RESOLUTION OF SINGULARITIES FOR A CLASS OF HILBERT MODULES

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ABSTRACT. Let \( \mathcal{M} \) be the completion of the polynomial ring \( \mathbb{C}[z] \) with respect to some inner product and for any ideal \( \mathcal{I} \subseteq \mathbb{C}[z] \), let \( [\mathcal{I}] \) be the closure of \( \mathcal{I} \) in \( \mathcal{M} \). For a homogeneous ideal \( \mathcal{I} \), the joint kernel of the submodule \( [\mathcal{I}] \subseteq \mathcal{M} \) is shown, after imposing some mild conditions on \( \mathcal{M} \), to be the linear span of the set of vectors

\[
\{ p_i(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m})K_{\mathcal{I}}(:, w)|_{w=0}, 1 \leq i \leq t, \}
\]

where \( K_{\mathcal{I}} \) is the reproducing kernel for the submodule \( [\mathcal{I}] \) and \( p_1, \ldots, p_t \) is some minimal "canonical set of generators" for the ideal \( \mathcal{I} \). The proof includes an algorithm for constructing this canonical set of generators, which is determined uniquely modulo linear relations, for homogeneous ideals. A short proof of the "Rigidity theorem" using the sheaf model for Hilbert modules over polynomial rings is given. We describe, via the monoidal transformation, the construction of a Hermitian holomorphic line bundle for a large class of Hilbert modules of the form \( [\mathcal{I}] \). We show that the curvature, or even its restriction to the exceptional set, of this line bundle is an invariant for the unitary equivalence class of \( [\mathcal{I}] \). Several examples are given to illustrate the explicit computation of these invariants.

1. Preliminaries

Beurling’s theorem describing invariant subspaces of the multiplication (by the coordinate function) operator on the Hardy space of the unit disc is essential to the Sz.-Nagy – Foias model theory and several other developments in modern operator theory. In the language of Hilbert modules, Beurling’s theorem says that all submodules of the Hardy module of the unit disc are equivalent. This observation, due to Cowen and Douglas [5], is peculiar to the case of one-variable operator theory. The submodule of functions vanishing at the origin of the Hardy module \( H^2(D^2) \) of the bi-disc is not equivalent to the Hardy module \( H^2(D^2) \). To see this, it is enough to note that the joint kernel of the adjoint of the multiplication by the two co-ordinate functions on the Hardy module of the bi-disc is 1-dimensional (it is spanned by the constant function 1) while the joint kernel of these operators restricted to the submodule is 2-dimensional (it is spanned by the two functions \( z_1 \) and \( z_2 \)).

There has been a systematic study of this phenomenon in the recent past [1][10] resulting in a number of “Rigidity theorems” for submodules of a Hilbert module \( \mathcal{M} \) over the polynomial ring \( \mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_m] \) of the form \( [\mathcal{I}] \) obtained by taking the norm closure of a polynomial ideal \( \mathcal{I} \) in the Hilbert module. For a large class of polynomial ideals, these theorems often take the form: two submodules \( [\mathcal{I}] \) and \( [\mathcal{J}] \) in some Hilbert module \( \mathcal{M} \) are equivalent if and only if the two ideals \( \mathcal{I} \) and \( \mathcal{J} \) are equal. We give a short proof of this theorem in a slightly more general set-up using

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the sheaf theoretic model developed earlier in [2]. Along the way, we find tractable invariants for Hilbert modules of the form \([\mathcal{H}]\).

Let \(\mathcal{H}\) be a Hilbert module of holomorphic functions on a bounded open connected subset \(\Omega\) of \(\mathbb{C}^m\) possessing a reproducing kernel \(K\). Assume that \(\mathcal{I} \subseteq \mathbb{C}[z]\) is the singly generated ideal \((p)\). Then the reproducing kernel \(K_{\mathcal{I}}(z)\) of \([\mathcal{I}]\) vanishes on the zero set \(V(\mathcal{I})\) and the map \(w \mapsto K_{\mathcal{I}}(z, w)\) defines a holomorphic Hermitian line bundle on the open set \(\Omega^*_w = \{w \in \mathbb{C}^m : \hat{w} \in \Omega \setminus V(\mathcal{I})\}\) which naturally extends to all of \(\Omega^*\). As is well known, the curvature of this line bundle completely determines the equivalence class of the Hilbert module \([\mathcal{I}]\) (cf. [4],[5]). However, if \(\mathcal{I} \subseteq \mathbb{C}[z]\) is not a principal ideal, then the corresponding line bundle defined on \(\Omega^*_w\) does not necessarily extend to all of \(\Omega^*\). Indeed, it was conjectured in [6] that the dimension of the joint kernel
\[
\{\gamma_w : (M_p - p(w))\gamma_w = 0, p \in \mathbb{C}[z]\}
\]
of the Hilbert module \([\mathcal{I}]\) at \(w\) is 1 for points \(w\) not in \(V(\mathcal{I})\), otherwise it is the codimension of \(V(\mathcal{I})\). Assuming that
\[
\begin{align*}
&\text{(a) } \mathcal{I} \text{ is a principal ideal or } \\
&\text{(b) } w \text{ is a smooth point of } V(\mathcal{I}),
\end{align*}
\]
Duan and Guo verify the validity of this conjecture in [11]. Furthermore if \(m = 2\) and \(\mathcal{I}\) is prime then the conjecture is valid.

Thus for any submodule \([\mathcal{I}]\) in a Hilbert module \(\mathcal{H}\), assuming that \(\mathcal{H}\) is in the Cowen-Douglas class \(B_1(\Omega^*)\) and the co-dimension of \(V(\mathcal{I})\) is greater than 1, it follows that \([\mathcal{I}]\) is in \(B_1(\Omega^*_w)\) but it doesn’t belong to \(B_1(\Omega^*)\). For example, \(H^2_0(\mathbb{D}^2)\) is in the Cowen-Douglas class \(B_1(\mathbb{D}^2 \setminus \{0,0\})\) but it does not belong to \(B_1(\mathbb{D}^2)\). To systematically study examples of submodules like \(H^2_0(\mathbb{D}^2)\), the following Definition from [2] will be useful.

Definition. Fix a bounded domain \(\Omega \subseteq \mathbb{C}^m\). A Hilbert module \(\mathcal{H} \subseteq \mathcal{O}(\Omega)\) over the polynomial ring \(\mathbb{C}[z]\) is said to be in the class \(B_1(\Omega)\) if
\[
\begin{align*}
&(\text{rk}) \text{ it possess a reproducing kernel } K \text{ (we don't rule out the possibility: } K(w, w) = 0 \text{ for } w \text{ in some closed subset } X \text{ of } \Omega) \text{ and } \\
&(\text{fin}) \text{ the dimension of } \mathcal{H}/m_{w,\mathcal{H}} \text{ is finite for all } w \in \Omega.
\end{align*}
\]
For Hilbert modules in \(B_1(\Omega)\), from [2], we have:

Lemma. Suppose \(\mathcal{H} \in B_1(\Omega)\) is the closure of a polynomial ideal \(\mathcal{I}\). Then \(\mathcal{H}\) is in \(B_1(\Omega^*)\) if the ideal \(\mathcal{I}\) is singly generated while if it is minimally generated by more than one polynomial, then \(\mathcal{H}\) is in \(B_1(\Omega^*_w)\).

This Lemma ensures that to a Hilbert module in \(B_1(\Omega)\), there corresponds a holomorphic Hermitian line bundle defined by the joint kernel for points in \(\Omega^*_w\). We will show that it extends to a holomorphic Hermitian line bundle on the “blow-up” space \(\hat{\Omega}^*\) via the monoidal transform under mild hypotheses on the zero set \(V(\mathcal{I})\). We also show that this line bundle determines the equivalence class of the module \([\mathcal{I}]\) and therefore its curvature is a complete invariant. However, computing it explicitly on all of \(\hat{\Omega}^*\) is difficult. In this paper we find invariants, not necessarily complete, which are easy to compute. One of these invariants is the curvature of the restriction of the line bundle on \(\hat{\Omega}^*\) to the exceptional subset of \(\Omega^*\).

A line bundle is completely determined by its sections on open subsets. To write down the sections, we use the decomposition theorem for the reproducing kernel [2] Theorem 1.4. The actual computation of the curvature invariant requires the explicit calculation of the norm of these sections. Thus it is essential to obtain explicit description of the eigenvectors \(K^{(i)}, 1 \leq i \leq d\), (as in the
decomposition Theorem\[2\] Theorem 1.4}) in terms of the reproducing kernel. We give two examples which, we hope, will motivate the results that follow. Let $H^2(\mathbb{D}^2)$ be the Hardy module over the bi-disc algebra. The reproducing kernel for $H^2(\mathbb{D}^2)$ is the Szegő kernel $\mathcal{S}(z, w) = \frac{1}{1-z_1\overline{w}_1 - z_2\overline{w}_2}$. Let $\mathcal{J}_0$ be the polynomial ideal $(z_1, z_2)$ and let $[\mathcal{J}_0]$ denote the minimal closed submodule of the Hardy module $H^2(\mathbb{D}^2)$ containing $\mathcal{J}_0$. Then the joint kernel of the adjoint of the multiplication operators $M_1$ and $M_2$ is spanned by the two linearly independent vectors: $z_1 = p_1(\partial\overline{z}_1, \partial\overline{z}_2)\mathcal{S}(z, w)_{|w_1=0=w_2}$ and $z_2 = p_2(\partial\overline{z}_1, \partial\overline{z}_2)\mathcal{S}(z, w)_{|w_1=0=w_2}$, where $p_1, p_2$ are the generators of the ideal $\mathcal{J}_0$. For a second example, take the ideal $\mathcal{J}_1 = (z_1 - z_2, z_2^2)$ and let $[\mathcal{J}_1]$ be the minimal closed submodule of the Hardy module $H^2_0(\mathbb{D}^2)$ containing $\mathcal{J}_1$. The joint kernel is not hard to compute. A set of two linearly independent vectors which span it are $p_1(\partial\overline{z}_1, \partial\overline{z}_2)\mathcal{S}(z, w)_{|w_1=0=w_2}$ and $p_2(\partial\overline{z}_1, \partial\overline{z}_2)\mathcal{S}(z, w)_{|w_1=0=w_2}$, where $p_1 = z_1 - z_2$ and $p_2 = (z_1 + z_2)^2$. Unlike the first example, the two polynomials $p_1, p_2$ are not the generators for the ideal $\mathcal{J}_1$ that were given at the start, nevertheless, they are easily seen to be a set of generators for the ideal $\mathcal{J}_1$ as well. This prompts the question:

Question. Let $\mathcal{M} \in \mathcal{B}_1(\Omega)$ be a Hilbert module and $\mathcal{I} \subseteq \mathcal{M}$ be a polynomial ideal. Assume without loss of generality that $0 \in \mathcal{V}(\mathcal{I})$. We ask

1. if there exists a set of polynomials $p_1, \ldots, p_t$ such that the vectors

$$p_1(\partial\overline{w}_1, \ldots, \partial\overline{w}_m)K_{[\mathcal{I}]}(z, w)_{|w=0}, \ i = 1, \ldots, t,$$

are a basis of the joint kernel at 0,

2. what conditions, if any, will ensure that the polynomials $p_1, \ldots, p_t$, as above, is a generating set for $\mathcal{I}$?

