomic Chern classes behave well under base change, i.e., when X is an $\mathcal{V}$-scheme, $\mathcal{V}'$ is a finite extension of $\mathcal{V}$ and $X'/\mathcal{V}'$ is the scheme obtained by base change to $\mathcal{V}'$, there is a commutative diagram*

$$\begin{array}{ccc} K_p(X) & \xrightarrow{c_j^p} & H^{2j-p}_{\mathrm{ms}}(X,j) \\ \downarrow & & \downarrow \\ K_p(X') & \xrightarrow{c_j^p} & H^{2j-p}_{\mathrm{ms}}(X',j). \end{array}$$

9. Comparison with other cohomology theories

In this section we compare our constructions with some other versions of syntomic cohomology and regulators and then also with étale cohomology and regulators.

As mentioned in the introduction, in [Gro94], Gros defines, for X smooth over $\mathcal{V}$ and $K = K_0$, rigid syntomic cohomology $H^i(X, s(n)_{X/K,\mathrm{rig}})$. When X is affine he further defines higher Chern classes into this cohomology. The main difference with our construction is that no attention to log singularities is given. We generalize his construction to the case $K \neq K_0$ as follows.

We first define complexes $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ following [Gro94, I.3.2]. As many of the details are similar to the constructions in sections 4–6 we allow ourselves to be a bit sketchy here. Let $j\colon X \hookrightarrow \overline{X}$ be a compactification of X over $\mathcal{V}$ and let $\overline{X} \hookrightarrow Y$ be a closed embedding of $\overline{X}$ into another $\mathcal{V}$-scheme which is smooth in a neighborhood of X. We have $\overline{X}_K^{\mathrm{an}} \subset]\overline{X}_\kappa[_{\hat{Y}}$.

Definition 9.1. The n-th filtered part of the rigid complex of X_κ relative to the data of $\overline{X}$ and Y is defines as

$$\mathrm{Fil}^n_{\overline{X}} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X},Y} := \varinjlim_U \mathbb{R}\Gamma(U, j_\kappa^\dagger \, \mathrm{Fil}^n_{\overline{X}} \Omega_U^\bullet) \, ,$$

where the limit is over all strict neighborhood of U of $]X_\kappa[_{\hat{Y}}$ in $]\overline{X}_\kappa[_{\hat{Y}}$ and

$$\mathrm{Fil}^n_{\overline{X}} \Omega_U^\bullet := I^n \to I^{n-1}\Omega_U^1 \to I^{n-2}\Omega_U^2 \to \cdots \, ,$$

with I the ideal defining $\overline{X}_K^{\mathrm{an}}$ in $]\overline{X}_\kappa[_{\hat{Y}}$ and letting $I^k = (1)$ for $k \leq 0$.

By results of Berthelot [Gro94, I.3.3, I.3.5] if we fix $\overline{X}$ then a smooth morphism of Y's will induce a quasi-isomorphism on these complexes, and if we forget Y and view the new complex as well defined in the derived category then a proper morphism of $\overline{X}$'s will again induced a quasi-isomorphism. Thus one can repeat the methods of section 4 and obtain complexes $\mathrm{Fil}^n_X \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ which are functorial complexes in X with natural maps (in the generalized sense) $\mathrm{Fil}^n_X \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ for each n.

Remark 9.2. Since we are assuming that X is smooth, we can always find at least one $\overline{X}$ and Y by just taking $Y = \overline{X}$. Note however, that we could be in this situation in general if we assumed that X was quasi-projective or if we generalized our construction to allow local embeddings of a covering of $\overline{X}$. Thus the entire theory can be developed for non-smooth schemes.

Definition 9.3. We define complexes $\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n)$ and $\widetilde{\mathbb{R}\Gamma}_{\mathrm{ms}}(X,n)$ by replacing in the definition of $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n)$ and $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)$ the map $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$ with the map $\mathrm{Fil}^n_X \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. The associated cohomologies will be denoted by $\tilde{H}^i_{\mathrm{syn}}(X,n)$ and $\tilde{H}^i_{\mathrm{ms}}(X,n)$.

Proposition 9.4. *Assume $K = K_0$. Then there exists a canonical isomorphism $\tilde{H}^i_{\mathrm{syn}}(X,n) \cong H^i(X, s(n)_{X/K,\mathrm{rig}})$, where the latter cohomology is the one defined by Gros.*

Proof. As in remark 8.7.3, when $K = K_0$ the construction of $\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n)$ simplifies to $\mathrm{Cone}(\mathrm{Fil}^n_X \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \xrightarrow{1-\phi/p^n} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))[-1]$. By choosing particular data this is easily seen to be quasi-isomorphic to the construction of Gros. □

Proposition 9.5. *There is a functorial commutative square of maps*

$$\begin{array}{ccc}
\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) & \longrightarrow & \widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n) \\
\downarrow & & \downarrow \\
\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) & \longrightarrow & \widetilde{\mathbb{R}\Gamma}_{\mathrm{ms}}(X,n).
\end{array}$$

In particular, we obtain a functorial map of cohomology theories $H^*_{\mathrm{syn}}(X,n) \to H^*(X, s(n)_{X/K,\mathrm{rig}})$.

Proof. The left vertical map has already been defined and the right vertical map is defined in exactly the same way. To construct the horizontal maps one only has to define maps $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathrm{Fil}^n_X \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)$. To that end, let $j\colon X \hookrightarrow \overline{X}$ be a compactification of X and let $i\colon X_K \hookrightarrow Y$ be a de Rham datum for X_K. Consider $U = X_K^{\mathrm{an}}$. Then we have $\mathrm{Fil}^n_{\overline{X}} \Omega^\bullet_U = \Omega^{\geqslant n}_U$. We can therefore obtain a map, in a similar manner to (5.2),

$$\begin{aligned}
&\mathbb{R}\Gamma(Y, \Omega^{\geqslant n}_{Y/K}\langle \log(Y - X_K)\rangle) \to \mathbb{R}\Gamma(Y, i_* \Omega^{\geqslant n}_{X_K/K}) \\
&\to \mathbb{R}\Gamma(X_K, \Omega^{\geqslant n}_{X_K/K}) \to \mathbb{R}\Gamma(U, \Omega^{\geqslant n}_U) \\
&\to \mathbb{R}\Gamma(U, j^\dagger_\kappa \Omega^{\geqslant n}_U) \to \mathrm{Fil}^n_{\overline{X}} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X},\overline{X}}.
\end{aligned}$$

Taking limits gives the required map. □

The following result will be needed in section 10

Lemma 9.6. *The restriction of $H^i_{\mathrm{ms}}(X,n) \to \tilde{H}^i_{\mathrm{ms}}(X,n)$ to $\mathrm{Ker}(H^i_{\mathrm{ms}}(X,n) \to H^i_{\mathrm{dR}}(X_K/K))$ is injective if $n \geq i$*

Proof. It is enough to show this at each finite level, so we may fix a Frobenius endomorphism φ and work with the associated modified complexes. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) & \xrightarrow{①} & H^{i-1}_{\mathrm{rig}}(X_\kappa/K) & \longrightarrow & H^i_{\mathrm{ms}}(X,n)_\varphi & \longrightarrow & \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) \\
\downarrow & & \| & & \downarrow & & \\
\mathrm{Fil}^n H^{i-1}_{\mathrm{rig}}(X_\kappa/K) & \xrightarrow{①} & H^{i-1}_{\mathrm{rig}}(X_\kappa/K) & \longrightarrow & \tilde{H}^i_{\mathrm{ms}}(X,n)_\varphi & &
\end{array}$$

where the map ① is $1-\varphi^*/q^n$. Under our assumption $\mathrm{Fil}^n H^{i-1}_{\mathrm{rig}}(X_\kappa/K)=0$, so the result follows from an easy diagram chase. $\square$

In [Niz99] and [Niz97] Nizioł defines another version of syntomic cohomology, this time based on the convergent cohomology of Ogus. This amounts to ignoring both logarithmic singularities and overconvergent singularities. We will need one of the versions of this definition.