We show that the answer to Question (1) is affirmative, that is, there is a natural basis for the joint eigenspace of the Hilbert module $[\mathcal{I}]$, which is obtained by applying a differential operator to the reproducing kernel $K_{[\mathcal{I}]}$ of the Hilbert module $[\mathcal{I}]$. Often, these differential operators encode an algorithm for producing a set of generators for the ideal $\mathcal{I}$ with additional properties. It is shown that there is an affirmative answer to Question (2) as well, if the ideal is assumed to be homogeneous. Thus if there were two sets of generators which serve to describe the joint kernel, as above, then these generators must be linear combinations of each other. We call such a generating set, a canonical set of generators.

The cardinality of the canonical set of generators is equal to that of a minimal generating set for the ideal $\mathcal{I}$. We note, in passing, that the well-known Grobner basis doesn’t possess this property and therefore is distinct from the canonical set of generators we construct here. The canonical set of generators provide an effective tool to determine if two ideals are equal. A number of examples are given to illustrate this.

While we follow very closely the methods of the book [3], we introduce one essential new ingredient in this paper, namely, an auxiliary vector space $\mathcal{V}_w(\mathcal{I})$ corresponding to the characteristic space $\mathcal{V}_w(\mathcal{I}), \mathcal{I} \subseteq \mathcal{C}[z], \ w \in \Omega$. We describe the characteristic space $\mathcal{V}_w(\mathcal{I})$, introduced in [3], in terms of the familiar Fock inner product on $\mathcal{C}[z]$. The quotient space $\mathcal{V}_w(\mathcal{I})/\mathcal{V}_w(\mathcal{I})$ then plays a significant role in explicitly identifying the joint kernel of the submodule $[\mathcal{I}] \subseteq \mathcal{H}$ for a Hilbert module $\mathcal{H}$ in $\mathcal{B}_1(\Omega)$.

Finally, this description of the joint kernel provides unitary invariants for the submodule $[\mathcal{I}]$ via the monoidal transform which provides a refinement of the ideas from [3].

It is possible to recast some of our results in the language of "micro-localization" of [13]. However, we have refrained from doing this. We expect to return to such an exposition at a future date.

In the following section, we describe the joint kernel for modules in $\mathcal{B}_1(\Omega)$ using the Fock inner product. In section [3] we construct the holomorphic Hermitian line bundle on the "blow-up "
space. In the last section, we provide explicit calculations for a family of examples introduced recently in [13].

Index of Notations.

- $\mathbb{C}[z]$ the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ of $m$-complex variables
- $m_w$ the maximal ideal of $\mathbb{C}[z]$ at the point $w \in \mathbb{C}^m$
- $\Omega^*$ the poly-disc $|z| < 1$, $1 \leq i \leq m$, $m \geq 1$
- $[\mathcal{F}]$ the completion of a polynomial ideal $\mathcal{I}$ in some Hilbert module
- $M_I$ the multiplication by the co-ordinate function $z_i$ on $[\mathcal{F}]$, $1 \leq i \leq m$
- $\mathcal{O}(U)$ space of holomorphic functions on an open set $U \subset \mathbb{C}^m$
- $K_{[\mathcal{F}]}$ the reproducing kernel of $[\mathcal{F}]$
- $a, |a|, \bar{a}$ the multi index $(a_1, \ldots, a_m)$, $|a| = \sum_{i=1}^m a_i$ and $a! = a_1! \ldots a_m!$
- $\prod_{i=1}^m (\frac{a_i}{m})$ for $a = (a_1, \ldots, a_m)$ and $k = (k_1, \ldots, k_m)$
- $z^a$ if $k_i \leq a_i$, $1 \leq i \leq m$.
- $\tilde{z}^a, \tilde{\partial}^a$ the Fock inner product at $0$ for all $\tilde{z} \in [\mathcal{F}]$
- $q(D)$ the differential operator $q(z^a, \bar{z}^a)$ for $q(z)$ of the form $M \in \mathcal{O}(\Omega)$
- $\mathcal{O}_w$ the sheaf of holomorphic functions on $\Omega$
- $\mathcal{O}_w \subset \mathcal{O}_0$ the sheaf corresponding to the Hilbert module $\mathcal{M} \subset \mathcal{O}_0(\Omega)$
- $V(\mathcal{F})$ the vector space $\{z \in \Omega : f(z) = 0 \text{ for all } f \in \mathcal{F}\}$, where $\mathcal{F} \subset \mathcal{O}(\Omega)$
- $\mathcal{O}_w$ the germs of holomorphic functions at $w$
- $\mathcal{V}_w(\mathcal{F})$ the characteristic space of $\mathcal{F} \subset \mathcal{O}_w$ at $w$
- $\mathcal{V}_w(\mathcal{F})$ for $q \in \mathbb{C}[z]$ such that $q(D)\{z\}_{w} = 0, f \in \mathcal{L}$

2. Calculation of Basis Vectors for the Joint Kernel

The Fock inner product of a pair of polynomials $p$ and $q$ is defined by the rule:

$$\langle p, q \rangle_0 = q^*(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m})p|_0, q^*(z) = \overline{q(\overline{z})}.$$  

The map $\langle \cdot, \cdot \rangle_0 : \mathbb{C}[z] \times \mathbb{C}[z] \rightarrow \mathbb{C}$ is linear in first variable and conjugate linear in the second and for $p = \sum_a a_za^a, q = \sum_b b_za^a$ in $\mathbb{C}[z]$, we have

$$\langle p, q \rangle_0 = \sum_a a!a \bar{b}_a$$

since $z^a(D)z^\beta|_{z=0} = a!$ if $a = \beta$ and 0 otherwise. Also, $\langle p, p \rangle_0 = \sum_a |a||a|^2 \geq 0$ and equals 0 only when $a_\alpha = 0$ for all $a$. The completion of the polynomial ring with this inner product is the well-known Fock space $L^2_a(\mathbb{C}^m, d\mu)$, that is, the space of all $\mu$-square integrable entire functions on $\mathbb{C}^m$, where

$$d\mu(z) = \pi^{-m} e^{-|z|^2} d\nu(z)$$

is the Gaussian measure on $\mathbb{C}^m$ ($d\nu$ is the usual Lebesgue measure).

The characteristic space (cf. [3] page 11]) of an ideal $\mathcal{I}$ in $\mathbb{C}[z]$ at the point $w$ is the vector space

$$\mathcal{V}_w(\mathcal{I}) := \{q \in \mathbb{C}[z] : q(D)p|_w = 0, p \in \mathcal{I}\} = \{q \in \mathbb{C}[z] : \langle p, q^* \rangle_w = 0, p \in \mathcal{I}\}.$$
The envelope of the ideal $\mathcal{I}$ at the point $w$ is defined to be the ideal

$$
\mathcal{I}_w^e := \{ p \in C[z] : q(D)p|_w = 0, q \in \mathcal{V}_w(\mathcal{I}) \} = \{ p \in C[z] : (p, q^*)_w = 0, q \in \mathcal{V}_w(\mathcal{I}) \}.
$$

It is known [3] Theorem 2.1.1, page 13] that $\mathcal{I} = \cap_{w \in V(\mathcal{I})} \mathcal{I}_w^e$. The proof makes essential use of the well known Krull's intersection theorem. In particular, if $V(\mathcal{I}) = \{ w \}$, then $\mathcal{I}_w^e = \mathcal{I}$.

The case, where the zero set $V(\mathcal{I})$ is a singleton. It is easy to verify this special case using the Fock inner product. We provide the details below after setting $w = 0$, without loss of generality.

Let $m_0$ be the maximal ideal in $C[z]$ at 0. By Hilbert's Nullstellensatz, there exists a positive integer $N$ such that $m_0^N \subseteq \mathcal{I}$. We identify $C[z]/m_0^N$ with span$_C \{ z^a : |a| < N \}$, which is the same as $(m_0^N)\perp$ in the Fock inner product. Let $\mathcal{V}_N$ be the vector space $\mathcal{I} \cap \text{span}_C \{ z^a : |a| < N \}$. Clearly $\mathcal{I}$ is the vector space (orthogonal) direct sum $\mathcal{V}_N \oplus m_0^N$. Let

$$
\widetilde{V} = \{ q \in C[z] : \deg q < N \text{ and } (p, q)_0 = 0, p \in \mathcal{V}_N \} = (m_0^N)\perp \mathcal{V}_N.
$$

Evidently, $\mathcal{V}_0(\mathcal{I}) = \widetilde{V} \setminus \{ 0 \}$, where $\widetilde{V} \setminus \{ 0 \} = \{ q \in V : q^* \in \widetilde{V} \}$. It is therefore clear that the definition of $\widetilde{V}$ is independent of $N$, that is, if $m_0^{N_1} \subseteq \mathcal{I}$ for some $N_1$, then $(m_0^N)\perp \mathcal{V}_N = (m_0^{N_1})\perp \mathcal{V}_N$. Thus

$$
\mathcal{I}_0^e = \{ p \in C[z] : \deg p < N \text{ and } (p, q^*)_0 = 0, q \in \mathcal{V}_0(\mathcal{I}) \} \oplus m_0^N = (m_0^N)\perp \mathcal{V}_N \oplus m_0^N = \mathcal{V}_N \oplus m_0^N
$$

showing that $\mathcal{I}_0^e = \mathcal{I}$.