Definition 9.7 (Nizioł [Niz97, Proof of Lemma 2.1])**.** Let X be a smooth quasi-projective $\mathcal{V}$-scheme. The f-cohomology of X with values in the sheaf $\mathcal{K}(n)$, $H^*_f(X,\mathcal{K}(n))$, is defined as the cohomology of the complex

$$\begin{aligned}
&\mathbb{R}\Gamma(X,\mathcal{S}^\bullet(n)) := \\
&\qquad \mathrm{Cone}(H((X_\kappa/\mathcal{V}_0)_{\mathrm{conv}},\mathcal{K}_{X_\kappa/\mathcal{V}_0}) \oplus H((X_\kappa/\mathcal{V})_{\mathrm{conv}},F^n_X) \\
&\qquad\quad \to H((X_\kappa/\mathcal{V}_0)_{\mathrm{conv}},\mathcal{K}_{X_\kappa/\mathcal{V}_0}) \oplus H((X_\kappa/\mathcal{V})_{\mathrm{conv}},\mathcal{K}_{X_\kappa/\mathcal{V}}))[-1].
\end{aligned}$$

Here, H denotes the derived functor of the global section functor, $\mathcal{K}_{X_\kappa/\mathcal{V}}$ is the canonical sheaf on the convergent topos and F^n_X is its standard filtration. The map defining the cone is given by 6.2.

Proposition 9.8. *There is a canonical map*

$$\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma(X,\mathcal{S}^\bullet(n)) ,$$

which is an isomorphism if X is proper.

Proof. It follows from [Ogu90, Theorem 0.6.6] that there are functorial maps (in the derived category) $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/L) \to H((X_\kappa/\mathcal{O}_L)_{\mathrm{conv}},\mathcal{K}_{X_\kappa/\mathcal{O}_L})$ for $L=K$ or K_0, which are quasi-isomorphisms if X is proper. One can check that these maps further induce maps $\mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \to H((X_\kappa/\mathcal{V})_{\mathrm{conv}},F^n_X)$. As in the proof of proposition 6.3 we see that $\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n)$ can be written as

$$\begin{aligned}
\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n) \cong{} & \mathrm{Cone}(\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \\
& \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K))[-1] ,
\end{aligned}$$

which makes the existence of the required map obvious. When X is proper the map is a quasi-isomorphism because each of its components is. $\square$

Proposition 9.9. *For X smooth and quasi-projective there is a functorial map of cohomology theories $H^*_{\mathrm{syn}}(X,n) \to H^*_f(X,\mathcal{K}(n))$ which is an isomorphism when X is proper and which commutes with Chern classes.*

Proof. To construct the map we simply compose $H_{\mathrm{syn}} \to \tilde{H}_{\mathrm{syn}} \to H_f$. We have shown both maps to be isomorphisms when X is proper. To show compatibility with Chern classes it is enough to check that the universal Chern classes in the cohomology of $\mathbf{B}_\bullet \mathrm{GL_n}$ are the same. But we know that $\mathbf{B}_\bullet \mathrm{GL_n}$ only has de Rham, rigid and convergent cohomologies in even degrees. This implies that the universal Chern classes coincide if their projection on de Rham cohomology do. But these projections are simply the corresponding universal de Rham Chern classes. Indeed, this is true by construction for H_{syn} and for H_f it follows from [Niz97, Lemma 2.2]. ☐

Finally, the comparison with Nizioł's cohomology allows us to connect our version of syntomic cohomology with étale cohomology. By [Niz99] and [Niz97, Cor. 3.1] there is a functorial map of cohomology theories $H_f^*(X, \mathcal{K}(n)) \to H_{\text{ét}}^*(X_K, \mathbb{Q}_p(n))$ which is compatible with Chern classes. Here ét denotes continuous étale cohomology as defined by Jannsen [Jan88].

Corollary 9.10. *For X smooth and quasi-projective there is a functorial map of cohomology theories $H_{\mathrm{syn}}^*(X, n) \to H_{\text{ét}}^*(X_K, \mathbb{Q}_p(n))$ which is compatible with Chern classes.*

To make the relation with étale cohomology even clearer, we note the following proposition.

Proposition 9.11. *Let X be smooth and projective. For all versions of syntomic cohomology (which are all the same in this case) The composed map*

$$\begin{aligned} &H_{\mathrm{dR}}^{i-1}(X_K/K)/\operatorname{Fil}^n H_{\mathrm{dR}}^{i-1}(X_K/K) \cong H_{\mathrm{rig}}^{i-1}(X_\kappa/K)/\operatorname{Fil}^n H_{\mathrm{dR}}^{i-1}(X_K/K) \\ &\qquad \xrightarrow{8.6.3} H_{\mathrm{syn}}^i(X, n) \to H_{\text{ét}}^i(X_K, \mathbb{Q}_p(n)) \to H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p(n)) \;, \end{aligned}$$

is 0, where the last map comes from the Hochschild-Serre spectral sequence. This and the spectral sequence in turn give a map

$$H_{\mathrm{dR}}^{i-1}(X_K/K)/\operatorname{Fil}^n H_{\mathrm{dR}}^{i-1}(X_K/K) \to H^1(\mathrm{Gal}(\bar{K}/K), H_{\text{ét}}^{i-1}(X_{\bar{K}}, \mathbb{Q}_p(n))) \;.$$

This map is the Bloch-Kato exponential map associated with the $\mathrm{Gal}(\bar{K}/K)$ representation $H_{\text{ét}}^{i-1}(X_{\bar{K}}, \mathbb{Q}_p(n))$.

Proof. This is an immediate consequence of [Niz99, Theorem 5.2] together with the observation that the map $l\colon H_f^1(\mathcal{V}, D) \to H^1(G_K, L(D))$ for an associated isocrystal D appearing there is just the Bloch-Kato map corresponding to the Galois representation $L(D)$. The same result is proved by Nekovář in [Nek98] and one need only assume properness instead of projectiveness. We did not try however to compare his version of syntomic cohomology with ours, though they are surely identical in the proper case. ☐

10. Regulators for functions

In this section we consider the cohomology $H^i_{\mathrm{ms}}(X,i)$ (so the degree equals the twist) of a smooth affine $\mathcal{V}$-scheme $X = \operatorname{Spec}(A)$. Fix a compactification $j\colon X \hookrightarrow \overline{X}$ over $\mathcal{V}$. Let $\mathcal{P} = \hat{\overline{X}}$. We get a rigid datum $(\overline{X}_\kappa, \mathcal{P})$. By the proof of proposition 1.10 in [Ber97] we see that the map

$$\Omega^\bullet_{A^\dagger,K} := \varinjlim_U \Gamma(U, \Omega^\bullet_U) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X}_\kappa,\mathcal{P}} ,$$

is a quasi-isomorphism, where the limit is over all strict neighborhoods of $]X_\kappa[_{\overline{X}_K}$. We remark that $\Omega^\bullet_{A^\dagger,K}$ is in fact the complex of differentials of the dagger algebra $A^\dagger$ used in the Monsky-Washnitzer cohomology [MW68, vdP86], but we will not need this fact here. Now fix a Frobenius endomorphism φ of degree q of X_κ. It follows from lifting theorems for dagger algebras ([Col85, Thm A-1] or [vdP86, Thm 2.4.4.ii]) that there is a lifting ϕ of φ to the dagger algebra $A^\dagger$. This implies that there are strict neighborhoods $U' \subset U''$ and a map $\phi\colon U' \to U''$ whose reduction is φ. It follows that (φ, ϕ) defines a map of the rigid triples from $(X_\kappa, \overline{X}_\kappa, \mathcal{P})$ to itself. Using proposition 4.9 we therefore obtain a commutative diagram

$$\begin{tikzcd}
\Omega^\bullet_{A^\dagger,K} \arrow[r, "\phi^*"] \arrow[d] & \Omega^\bullet_{A^\dagger,K} \arrow[d] \\
\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K) \arrow[r, "\varphi^*"] & \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K),
\end{tikzcd}$$

where the vertical maps are (generalized) quasi-isomorphisms.