Let $\mathcal{M}$ be a submodule of an analytic Hilbert module $\mathcal{H}$ on $\Omega$ such that $\mathcal{M} = [\mathcal{I}]$, closure of the ideal $\mathcal{I}$ in $\mathcal{H}$ with $V(\mathcal{I}) = \{ 0 \}$. It is known that $\mathcal{V}_0(\mathcal{I}) = \mathcal{V}_0(\mathcal{M})$ (cf. [2]). Since $\mathcal{M} \subseteq \mathcal{M}_0^e := \{ f \in \mathcal{H} : q(D)f|_0 = 0 \text{ for all } q \in \mathcal{V}_0(\mathcal{M}) \}$, it follows that

$$
\dim \mathcal{H} / \mathcal{M}_0^e \leq \dim \mathcal{H} / \mathcal{M} = \dim C[z]/\mathcal{I} \leq \dim C[z]/m_0^N \\
\leq \sum_{k=0}^{N-1} \binom{k + m - 1}{m - 1} < +\infty.
$$

Therefore, from [10] Corollary 2.8], we have $\mathcal{M}_0^e \cap C[z] = \mathcal{I}_0^e$ and $\mathcal{M} \cap C[z] = \mathcal{I}$, and hence

$$
\mathcal{M}_0^e = [\mathcal{I}_0^e] = [\mathcal{I}] = \mathcal{M}.
$$

(2.1)

Assumption: We assume that the Hilbert module $\mathcal{M}$ is (i) the completion with respect to some inner product of the ideal $\mathcal{I} \subseteq C[z]$ and that (ii) it is in the class $\mathcal{B}_1(\Omega)$.

For notational convenience, in the following discussion, we let $K$ be the reproducing kernel of $\mathcal{M} = [\mathcal{I}]$, instead of $K_{[\mathcal{I}]}$.

To describe the joint kernel $\cap_{i=1}^m \ker(M_j(w))$ at $w$, of the adjoint of the multiplication operator, using the characteristic space $\mathcal{V}_w(\mathcal{I})$, we define the auxilary space

$$
\mathcal{V}_w(\mathcal{I}) = \{ q \in C[z] : \frac{\partial q}{\partial z_i} \in \mathcal{V}_w(\mathcal{I}), 1 \leq i \leq m \}.
$$
Clearly $V(m_w, \mathcal{J}) \setminus V(\mathcal{J}) = \{w\}$ and $V_w(m_w, \mathcal{J}) = \tilde{V}_w(\mathcal{J})$. Therefore,
\[
\dim \bigcap_{j=1}^{m} \ker(M_j - w) = \dim \mathcal{J}/m_w, \mathcal{M} = \dim \mathcal{J}/m_w, \mathcal{J} = \sum_{\lambda \in \mathcal{V}(m_w, \mathcal{J}) \setminus V(\mathcal{J})} \dim \tilde{V}_j(m_w, \mathcal{J})/\tilde{V}_j(\mathcal{J}) = \dim \tilde{V}_w(\mathcal{J})/\tilde{V}_w(\mathcal{J}).
\] (2.2)

For the second and the third equalities, see [3 Theorem 2.2.5 and 2.1.7]. Since $\tilde{V}_w(\mathcal{J})$ is a subspace of the inner product space $C[z]$, we will often identify the quotient space $\tilde{V}_w(\mathcal{J})/\tilde{V}_w(\mathcal{J})$ with the subspace of $\tilde{V}_w(\mathcal{J})$ which is the orthogonal complement of $\tilde{V}_w(\mathcal{J})$ in $\tilde{V}_w(\mathcal{J})$. Equation (2.2) motivates the following straightforward lemma describing the basis of the joint kernel of the adjoint of the multiplication operator at a point in $\Omega$. This answers the question [1] of the introduction.

**Lemma 2.1.** Fix $w_0 \in \Omega$ and polynomials $q_1, \ldots, q_t$. Let $\mathcal{J}$ be a polynomial ideal and $K$ be the reproducing kernel corresponding the Hilbert module $[\mathcal{J}]$, which is assumed to be in $\mathcal{B}_1(\Omega)$. Then the vectors
\[
q_1(\bar{D})K(\cdot, w)|_{w=w_0}, \ldots, q_t(\bar{D})K(\cdot, w)|_{w=w_0}
\]
form a basis of the joint kernel at $w_0$ of the adjoint of the multiplication operator if and only if the classes $[q_1], \ldots, [q_t]$ form a basis of $\tilde{V}_w(\mathcal{J})/\tilde{V}_w(\mathcal{J})$.

**Proof.** Without loss of generality we assume $0 \in \Omega$ and $w_0 = 0$.

Claim 1: For any $q \in C[z]$, the vector $q(\bar{D})K(\cdot, w)|_{w=0} \neq 0$ if and only if $q^* \notin \mathcal{V}_0(\mathcal{J})$.

Using the reproducing property $f(w) = (f, K(\cdot, w))$ of the kernel function $K$, it is easy to see (cf. [7]) that
\[
\partial^\alpha f(w) = (f, \partial^\alpha K(\cdot, w)), \alpha \in \mathbb{Z}^+_m, \ w \in \Omega, \ f \in \mathcal{M}.
\]

Thus
\[
\partial^\alpha f(w)|_{w=0} = (f, \partial^\alpha K(\cdot, w)|_{w=0} = (f, \bar{D}^\alpha w)|_{w=0} = \frac{\partial^\beta K(z, 0)}{\beta!} w^\beta|_{w=0} = \sum_{\beta \geq \alpha} \left( f, \frac{\partial^\beta K(z, 0)}{\beta!} w^\beta \right)|_{w=0} = (f, \partial^\alpha K(\cdot, w)|_{w=0}).
\]

So for $f \in \mathcal{M}$ and a polynomial $q = \sum a_\alpha z^\alpha$, we have
\[
(f, q(\bar{D})K(\cdot, w)|_{w=0} = (f, q, \partial^\alpha K(\cdot, w)|_{w=0} = \sum_a a_\alpha (f, \partial^\alpha K(\cdot, w)|_{w=0} = \sum_a \bar{a}_\alpha (f, \partial^\alpha K(\cdot, w)|_{w=0} = q^*(D)f|_{w=0}.
\] (2.3)

This proves the claim.

Claim 2: For any $q \in C[z]$, the vector $q(\bar{D})K(\cdot, w)|_{w=0}$ is in $\cap_{j=1}^{m} \ker M_j^* q^* + \frac{\partial q^*}{\partial z_j}(D)f|_{w=0} = \frac{\partial q^*}{\partial z_j}(D)f|_{w=0}$ verifying the claim.
As a consequence of claims 1 and 2, we see that \( q(\bar{D})K(\cdot, w)|_{w=0} \) is a non-zero vector in the joint kernel if and only if the class \([q^*] \) in \( \overline{\mathcal{V}}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I}) \) is non-zero.

Pick polynomials \( q_1, \ldots, q_t \). From the equation (2.2) and claim 2, it is enough to show that \( q_i(\bar{D})K(\cdot, w)|_{w=0}, \ldots, q_t(\bar{D})K(\cdot, w)|_{w=0} \) are linearly independent if and only if \([q_i^*], \ldots, [q_t^*] \) are linearly independent in \( \overline{\mathcal{V}}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I}) \). But from claim 1 and equation (2.3), it follows that

\[
\sum_{i=1}^t \bar{q}_i q_i(\bar{D})K(\cdot, w)|_{w=0} = 0
\]

if and only if \( \sum_{i=1}^t \alpha_i [q_i^*] = 0 \) in \( \overline{\mathcal{V}}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I}) \) for \( \alpha_i \in \mathbb{C}, 1 \leq i \leq t \). This completes the proof. \( \square \)

**Remark 2.2.** The ‘if’ part of the Lemma can also be obtained from the decomposition Theorem [2, Theorem 1.4]:

For a module \( \mathcal{M} \) in the class \( \mathcal{B}_1(\Omega) \), let \( \mathcal{F}_{w}^\mathcal{M} \) be the subsheaf of the sheaf of holomorphic functions \( \mathcal{O}_\Omega \) whose stalk \( \mathcal{F}_{w}^\mathcal{M} \) at \( w \in \Omega \) is

\[
\{(f_1)_{w} \mathcal{O}_w + \cdots + (f_n)_{w} \mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\},
\]
and the characteristic space at \( w \in \Omega \) is the vector space

\[
\mathcal{V}_w(\mathcal{F}_{w}^\mathcal{M}) = \{ q \in \mathbb{C}[\mathbb{Z}] : q(D)|_w = 0, f_w \in \mathcal{F}_{w}^\mathcal{M} \}.
\]

Since

\[
\dim \mathcal{F}_{0}^\mathcal{M}/m_0 \mathcal{F}_{0}^\mathcal{M} = \dim \bigcap_{j=1}^m \ker M_j = \dim \overline{\mathcal{V}}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I}) = t,
\]

there exists a minimal set of generators \( g_1, \ldots, g_r \) of \( \mathcal{F}_{0}^\mathcal{M} \) and a \( r > 0 \) such that

\[
K(\cdot, w) = \sum_{i=1}^t g_i(w)K^{(i)}(\cdot, w) \quad \text{for all } w \in \Delta(0; r)
\]

for some choice of anti-holomorphic functions \( K^{(1)}, \ldots, K^{(t)} : \Delta(0; r) \to \mathcal{M} \). For \( q_i^* \in \overline{\mathcal{V}}_0(\mathcal{I}), 1 \leq i \leq t \), the formula

\[
q(D)(z^a g) = \sum_{k \leq a} \left( \begin{array}{c} a \\ k \end{array} \right) z^{a-k} \frac{\partial |k|}{\partial z^k} (D)(g)
\]

(2.4)
gives

\[
q_i(\bar{D})K(\cdot, w)|_{w=0} = \sum_{j=1}^t \{ K^{(j)}(\cdot, w)|_{w=0} \} \{ q_i(\bar{D})g_j(w)|_{w=0} \}
\]

\[
= \sum_{j=1}^t \{ K^{(j)}(\cdot, w)|_{w=0} \} \{ q_i, g_j \}_0.
\]

Since \( \mathcal{V}_w(\mathcal{F}) = \mathcal{V}_w(\mathcal{M}) = \mathcal{V}_w(\mathcal{F}_{w}^\mathcal{M}) \) and the matrix \( \{\{q_i, g_j \}_0\} \) is invertible, the proof is complete.