Now choose any de Rham datum $X_K \hookrightarrow Y$ for X_K. By Hodge theory [Del71, Corollaire 3.2.13.ii] we see that the space

$$\Omega^i(X_K/K)_{\log} := H^0(Y, \Omega^i_{Y/K}\langle \log(Y - X_K)\rangle)$$

is independent of the choice of Y and is isomorphic to $\operatorname{Fil}^i H^i_{\mathrm{dR}}(X_K/K) = H^i(Y, \Omega^{\geqslant i}_{Y/K}\langle \log(Y - X_K)\rangle)$.

Proposition 10.1. *Let X and ϕ be as above.*

1. *There is a canonical isomorphism*

(10.1)
$$H^i_{\mathrm{ms}}(X,i) \cong \varinjlim_k \Big\{ (\omega, h)\colon \omega \in \Omega^i(X_K/K)_{\log},$$
$$h \in \Omega^{i-1}_{A^\dagger,K}/d\Omega^{i-2}_{A^\dagger,K}, \quad dh = \left(1 - \left(\frac{\phi^*}{q^i}\right)^k\right)\omega \Big\} ,$$

where we abusively identified ω *with its image in* $\Omega^i_{A^\dagger,K}$. *The connecting map between level* k *and level* km *is given by*

$$(\omega,h)\mapsto\left(\omega,\sum_{s=0}^{m-1}(\phi^*/q^n)^{sk}h\right). \tag{10.2}$$

2. *The cup product* $H^i_{\mathrm{ms}}(X,i)\times H^j_{\mathrm{ms}}(X,j)\to H^{i+j}_{\mathrm{ms}}(X,i+j)$ *is given in level* k *by the formula*

$$\begin{aligned}(\omega_1,h_1)\cup(\omega_2,h_2)=\Bigg(&\omega_1\wedge\omega_2,\\ &h_1\wedge\left(\gamma+(1-\gamma)\left(\frac{\phi^*}{q^j}\right)^k\right)\omega_2\\ &+(-1)^i\left(\left((1-\gamma)+\gamma\left(\frac{\phi^*}{q^i}\right)^k\right)\omega_1\right)\wedge h_2\Bigg).\end{aligned}\tag{10.3}$$

3. *When* $i>\operatorname{reldim}X$ *the isomorphism* $H^{i-1}_{\mathrm{rig}}(X_\kappa/K)\to H^i_{\mathrm{ms}}(X,i)$ *of* (8.5) *is given by the formula*

$$u\in H^{i-1}_{\mathrm{rig}}(X_\kappa/K)\subset\Omega^{i-1}_{A^\dagger,K}/d\Omega^{i-2}_{A^\dagger,K}\mapsto\big((0,(1-(\phi^*/q^i)^k)u)\big)_{k>0}.$$

Proof. Using lemma 3.4 (see also remark 5.2) we find a map

$$\begin{aligned}&\operatorname{Cone}\left(1-\left(\frac{\phi^*}{q^n}\right)^k:\Gamma(Y,\Omega^{\geqslant i}_{Y/K}\langle\log(Y-X_K)\rangle)\to\Omega^\bullet_{A^\dagger,K}\right)[-1]\\ &\qquad\to\operatorname{Cone}\left(1-\left(\frac{\varphi^*}{q^n}\right)^k:\ \mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\to\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)\right)[-1].\end{aligned}$$

Writing out the long cohomology sequences we saw that the maps on rigid cohomology are always isomorphisms, and Hodge theory shows that the maps $\Gamma(Y,\Omega^j_{Y/K}\langle\log(Y-X_K)\rangle)\to\mathrm{Fil}^i H^j_{\mathrm{dR}}(X_K/K)$ are isomorphisms for $j=i,i-1$ (for $i-1$ both sides are 0). It follows that the map on the degree i cohomology is an isomorphism and together with (8.3) the first part follows. The second part follows easily from remark 8.7.2. The last part is straightforward (compare remark 8.7.3). ∎

In precisely the same situation we can also compute the modified Gros style syntomic cohomology. Since $\overline{X}$ is already smooth in a neighborhood of X the induced filtration on the complex of differential is obtained by equating the terms of low degree to 0. The proof of proposition 1.10 in [Ber97] also gives that $\mathrm{Fil}^n_{\overline{X}}\,\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K)_{\overline{X},\overline{X}}$ is quasi-isomorphic to $\Omega^{\geqslant n}_{A^\dagger,K}$. This makes the next proposition obvious

Proposition 10.2. *Let X and ϕ be as above. There is a canonical isomorphism*

(10.4)
$$\tilde{H}^i_{\mathrm{ms}}(X,i) \cong \varinjlim_k \Big\{(\omega,h)\colon \omega \in \Omega^i_{A^\dagger,K},\quad h \in \Omega^{i-1}_{A^\dagger,K}/d\Omega^{i-2}_{A^\dagger,K},\quad dh = \left(1-\left(\frac{\phi^*}{q^i}\right)^k\right)\omega\Big\}\ ,$$

and the connecting maps and cup products are given by exactly the same formulas as in proposition 10.1. *The map $H^i_{\mathrm{ms}}(X,i) \to \tilde{H}^i_{\mathrm{ms}}(X,i)$ is simply given by sending (ω,h) to (ω,h), where ω is seen as an algebraic form with logarithmic singularities on the left and as a rigid form on the right.*

Proposition 10.3. *For $X = \operatorname{Spec}(A)$, φ and ϕ as above, The composed map*

$$A^\times \to K_1(X) \xrightarrow{c_1^1} H^1_{\mathrm{ms}}(X,1)\ ,$$

is given as follows: Let $f \in A^\times$ and let $\bar{f}$ be its reduction. As $\bar{f}$ is defined over some finite field, there is some power of φ, say φ^k, of degree q^k, such that $\bar{f}\circ\varphi^k = \bar{f}^{q^k}$. It follows that $f\circ\phi^k \equiv f^{q^k} \pmod{\mathfrak{p}}$ and therefore that the rigid function

$$f_0 := \frac{f^{q^k}}{f\circ\phi^k}$$

satisfies $\log f_0 \in \Omega^0_{A^\dagger,K}$. With all that, under the isomorphism (10.1) *the cohomology class $c_1^1(f)$ is given in degree k by*

$$\left(\operatorname{dlog} f, \frac{\log f_0}{q^k}\right)\ .$$

The exact same result is true with H_{ms} replaced by $\tilde{H}_{\mathrm{ms}}$.

Proof. First we notice that the last statement of the proposition follows immediately by proposition 10.2. Conversely, by lemma 9.6 and the fact that H_{ms} Chern classes map to de Rham Chern classes under $H^1_{\mathrm{ms}}(X,1) \to F^1H^1_{\mathrm{dR}}(X_K/K)$, the proposition will be true for H_{ms} if it is true for $\tilde{H}_{\mathrm{ms}}$, and moreover this equivalence is true for each X and ϕ separately. Next one easily checks that the statement of the proposition is compatible with the transition maps (10.2). This means that by replacing ϕ by ϕ^k we may always assume $k=1$. We will abuse the notation to write $c_1^1(f)$ for the component of the regulator in the $k=1$ level of the direct limit, which is defined under our assumption. We start with the case $X = \mathbb{G}_m$ (so $A = \mathcal{V}[T,T^{-1}]$), $f = T$ and ϕ is defined by $\phi^*(T) = T^q$. Since the modified syntomic Chern class lifts the de Rham Chern class, which for T is just $\operatorname{dlog} T$, we see that

$$c_1^1(T) = (h, \operatorname{dlog} T), \quad \text{where} \quad dh = \operatorname{dlog} T - \frac{1}{q}\operatorname{dlog}\phi^*(T) = 0\ .$$