**Remark 2.3.** We give details of the case where the ideal \( \mathcal{I} \) is singly generated, namely \( \mathcal{I} = \langle p \rangle \). From [3], it follows that the reproducing kernel \( K \) admits a global factorization, that is, \( K(z, w) = p(z)\bar{p}(z, w)\bar{w} \bar{p}(w) \) for \( z, w \in \Omega \) where \( \bar{p}(z, w) \neq 0 \) for all \( w \in \Omega \). So we get \( K_0(\cdot, w) = p(\cdot)\bar{p}(\cdot, w) \) for all \( w \in \Omega \). We use Lemma 2.1 to write down this section in term of the reproducing kernel. Let \( 0 \in V(\mathcal{I}) \). Let \( q_0 \) be the lowest degree term in \( p \). We claim that \([q_0^*] \) gives a non-trivial class in \( \overline{\mathcal{V}}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I}) \). All partial derivatives of \( q_0^* \) have degree less than that of \( q_0^* \). Hence

\[
q_0^*(D)(z^a g)|_0 = \frac{\partial a q_0^*}{\partial z^a} (D)(p)|_0 = 0, \quad a, |a| > 0.
\]
from \( \mathcal{A} \). Consequently, \( \frac{\partial \Phi_i^0}{\partial z_i} \in \mathcal{V}_0(\mathcal{I}) \) for all \( i, 1 \leq i \leq m \), that is, \( q_i^0 \in \mathcal{V}_0(\mathcal{I}) \). Also as the lowest degree of \( p - q_0 = 0 \) is strictly greater than that of \( q_0 \),

\[
q_i^0(D)p|_{\tilde{w}_0} = q_i^0(D)(p - q_0)|_{\tilde{w}_0} = q_i^0(D)q_0|_{\tilde{w}_0} = \|q_0\|^2_{\tilde{w}_0} > 0
\]

This shows that \( q_i^0 \notin \mathcal{V}_0(\mathcal{I}) \) and hence its class in \( \mathcal{V}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I}) \) is non-trivial. Therefore, from the proof of Lemma 2.1, we have

\[
q_0(\tilde{D})K(\cdot, w)|_{\tilde{w}_0} = K(\cdot, w)|_{\tilde{w}_0} = \|q_0\|^2_{\tilde{w}_0} K(\cdot, w)|_{\tilde{w}_0}.
\]

Let \( q_{w_0} \) denote term of the lowest degree in the expansion of \( p \) around \( w_0 \). Then

\[
K(\cdot, w)|_{w = w_0} = \begin{cases} K(\cdot, w)|_{w = w_0} & \text{if } w_0 \notin \mathcal{V}(\mathcal{I}) \cap \Omega \\ \frac{K(\cdot, w)|_{w = w_0}}{\|q_{w_0}\|^2_{\tilde{w}_0}} & \text{if } w_0 \in \mathcal{V}(\mathcal{I}) \cap \Omega. \end{cases} \tag{2.5}
\]

For a fixed set of polynomials \( q_1, \ldots, q_t \), the next lemma provides a sufficient condition for the classes \([q_1^t], \ldots, [q_t^t]\) to be linearly independent in \( \mathcal{V}_{w_0}(\mathcal{I})/\mathcal{V}_{w_0}(\mathcal{I}) \). The methods of the two different proofs given below will be used repeatedly in the sequel.

**Lemma 2.4.** Let \( q_1, \ldots, q_t \) be linearly independent polynomials in the polynomial ideal \( \mathcal{I} \) such that \( q_1^t, \ldots, q_t^t \in \mathcal{V}_{w_0}(\mathcal{I}) \). Then \([q_1^t], \ldots, [q_t^t]\) are linearly independent in \( \mathcal{V}_{w_0}(\mathcal{I})/\mathcal{V}_{w_0}(\mathcal{I}) \).

**First Proof.** Suppose \( \sum_{i=1}^t \alpha_i [q_i] = 0 \) in \( \mathcal{V}_{w_0}(\mathcal{I})/\mathcal{V}_{w_0}(\mathcal{I}) \) for some \( \alpha_i \in \mathbb{C} \), \( 1 \leq i \leq t \). Thus \( \sum_{i=1}^t \alpha_i q_i^t = q \) for some \( q \in \mathcal{V}_{w_0}(\mathcal{I}) \). Taking the inner product of \( \sum_{i=1}^t \alpha_i q_i^t \) with \( q_j \) for a fixed \( j \), we get

\[
\sum_{i=1}^t \alpha_i (q_j, q_i)_{w_0} = \left( \sum_{i=1}^t \alpha_i q_i^t \right) (D)q_j|_{w_0} = q(D)q_j|_{w_0} = 0.
\]

The Grammian \( ([q_j, q_i]_{w_0})_{i,j=1}^t \) of the linearly independent polynomials \( q_1, \ldots, q_t \) is non-singular. Thus \( \alpha_i = 0, 1 \leq i \leq t \), completing the proof.

**Second Proof.** If \([q_1^t], \ldots, [q_t^t]\) are not linearly independent, then we may assume without loss of generality that \([q_1^t] = \sum_{i=2}^t \alpha_i [q_i^t]\) for \( \alpha_1, \ldots, \alpha_t \in \mathbb{C} \). Therefore \([q_1^t - \sum_{i=2}^t \alpha_i p_i^t] = 0 \) in the quotient space \( \mathcal{V}_{w_0}(\mathcal{I})/\mathcal{V}_{w_0}(\mathcal{I}) \), that is, \( q_1 - \sum_{i=2}^t \alpha_i q_i^t \in \mathcal{V}_{w_0}(\mathcal{I}) \). So, we have

\[
(q_1 - \sum_{i=2}^t \alpha_i q_i^t)(D)q|_{w_0} = 0 \quad \text{for all } q \in \mathcal{I}.
\]

Taking \( q = q_1 - \sum_{i=2}^t \alpha_i q_i \) we have \( \|q_1 - \sum_{i=2}^t \alpha_i q_i\|^2_{w_0} = 0 \). Hence \( q_1 = \sum_{i=2}^t \alpha_i q_i \) which is a contradiction. \( \square \)

In the rest of this section, we continue our discussion assuming \( w_0 = 0 \), however the results, properly translated, remain valid in general. Suppose \{\( p_1, \ldots, p_t \)\} is a minimal set of generators for \( \mathcal{I} \). Let \( \mathcal{M} \) be the completion of \( \mathcal{I} \) with respect to some inner product induced by a positive definite kernel. We recall from [9] that \( \text{rank}_{\mathbb{C}[\Omega]} \mathcal{M} = t \). Let \( w_0 \) be a fixed but arbitrary point in \( \Omega \). We ask if there exist a choice of generators \( q_1, \ldots, q_t \) such that the vectors \( q_1(\tilde{D})K(\cdot, w)|_{0}, \ldots, q_t(\tilde{D})K(\cdot, w)|_{0} \) form a basis for \( \cap_{j=1}^t \ker M_j^* \). We isolate some instances where the answer is affirmative. However, this is not always possible (see remark 2.12). From [9] Lemma 5.11, Page 89, we have

\[
\dim \cap_{j=1}^n \ker M_j^* = \dim \mathcal{M}/m_0 \mathcal{M} = \dim \mathcal{M} \otimes_{\mathbb{C}[\Omega]} \mathbb{C}_0 \leq \text{rank}_{\mathbb{C}[\Omega]} \mathcal{M} \cdot \dim \mathbb{C}_0 \leq t,
\]
Lemma 2.5. Let \( p_1, \ldots, p_t \) be a minimal set of generators for an ideal \( \mathcal{I} \subset \mathbb{C}[z] \). Assume that \( p_1, \ldots, p_t \) are homogeneous polynomials not necessarily of the same degree. Let \( \mathcal{M} \in \mathcal{B}_1(\Omega) \) be of the form \([\mathcal{I}]\). Then the germs \( p_1, \ldots, p_t \) at 0 form a minimal set of generators for \( \mathcal{I} \).

Proof. For \( 1 \leq i \leq t \), let \( \deg p_i = \alpha_i \). Without loss of generality we assume that \( \alpha_i \leq \alpha_{i+1}, 1 \leq i \leq t-1 \). Suppose the germs \( p_1, \ldots, p_t \) are not minimal. Then we have

\[
 p_k = \sum_{\alpha_i \leq \alpha_k} \phi_{i, \alpha_k - \alpha_i} p_i,
\]

where \( \phi_{i, \alpha_k - \alpha_i} \) is the Taylor polynomial of degree \( \alpha_k - \alpha_i \) of the holomorphic function \( \phi_i \). Therefore \( p_1, \ldots, p_t \) can not be a minimal set of generators for the ideal \( \mathcal{I} \). This contradiction completes the proof. \( \square \)

Consider the ideal \( \mathcal{I} \) generated by the polynomials \( z_1 + z_2 + z_1^2 + z_2^2 - z_1 - z_2 \). We will see later that the joint kernel at 0, in this case is spanned by the independent vectors \( p(D)K(\cdot, w)|_{w=0}, q(D)K(\cdot, w)|_{w=0} \), where \( p = z_1 + z_2 \) and \( q = (z_1 - z_2)^2 \). Therefore any vectors in the joint kernel is of the form \((\alpha p + \beta q)\bar{K}(\cdot, w)|_{w=0} \) for some \( \alpha, \beta \in \mathbb{C} \). It then follows that \( \alpha p + \beta q \) and \( \alpha'p + \beta'q \) can not be a set of generators of \( \mathcal{I} \) for any choice of \( \alpha, \beta, \alpha', \beta' \in \mathbb{C} \). However in certain cases, this is possible. We describe below the case where \( \{p_1(D)K(\cdot, w)|_{w=0}, \ldots, p_t(D)K(\cdot, w)|_{w=0}\} \) forms a basis for \( \cap_{j=1}^m \ker M_j^* \) for an obvious choice of generating set in \( \mathcal{I} \).