The proposition in this case amounts to the statement that $h = 0$. To see this we use the involution $\tau\colon A \to A$ defined by $\tau(T) = T^{-1}$. As the Chern class is functorial, and as τ commutes with ϕ, we see that

$$c_1^1(T^{-1}) = \tau^*(h, \operatorname{dlog} T) = (\tau^* h, \tau^* \operatorname{dlog} T) = (h, -\operatorname{dlog} T)\ .$$

As c_1^1 is a group homomorphism, we have $(0,0) = c_1^1(T^{-1} \cdot T) = (2h, 0)$, which proves what we wanted in this case. It follows that in this case the proposition is also proved for $\tilde{H}_{\mathrm{ms}}$ (note however that the proof would not have worked if we used $\tilde{H}_{\mathrm{ms}}$ directly because knowing the cohomology class of ω is not sufficient to determine it uniquely without the assumption of log singularities).

Now we prove the proposition for a general X and ϕ on $\tilde{H}_{\mathrm{ms}}$, hence again for H_{ms} as well. Let $Y = \overline{X} \times \mathbb{P}^1$ and let $\overline{\overline{X}}$ be the closure of the image of X in Y under the map $x \mapsto (x, f(x))$. Then $(\overline{\overline{X}}, Y)$ give a data for the computation of $\tilde{H}^1_{\mathrm{ms}}(X, 1)$. Let $U' \subset \overline{X}_K$ be a strict neighborhood of $]X_\kappa[_{\hat{\overline{X}}}$ on which ϕ and f_0 are defined. Let U be a strict neighborhood of $]X_\kappa[_{\hat{Y}}$ in $]\overline{\overline{X}}_\kappa[_{\hat{Y}}$ contained in the preimage of $]\overline{\overline{X}}_\kappa[_{\hat{Y}} \cap (U' \times \mathbb{G}_m)$ under the map $(x, T) \mapsto (\phi(x), T^q)$. The existence of such a U is guaranteed by lemma 4.3 and the assumption that on X_κ we have $\bar{f}(\varphi(x)) = (\bar{f}(x))^q$. Then $\tilde{H}^1_{\mathrm{ms}}(X, 1)$ is realized as the first cohomology of the complex

$$\operatorname{Cone}(1 - \phi^*/q\colon \mathbb{R}\Gamma(U, j^\dagger \operatorname{Fil}^1_{\overline{\overline{X}}} \Omega_U^\bullet) \to \mathbb{R}\Gamma(U, j^\dagger \Omega_U^\bullet))\ . \tag{10.5}$$

The projection from U to the $\overline{X}$ component induces the identity map from (10.4) to this cohomology. The projection to $\mathbb{P}^1$ similarly induces f^*. To prove the proposition it is therefore enough to prove that the pullback of $(\operatorname{dlog} f, \log(f_0)/q)$ under the first projection is the same as the pullback of $(\operatorname{dlog} T, 0)$ under the second projection, and it is enough to do this on the cohomology of global sections, i.e., with Γ replacing $\mathbb{R}\Gamma$ in (10.5), because both maps factor through this group. It thus remains to show that $(\operatorname{dlog}(f(x)), \log(f_0(x))/q) - (\operatorname{dlog} T, 0) = d(H) = (dH, (1 - \phi^*/q)H)$, with H a rigid function defined on some strict neighborhood contained in U and vanishing on the set $\{(x, T)\colon f(x) = T\}$. It is easy to see that the function $H(x, T) = \log(f(x)/T)$ works and this completes the proof. □

Remark 10.4. In the second part of the proof the use of the Gros style cohomology is essential.

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