Lemma 2.6. Suppose that \( \{p_1, \ldots, p_t\} \) is a minimal set of generators for the homogeneous ideal \( \mathcal{I} \subset \mathbb{C}[z] \), and that \( p_1, \ldots, p_t \) be homogeneous polynomials of same degree. Let \( K \) be the reproducing kernel corresponding the Hilbert module \([\mathcal{I}]\), which is assumed to be in \( \mathcal{B}_1(\Omega) \). Then the set

\[
 \{p_1(D)K(\cdot, w)|_{w=0}, \ldots, p_t(D)K(\cdot, w)|_{w=0}\}
\]

forms a basis for \( \cap_{j=1}^m \ker M_j^* \).

Proof. For \( 1 \leq i \leq t \), let \( \deg p_i = k \). It is enough to show, using Lemma 2.1, 2.4 and 2.5 that the polynomials \( p_i^1, \ldots, p_i^m \) are in \( \mathcal{V}_0(\mathcal{I}) \). The degree of \( \frac{\partial^{m} p_i^j(D)}{\partial z^{j}} \) is at most \( k - 1, 1 \leq i \leq t, 1 \leq j \leq m \). The term of lowest degree in each polynomial \( p_i \) in the ideal \( \mathcal{I} \) is at least \( k \). It follows that \( \frac{\partial^{m} p_i^j(D)}{\partial z^{j}}|_{z=0} = 0, p \in \mathcal{I}, 1 \leq i \leq t, 1 \leq j \leq m \). This completes the proof. \( \square \)

Example 2.7. Let \( \mathcal{M} \) be an analytic Hilbert module over \( \Omega \subset \mathbb{C}^m \), and \( \mathcal{M}_n \) be a submodule of \( \mathcal{M} \) of the form \([\mathcal{I}]\), where

\[
 \mathcal{I} = \langle z_1^{\alpha_1} \ldots z_m^{\alpha_m} : \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| = \sum_{i=1}^m \alpha_i = n \rangle.
\]

Let \( \mathcal{K}_n \) be the reproducing kernel corresponding to \( \mathcal{M}_n \). We have

\( 1 \) \( \mathcal{M}_n = \{ f \in \mathcal{M} : \partial^\alpha f(0) = 0, \text{ for } \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| \leq n - 1 \} \)
and Lemma 2.4, it is enough to show that we have to show (using Lemma 2.1) that
deal with the simple case of two generators.

Suppose the ideal \( I \subseteq \mathbb{C}[z] \) is minimal set of generators for \( I \). We begin with the simple case of two generators.

\[
\text{Proof. Let } \deg p_1 = k \text{ and } \deg p_2 = k + n \text{ for some } n \geq 1. \text{ The set } \{p_1, p_2 + (\sum_{i=0}^{m} \gamma_i z^i)p_1\} \text{ is a minimal set of generators for } I, \gamma_i \in \mathbb{C} \text{ where } i = (i_1, \ldots, i_m) \text{ and } |i| = i_1 + \ldots + i_m. \text{ We will take } q_1 = p_1 \text{ and find constants } \gamma_i \in \mathbb{C} \text{ such that }
\]
\[
q_2 = p_2 + (\sum_{|i|=n} \gamma_i z^i)p_1.
\]

We have to show (using Lemma 2.1) that \([\{q_1^*, q_2^*\}]\) is a basis in \( \tilde{V}_0(I)/V_0(I) \). From equation (2.2) and Lemma 2.4, it is enough to show that \( q_2^* \) is a in \( \tilde{V}_0(I) \). To ensure that \( \frac{\partial q_2^*}{\partial z^a} \in V_0(I), 1 \leq k \leq m \), we need to check:

\[
\frac{\partial |a| q_2^*}{\partial z^a}(D)p_i|_{w=0} = (p_i, \frac{\partial |a| q_2^*}{\partial z^a})|_{0} = 0
\]

for all multi-index \( a = (a_1, \ldots, a_m) \) with \( 1 \leq |a| \leq n \) and \( \ell = 1, 2 \). For \( |a| > n \), these conditions are evident. Since the degree of the polynomial \( q_2 \) is \( k + n \), we have \( (p_2, \frac{\partial |a| q_2^*}{\partial z^a})|_{0} = 0, 1 \leq |a| \leq n \). If \( n > 1 \), then \( (p_1, \frac{\partial |a| q_2^*}{\partial z^a})|_{0} = 0, 1 \leq |a| < n \). To find \( \gamma_i, i = (i_1, \ldots, i_m) \), we solve the equation \( (p_1, \frac{\partial |a| q_2^*}{\partial z^a})|_{0} = 0 \) for all \( a \) such that \( |a| = n \). By the Leibnitz rule,

\[
\frac{\partial |a| q_2^*}{\partial z^a} = \frac{\partial |a| p_2^*}{\partial z^a} + \sum_{\nu \leq a} \binom{a}{\nu} \frac{\partial |a| \gamma_i}{\partial z^\nu} \frac{\partial |\nu| p_2^*}{\partial z^\nu} = \frac{\partial |a| p_2^*}{\partial z^a} + \sum_{\nu \leq a} \binom{a}{\nu} \left( \sum_{i=0}^{n} \frac{i!}{(i-\nu+1)!} \frac{\partial |\nu| p_2^*}{\partial z^\nu} \right) z^{i-\nu+1}. \]

Since \( \frac{\partial |a| p_2^*}{\partial z^a}(D)p_i|_{w=0} = 0 \), we obtain (using (2.4))

\[
0 = (\frac{\partial |a| p_2^*}{\partial z^a} + \sum_{\nu \leq a} \binom{a}{\nu} \left( \sum_{i=0}^{n} \frac{i!}{(i-\nu+1)!} \frac{\partial |\nu| p_2^*}{\partial z^\nu} \right) z^{i-\nu+1}) (D)p_i|_{w=0} = (p_1, \frac{\partial |a| p_2^*}{\partial z^a}) + \sum_{r=0}^{n} \sum_{|i|=n} A_{ai}(r) \gamma_i,
\]

where given the multi-indices \( a, i \),

\[
A_{ai}(r) = \begin{cases} 0 & |\nu| = r, \nu \leq a, i \geq a - \nu; \\ \sum_{\nu \leq a} \binom{a}{\nu} \frac{i!}{(i-\nu+1)!} \frac{\partial |\nu| p_1^*}{\partial z^\nu} \frac{\partial |\nu| p_1^*}{\partial z^\nu} & \text{otherwise.} \end{cases}
\]
Let $A(r) = \{A_{ij}(r)\}$ be the $(n + m - 1) \times (n + m - 1)$ matrix in colexicographic order on $\alpha$ and $i$. Let $A = \sum_{r=0}^{n} A(r)$ and $\gamma_n$ be the $(n + m - 1)$ \times 1 column vector $(\gamma_n)_i = n$. Thus the equation (2.8) is of the form

$$\tilde{A}\gamma_n = \Gamma,$$

where $\Gamma$ is the $(n + m - 1) \times 1$ column vector $(-\langle p_1, \frac{\partial}{\partial z^\alpha}\rangle_0)_|\alpha|=n$. Invertibility of the coefficient matrix $A$ then guarantees the existence of a solution to the equation (2.8). We show that the matrix $A(r)$ is non-negative definite and the matrix $A(0)$ is diagonal:

$$A(0)_{ai} = \begin{cases} \| p_1 \|^2_0 & \text{if } \alpha = i \\ 0 & \text{if } \alpha \neq i. \end{cases}$$

Fix a $r$, $1 \leq r \leq n$. To prove that $A(r)$ is non-negative definite, we show that it is the Grammian with respect to Fock inner product at 0. To each $\mu = (\mu_1, \ldots, \mu_m)$ such that $|\mu| = n - r$, we associate a $1 \times (n + m - 1)$ tuple of polynomials $X^r_\mu$, defined as follows

$$X^r_\mu(\beta) = \begin{cases} \mu! (\beta - \mu) \frac{\partial^{\beta - \mu} p_1}{\partial z^{\beta - \mu}} & \text{if } \beta \geq \mu \\ 0 & \text{otherwise}, \end{cases}$$

where $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| = n$ ($\beta \geq \mu$ if and only if $\beta_i \geq \mu_i$ for all $i$). By $X^r_\mu : (X^r_\mu)'$, we denote the $(n + m - 1) \times (n + m - 1)$ matrix whose $ai$-th element is $\langle X^r_\mu(\alpha), X^r_\mu(i) \rangle_0$, $|\alpha| = n = |i|$. We note that

$$\sum_{|\mu|=n-r} \frac{1}{\mu!} (X^r_\mu)'(X^r_\mu)'_{ai} = \sum_{|\mu|=n-r} \frac{1}{\mu!} (X^r_\mu(\alpha), X^r_\mu(i))_0$$

$$= \sum_{|\mu|=n-r, \alpha \geq \mu, i \geq \mu} \frac{1}{\mu!} \mu! \left( \begin{array}{c} \alpha \\ \alpha - \mu \end{array} \right) \frac{\partial^{\alpha-\mu} p_1}{\partial z^{\alpha-\mu}} \mu! \left( \begin{array}{c} i \\ i - \mu \end{array} \right) \frac{\partial^{i-\mu} p_1}{\partial z^{i-\mu}}$$

$$= \sum_{|\nu|=r, \nu \leq \alpha, \gamma \geq \alpha - \nu} (\alpha - \nu)! \left( \begin{array}{c} \gamma \\ \nu \end{array} \right) \left( \begin{array}{c} i \\ i - \gamma + \nu \end{array} \right) \frac{\partial^{\alpha-\nu} p_1}{\partial z^{\alpha-\nu}} \frac{\partial^{i-\gamma+n} p_1}{\partial z^{i-\gamma+n}}$$

$$= A_{ai}(r).$$

Since $X^r_\mu : (X^r_\mu)'$ is the Grammian of the vector tuple $X^r_\mu$, it is non-negative definite. Hence $A(r) = \sum_{|\mu|=n-r} \frac{1}{\mu!} (X^r_\mu)'(X^r_\mu)'$ is non-negative definite. Therefore $A$ is positive definite and hence equation (2.8) admits a solution, completing the proof.

Let $\mathcal{I}$ be a homogeneous polynomial ideal. As one may expect, the proof in the general case is considerably more involved. However the idea of the proof is similar to the simple case of two generators. Let $\{p_1, \ldots, p_v\}$ be a minimal set of generators consisting of homogeneous polynomials for the ideal $\mathcal{I}$. We arrange the set $\{p_1, \ldots, p_v\}$ in blocks of polynomials $P^1, \ldots, P^k$ according to ascending order of their degree, that is,

$$\{P^1, \ldots, P^k\} = \{p_1^1, \ldots, p_{u_1}^1, p_2^2, \ldots, p_{u_2}^2, \ldots, p_i^i, \ldots, p_{u_i}^i, \ldots, p_k^k, \ldots, p_{u_k}^k\},$$

where each $P^i = \{p_1^i, \ldots, p_{u_i}^i\}$, $1 \leq l \leq k$ consists of homogeneous polynomials of the same degree, say $n_i$ and $n_{i+1} > n_i$, $1 \leq l \leq k - 1$. As before, for $l = 1$, we take $q_1^1 = p_1^1$, $1 \leq j \leq u_1$ and for $l \geq 2$ take

$$q_j^l = p_j^l + \sum_{f=1}^{l-1} \sum_{s=1}^{u_f} \gamma_{ij}^f p_s^l,$$

where $\gamma_{ij}^f(z) = \sum_{|i|=n_i-n_f} \gamma_{ij}^f(z_i) z^i$. 


Each $\gamma_{ij}^{fs}$ is a polynomial of degree $n_1 - n_f$ for some choice of $\gamma_{ij}^{fs}(i)$ in $\mathbb{C}$. So we obtain another set of polynomials $\{Q^1, \ldots, Q^k\}$ with $Q^l = \{q_{ij}^l, \ldots, q_{ij}^0\}, 1 \leq l \leq k$, satisfying the the same property as the set of polynomials $\{P^1, \ldots, P^k\}$. From Lemma 2.1 and 2.4 it is enough to check $q_{ij}^{fs}$ is in $\tilde{V}_0(\mathcal{G})$. This condition yields a linear system of equation as in the proof of Proposition 2.8 except that the co-efficient matrix is a block matrix with each block similar to $A$ defined by the equation (2.7). For $q_{ij}^{fs}$ in $\tilde{V}_0(\mathcal{G})$, the constants $\gamma_{ij}^{fs}(i)$ must satisfy:

\[
0 = \frac{\partial |a|q_{ij}^{fs}}{\partial z^a}(D)p_{ij}^{e0}
= (p_{ij}^e, \frac{\partial |a|p_{ij}^l}{\partial z^a})_0 + \sum_{j=1}^{l-1} \sum_{s=1}^{u_j} \sum_{\alpha \geq \nu} (a) \sum_{|i|=n_1-n_f,i \geq \nu} \gamma_{ij}^{fs}(i) \frac{i!}{(i-\alpha+\nu)!} \frac{\partial \gamma_{ij}^{fs}(i)^{i-\alpha+\nu}}{\partial z^{i-\alpha+\nu}}p_{ij}^{e0} \frac{\partial |a|p_{ij}^s}{\partial z^a}0
\]

All the terms in the equation are zero except when $|d| = n_l - n_d, 1 \leq d \leq l - 1$. For $e = d = f$, we have the equations

\[
-(p_{ij}^d, \frac{\partial |a|p_{ij}^l}{\partial z^a})_0 = \sum_{s=1}^{u_d} \sum_{r=0}^{m_1} \sum_{|i|=n_1-n_d} (A_{st}^d(r))_{ai} \gamma_{ij}^{fs}(i), \quad (2.11)
\]

where

\[
(A_{st}^d(r))_{ai} = \sum_{s=1}^{u_d} \sum_{r=0}^{m_1} (A_{st}^d(r))_{ai}
\]

Let $A_{st}^d(r)$ be the $\begin{pmatrix} n_1-n_d-m+1 \\ m-1 \end{pmatrix} \times \begin{pmatrix} n_1-n_d-m+1 \\ m-1 \end{pmatrix}$ matrix whose $ai$-th element is $(A_{st}^d(r))_{ai}$. We consider the block-matrix $A^d(r) = (A_{st}^d(r)), 1 \leq s, t \leq u_d$.

Fix a $r, 1 \leq r \leq n_l - n_d$. To each $\mu = (\mu_1, \ldots, \mu_m)$ such that $|\mu| = n_l - n_d - r$, associate a $1 \times (n_1-n_d-m+1)$ tuple of polynomials $X_{\mu}^{d}$, where

\[
X_{\mu}^{d}(\beta) = \begin{cases} \mu!(\beta - \mu) \frac{\partial \gamma_{ij}^{fs}}{\partial z^a} & \text{if } \beta \geq \mu \\ 0 & \text{otherwise,} \end{cases}
\]

where $\beta = (\beta_1, \ldots, \beta_m)$ and $|\beta| = n_l - n_d$. Let $X_{\mu}^{d} = (X_{\mu}^{d1}, \ldots, X_{\mu}^{d(n_1-n_d)})$. Using same argument as in (2.9) and (2.10), we see that the matrix

\[
A^d(r) = \sum_{|\mu| = n_l - n_d} \frac{1}{\mu!}(X_{\mu}^{d}(X_{\mu}^{d})^t)
\]

is non-negative definite when $r \geq 0$ and $A^d(0)$ is positive definite. Thus $A^d = \sum_{r=0}^{n_1-n_d} A^d(r)$ is positive definite. Let $\gamma_{ij}^{fs} = ((\gamma_{ij}^{fs}(i))_{|i|=n_1-n_d}, \ldots, (\gamma_{ij}^{fs(n_1-n_d)}(i))_{|i|=n_1-n_d})^t$, where each $(\gamma_{ij}^{fs}(i))_{|i|=n_1-n_d}$ is a $\begin{pmatrix} n_1-n_d+m-1 \\ m-1 \end{pmatrix} \times 1$ column vector. Define

\[
\Gamma_{ij}^{d} = ((-p_{ij}^{d1}, \frac{\partial |a|p_{ij}^{l}}{\partial z^a})_0)_{|a|=n_1-n_d}, \ldots, ((-p_{ij}^{d|a|}, \frac{\partial |a|p_{ij}^{l}}{\partial z^a})_0)_{|a|=n_1-n_d})
\]

The equation (2.11) then takes the form $A_{ij}^{d}\gamma_{ij}^{fs} = \Gamma_{ij}^{d}$, which admits a solution (as $A^d$ is invertible) for each $d, l$ and $j$. Thus we have proved the following theorem.
Theorem 2.9. Let $\mathcal{I} \subset \mathbb{C}[z]$ be a homogeneous ideal and \{p_1, \ldots, p_v\} be a minimal set of generators for $\mathcal{I}$ consisting of homogeneous polynomials. Let $K$ be the reproducing kernel corresponding to the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathcal{B}_1(\Omega)$. Then there exists a set of generators $q_1, \ldots, q_v$ for the ideal $\mathcal{I}$ such that the set \{q_i(D)K(\cdot, w)_{w=0} : 1 \leq i \leq v\} is a basis for $\\bigoplus_{i=1}^v \ker M^*_i$.

We note that the new set \{q_1, \ldots, q_v\} of generators for $\mathcal{I}$ is more or less “canonical!” It is uniquely determined modulo a linear transformation as shown below.

Let $\mathcal{I} \subset \mathbb{C}[z]$ be an ideal. Suppose there are two sets of homogeneous polynomials \{p_1, \ldots, p_v\} and \{\tilde{p}_1, \ldots, \tilde{p}_v\} both of which are minimal set of generators for $\mathcal{I}$. Theorem 2.9 guarantees the existence of a new set of generators \{q_1, \ldots, q_v\} and \{\tilde{q}_1, \ldots, \tilde{q}_v\} corresponding to each of these generating sets with additional properties which ensure that the equality

$$[\tilde{q}_i^*] = \sum_{j=1}^v a_{ij} [q_j^*], 1 \leq i \leq v$$

holds in $\mathcal{V}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I})$ for some choice of complex constants $a_{ij}$, $1 \leq i, j \leq v$. Therefore $\tilde{q}_i - \sum_{i=1}^v a_{ij} q_j \in \mathcal{V}_0(\mathcal{I})$. Since $q_i - \sum_{i=1}^v a_{ij} q_j$ is in $\mathcal{I}$, we have

$$0 = (q_i - \sum_{i=1}^v a_{ij} q_j)(D) (q_i - \sum_{i=1}^v a_{ij} q_j) = \|q_i - \sum_{i=1}^v a_{ij} q_j\|_0^2, 1 \leq i \leq v,$$

and hence $\tilde{q}_i = \sum_{i=1}^v a_{ij} q_j$, $1 \leq i \leq v$. We have therefore proved the following.

Proposition 2.10. Let $\mathcal{I} \subset \mathbb{C}[z]$ be a homogeneous ideal. If \{q_1, \ldots, q_v\} is a minimal set of generators for $\mathcal{I}$ with the property that \{[q_i^*] : 1 \leq i \leq v\} is a basis for $\mathcal{V}_0(\mathcal{I})/\mathcal{V}_0(\mathcal{I})$, then $q_1, \ldots, q_v$ is unique up to a linear transformation.

We end this section with the explicit calculation of the joint kernel for a class of submodules of the Hardy module which illustrate the methods of Proposition 2.8.

Example 2.11. Let $p_1, p_2$ be the minimal set of generators for an ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$. Assume that $p_1, p_2$ are homogeneous, deg $p_2 = \text{deg} p_1 + 1$ and $V(\mathcal{I}) = \{0\}$. As in Proposition 2.8 set $q_1 = p_1$ and $q_2 = p_2 + (\gamma_1 z_1 + \gamma_0 z_2)p_1$ subject to the equations

$$\begin{pmatrix} \|\partial_1 p_1\|_0^2 + \|p_1\|_0^2 & \langle \partial_2 p_1, \partial_1 p_1 \rangle_0 \\ \langle \partial_1 p_1, \partial_2 p_1 \rangle_0 & \|\partial_2 p_1\|_0^2 + \|p_1\|_0^2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_0 \end{pmatrix} = -\begin{pmatrix} \langle p_1, \partial_1 p_2 \rangle_0 \\ \langle p_1, \partial_2 p_2 \rangle_0 \end{pmatrix}$$

(2.12)

In this special case, the invertibility of the coefficient matrix follows from the positivity (Cauchy-Schwarz inequality) of its determinant

$$\|p_1\|_0^2 + \|\partial_1 p_1\|_0^2 \|p_2\|_0^2 + \|\partial_2 p_1\|_0^2 \|p_1\|_0^2 + \|\partial_2 p_2\|_0^2 \|p_1\|_0^2 + \|\partial_2 p_2\|_0^2 \|p_2\|_0^2 - (\langle \partial_1 p_1, \partial_2 p_2 \rangle_0)^2.$$ 

Specifically, if the ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ is generated by $z_1 + z_2$ and $z_2^2$. We have $V(\mathcal{I}) = \{0\}$. The reproducing kernel $K$ for $[\mathcal{I}] \subset H^2(\mathbb{D}^2)$ is

$$K_{[\mathcal{I}]}(z, w) = \frac{1}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)} - \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2) - 1}{2} = \frac{(z_1 + z_2)(\bar{w}_1 + \bar{w}_2)}{2} + \sum_{i+j \geq 2} z_1^i z_2^j \bar{w}_1^j \bar{w}_2^j.$$ 

The vector $\partial_2 K_{[\mathcal{I}]}(z, w)|_{w=0} = 2z_2^2$ is not in the joint kernel of $P_{[\mathcal{I}]}(M_1^*, M_2^*)|_{[\mathcal{I}]}$ since $M_2^*(z_2^2) = z_2$ and $P_{[\mathcal{I}]} z_2 = (z_1 + z_2)/2 \neq 0$. However, from equation (2.12), we have $q_1 = z_1 + z_2$ and $q_2 = (z_1 - z_2)^2,$
we see that $q_1, q_2$ generate the ideal $\mathcal{I}$ and $\{ (\partial_1 + \partial_2)K(\cdot, w)|_{0}, (\partial_1 - \partial_2)^2K(\cdot, w)|_{0} \}$ forms a basis of the joint kernel.

**Remark on Example 2.11** Let $\mathcal{I}$ be the ideal generated by $z_1 + z_2$ and $z_2^2$ and $\hat{\mathcal{I}}$ be the ideal generated by $z_1$ and $z_2$. Since $z_1 + z_2$ is not a linear combination of $z_1$ and $z_2^2$, it follows (Proposition 2.10) that $\hat{\mathcal{I}} \neq \mathcal{I}$.

**Remark 2.12.** If the generators of the ideal are not homogeneous then the conclusion of Theorem 2.9 is not valid. For instance, take the ideal $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ generated by $z_1(1 + z_1), z_1(1 - z_2), z_2 f$ which is also minimal for $\mathcal{I}$. We have $V(\mathcal{I}) = \{0\}$. We note that the stalk $\mathcal{I}_0^{\mathcal{I}}$ at 0 is generated by $z_1$ and $z_2^2$. Similar calculations, as above, shows that $\{ \partial_1 K(\cdot, w)|_{0}, \partial_2^2 K(\cdot, w)|_{0} \}$ is a basis of $\Gamma_{j=1}^n \ker M_j^\mathcal{I}$. But $z_1$ and $z_2^2$, or any linear combination of them, can not be a set of generators for $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ which is of rank 3. On the other hand, let $\mathcal{I}$ be the ideal generated by $z_1 + z_2 + z_1^2, z_2 - z_1^2$ which is minimal and $V(\mathcal{I}) = \{0\}$. In this case $\{ \partial_1 + \partial_2 K(\cdot, w)|_{0}, (\partial_1 - \partial_2)^2 K(\cdot, w)|_{0} \}$ is a basis of $\Gamma_{j=1}^n \ker M_j^\mathcal{I}$. But $z_1 + z_2$ and $(z_1 - z_2)^2$ is not a generating set for the stalk at 0.

3. Resolution of Singularities

We will use the familiar technique of ‘resolution of singularities’ and construct the blow-up space of $\hat{\Omega}$ along an ideal $\mathcal{I}$, which will be denoted by $\Omega$. There is a map $\pi : \Omega \to \hat{\Omega}$ which is biholomorphic on $\Omega \setminus \pi^{-1}(V(\mathcal{I}))$. However, in general, $\hat{\Omega}$ need not even be a complex manifold. Abstractly, the inverse image sheaf of $\mathcal{I}^{\mathcal{I}}$ under $\pi$ is locally principal and therefore corresponds to a line bundle on $\Omega$. Here, we explicitly construct a holomorphic line bundle, via the monoidal transformation, on $\pi^{-1}(w_0)$, $w_0 \in V(\mathcal{I})$, and show that the equivalence class of these Hermitian holomorphic vector bundles are invariants for the Hilbert module $\mathcal{M}$.

In the paper [8], submodules of functions vanishing at the origin of $H^{1, 0}(\mathbb{D}^2)$ were studied using the blow-up space $\mathbb{D}^2 \setminus (0, 0) \cup \mathbb{P}^1$ of the bi-disc. This is also known as the quadratic transform. However, this technique yields useful information only if the generators of the submodule are homogeneous polynomials of same degree. The monoidal transform, as we will see below, has wider applicability.

Let $L : \mathcal{M}_1 \to \mathcal{M}_2$ be a module map between two Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$ in the class $\mathcal{B}_1(\Omega)$. Define the map $\mathcal{F}^L : \mathcal{I}_{\mathcal{M}_1}(V) \to \mathcal{I}_{\mathcal{M}_2}(V)$ by setting

$$\mathcal{F}^L = \sum_{i=1}^n f_i |_{\mathcal{V}} g_i := \sum_{i=1}^n L f_i |_{\mathcal{V}} g_i, \text{ for } f_i \in \mathcal{M}_1, g_i \in \mathcal{O}(V), n \in \mathbb{N}.$$ 

The map $\mathcal{F}^L$ is well defined: if $\sum_{i=1}^n f_i |_{\mathcal{V}} g_i = \sum_{i=1}^n \bar{f}_i |_{\mathcal{V}} \bar{g}_i$, then $\sum_{i=1}^n L f_i |_{\mathcal{V}} g_i = \sum_{i=1}^n L \bar{f}_i |_{\mathcal{V}} \bar{g}_i$.

Suppose $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$ via the unitary module map $L$. Now, it is easy to verify that $(\mathcal{F}^L)^{-1} = \mathcal{F}^{L^*}$. It then follows that $\mathcal{I}_{\mathcal{M}_1}$ is isomorphic, as a sheaf of modules over $\mathcal{O}_\Omega$, to $\mathcal{I}_{\mathcal{M}_2}$ via the map $\mathcal{F}^L$.

Let $K_l$ be the reproducing kernel corresponding to $\mathcal{M}_i$, $i = 1, 2$. We assume that the dimension of the zero sets $X_i = V(\mathcal{M}_i)$ of the modules $\mathcal{M}_i$, $i = 1, 2$, is less or equal to $m - 2$. Recall that the stalk $\mathcal{F}_w^{\mathcal{M}_1}$ is $\mathcal{O}_w$ for $w \in \Omega \setminus X_i$, $i = 1, 2$. Let $X = X_1 \cup X_2$ and assume that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are in $\mathcal{B}_1(\Omega) \setminus X$. From [2, Lemma 1.3] and [7, Theorem 3.7], it follows that there exists a non-vanishing holomorphic function $\phi : \Omega \setminus X \to \mathbb{C}$ such that $L K_1(\cdot, w) = \phi(w) K_2(\cdot, w)$, $L^* f = \phi f$ and $K_1(z, w) = \phi(z) K_2(z, w) \phi(w)$. The function $\psi = 1/\phi$ on $\Omega \setminus X$ (induced by the inverse of $L$, that is, $L^*$) is holomorphic. Since $\dim X \leq m - 2$, by Hartog’s theorem (cf. [13, Page 196]) there is a unique extension of $\phi$ to $\Omega$ such that $\phi$ is non-vanishing on $\Omega$ ($\psi$ have an extension to $\Omega$ and $\phi \psi = 1$ on
We point out that by the decomposition theorem for some choice of anti-holomorphic functions $K_i$ of corresponding reproducing kernel. Let $w$ be the submodule in the Hardy module $(\mathcal{H}, \mathcal{M})$. In other words, setting $I = \{p \in \mathbb{C}[z] : q(D)p|_{u_0} = 0\}$ for all $q \in \mathcal{V}_{u_0}(\mathcal{H})$, as in [3], we see that $\mathcal{I}^e_{u_0} = \mathcal{I}^e_{u_0}$ for all $u_0 \in \Omega$. The proof is now complete since $\mathcal{I} = \cap_{u_0 \in \Omega} \mathcal{I}^e_{u_0}$ (cf. [3 Corollary 2.1.2]).

**Example 3.2.** For $j = 1, 2$, let $\mathcal{I}_j \subset \mathbb{C}[z_1, \ldots, z_m]$, $m > 2$, be the ideals generated by $z_1^n$ and $z_1^k z_2^{n-k}$. Let $[\mathcal{I}_j]$ be the submodule in the Hardy module $H^2(\mathbb{D}^m)$. From Theorem 3.1 it follows that $[\mathcal{I}_1]$ is equivalent to $[\mathcal{I}_2]$ if and only if $\mathcal{I}_1 = \mathcal{I}_2$. We conclude, using Proposition 2.10, that these two ideals are same only if $k_1 = k_2$.

3.1. The Monoidal Transformation. Let $\mathcal{M} = [\mathcal{I}]$ be a Hilbert module in $\mathcal{B}_1(\Omega)$ for some polynomial ideal $\mathcal{I}$. Assume that the dimension of the zero set $V(\mathcal{I})$ is at most $m - 2$. Let $K$ denote the corresponding reproducing kernel. Let $w_0 \in V(\mathcal{M})$. Set

$$t = \dim \mathcal{F}_w / m_{w_0} \mathcal{F}_w = \dim \cap_{j=1}^m \ker(M_j - w_0)^* = \dim \mathcal{V}_{w_0}(\mathcal{I}) / \mathcal{V}_{w_0}(\mathcal{I}).$$

By the decomposition Theorem [2 Theorem 1.4], there exists a minimal set of generators $g_1, \ldots, g_t$ of $\mathcal{F}_w$ and a $r > 0$ such that

$$K(\cdot, w) = \sum_{i=1}^t g_i(w) K(i)(\cdot, w), w \in \Delta(w_0; r)$$

(3.1)

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(t)} : \Delta(w_0; r) \to \mathcal{M}$.

Assume that $Z := V(g_1, \ldots, g_t) \cap \Omega$ be a singularity free analytic subset of $\mathbb{C}^m$ of codimension $t$. We point out that $Z$ depends on $\mathcal{M}$ as well as $w_0$. Define

$$\tilde{\Delta}(w_0; r) := \{ (\cdot, z(\cdot)) \in \Delta(w_0; r) \times \mathbb{P}^{t-1} : u_i g_i(w) - u_j g_j(w) = 0, 1 \leq i, j \leq t \}.$$

Here the map $\pi : C^1 \setminus \Omega \to \mathbb{P}^{t-1}$ is given by $\pi(u) = (u_1 : \ldots : u_t)$, the corresponding projective coordinate. The space $\tilde{\Delta}(w_0; r)$ is the monoidal transformation with center $Z$ ([12] page 241)]. Consider
the map \( p := \text{pr}_1 : \Delta(w_0; r) \to \Delta(w_0; r) \) given by \((w, \pi(z)) \mapsto w \). For \( w \in Z \), we have \( p^{-1}(w) = \{w\} \times \mathbb{P}^{t-1} \). This map is holomorphic and proper. Actually \( p : \Delta(w_0; r) \setminus p^{-1}(Z) \to \Delta(w_0; r) \setminus Z \) is biholomorphic with \( p^{-1} : w \to (w, (g_1(w) : \ldots : g_t(w))) \). The set \( E(\mathcal{M}) := p^{-1}(Z) \) which is \( Z \times \mathbb{P}^{t-1} \), is called the exceptional set.

We describe a natural line bundle on the blow-up space \( \Delta(w_0; r) \). Consider the open set \( U_1 = (\Delta(w_0; r) \times \{1 \neq 0\}) \cap \Delta(w_0; r) \). Let \( \frac{w}{w} = \theta_1 \), \( 2 \leq j \leq t \). On this chart \( g_j(w) = \theta_j g_j(w) \). From the decomposition given in the equation \([3.1]\), we have

\[
K(\cdot, w) = \frac{\phi_1(w)}{\theta_1} \cdot \Delta(\omega_1; \omega) + \sum_{j=2}^{t} \theta_j K^{(j)}(\cdot, w).
\]

This decomposition then yields a section on the chart \( U_1 \) of the line bundle on the blow-up space \( \Delta(w_0; r) \):

\[
s_1(w, \theta) = K^{(1)}(\cdot, w) + \sum_{j=2}^{t} \theta_j K^{(j)}(\cdot, w).
\]

The vectors \( K^{(j)}(\cdot, w) \) are not uniquely determined. However, there exists a canonical choice of these vectors starting from a basis \( \{v_1, \ldots, v_t\} \) of the joint kernel \( \cap_{i=1}^{n} \ker(M_j - w_j)^{*} \):

\[
K(\cdot, w) = \sum_{j=1}^{t} \theta_j g_j(w) \cdot \Delta(w, \bar{w}_0)v_j, \ w \in \Delta(w_0; r)
\]

for some \( r > 0 \) and generators \( g_1, \ldots, g_t \) of the stalk \( \mathcal{F}_{w_0}^{M} \). Thus we obtain the canonical choice \( K^{(j)}(\cdot, w) = P(w, \bar{w}_0)v_j, 1 \leq j \leq t \) (cf. [2, Section 6]). Let \( \mathcal{L}(\mathcal{M}) \) be the line bundle on the blow-up space \( \Delta(w_0; r) \) determined by the section \( (w, \theta) \mapsto s_1(w, \theta) \), where

\[
s_1(w, \theta) = \Delta(w, \bar{w}_0)v_1 + \sum_{j=2}^{t} \theta_j P(w, \bar{w}_0)v_j, (w, \theta) \in U_1.
\]

Let \( \mathcal{M} \) be a second Hilbert module in \( \mathcal{B}_1(\Omega) \), which is again the closure of some polynomial ideal \( \mathcal{I} \) but with respect to a second inner product. Assume that \( \mathcal{M} \) is equivalent to \( \mathcal{M} \) via a unitary module map \( L \), that is, \( LK(\cdot, w) = \varphi(w) \cdot \bar{K}(\cdot, w), w \in \Omega \) for some nonzero holomorphic function \( \phi \) on \( \Omega \). In the proof of Theorem 1.10 in [2], we have shown that \( LP(w, \bar{w}_0) = \bar{P}(w, \bar{w}_0)L \).

Thus

\[
\varphi(w) \cdot \bar{K}(\cdot, w) = LK(\cdot, w) = \sum_{j=1}^{t} \theta_j g_j(w) \cdot \Delta(w, \bar{w}_0)v_j = \sum_{j=1}^{t} \theta_j g_j(w) \cdot \bar{P}(w, \bar{w}_0)Lv_j.
\]

Therefore,

(i) \( s_1(w, \theta) = \Delta(\omega_1; \omega) \cdot \bar{K}(\cdot, w) + \sum_{j=2}^{t} \theta_j \bar{P}(w, \bar{w}_0)Lv_j \) and

(ii) \( Ls_1(w, \theta) = \varphi(w)s_1(w, \theta) \).

Hence the line bundles \( \mathcal{L}(\mathcal{M}) \) and \( \mathcal{L}(\mathcal{M}) \) are equivalent as Hermitian holomorphic line bundle on \( \Delta(w_0; r)^{\prime} = \{(w, \pi(u)) : (w, \pi(u)) \in \Delta(w_0; r)^{\prime}\} \). Since the vectors \( K(\cdot, w), 1 \leq j \leq t \), are linearly independent [2, Theorem 1.5], it follows that \( \mathcal{V}(\mathcal{M}) \cap \Delta(w_0; r) = Z \). Thus if \( w \in \Delta(w_0; r) \setminus Z \), then \( g_i(w) \neq 0 \) for some \( i, 1 \leq i \leq t \). Hence \( s_1(w, \theta) = \Delta(\omega_1; \omega) = \sum_{j=1}^{t} \theta_j g_j(w) \cdot \bar{P}(w, \bar{w}_0)Lv_j \). Therefore the restriction of the bundle \( \mathcal{L}(\mathcal{M}) \) to \( \Delta(w_0; r) \setminus p^{-1}(Z) \) is the pull back of the Cowen-Douglas bundle for \( \mathcal{M} \) on \( \Delta(w_0; r) \setminus Z \) via the biholomorphic map \( \pi \) on \( \Delta(w_0; r) \setminus p^{-1}(Z) \). We have therefore proved the following Theorem.
Theorem 3.3. Let $\mathcal{M}$ and $\mathcal{N}$ be two Hilbert modules of the form $[\mathcal{S}]$ and $[\mathcal{H}]$ ($\mathcal{S}$, $\mathcal{H}$ are polynomial ideals), respectively. Assume that they are in $\mathcal{B}_1(\Omega)$ and that the dimension of the zero set of these modules is at most $m - 2$. Then $\mathcal{M}$ and $\mathcal{N}$ are equivalent if and only if the line bundles $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\mathcal{N})$ are equivalent as Hermitian holomorphic line bundles on $\Delta(w_0; r)^\circ$.

Although, in general, $Z$ need not be a complex manifold, the restriction of $s_1$ to $p^{-1}(w_0)$ for $w_0 \in Z$ determines a holomorphic line bundle on $p^{-1}(w_0)^\circ := \{(w_0, \pi(\bar{u})) : (\bar{w}_0, \pi(u)) \in p^{-1}(w_0)\}$, which we denote by $\mathcal{L}_0(\mathcal{M})$. Thus $s_1 = s_1(w, \theta)|_{\{w_0\times \theta \neq 0\}}$ is given by the formula

$$s_1(\theta) = K^{(1)}(\cdot, w_0) + \sum_{j=2}^t \bar{\theta}^j K^{(j)}(\cdot, w_0).$$

Since the vectors $K^{(j)}(\cdot, w_0)$, $1 \leq j \leq t$, are uniquely determined by the generators $g_1, \ldots, g_t$, we infer that $s_1$ is well defined.

The following theorem follows from the one we have just proved. All we have to do is to restrict the line bundles to suitable subsets of the exceptional set. However, the details given in the proof below will be useful in studying explicit examples in the last section.

Theorem 3.4. Let $\mathcal{M}$ and $\mathcal{N}$ be two Hilbert modules of the form $[\mathcal{S}]$ and $[\mathcal{H}]$ ($\mathcal{S}$, $\mathcal{H}$ are polynomial ideals), respectively. Assume that they are in $\mathcal{B}_1(\Omega)$ and that the dimension of the zero set of these modules is at most $m - 2$. If the modules $\mathcal{M}$ and $\mathcal{N}$ are equivalent, then the corresponding bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\mathcal{N})$ they determine on the projective space $p^{-1}(w_0)^\circ$ for $w_0 \in Z$, are equivalent as Hermitian holomorphic line bundles.

Proof. Let $L: \mathcal{M} \to \mathcal{N}$ be the unitary module map and $K$ and $\tilde{K}$ be the reproducing kernels corresponding to $\mathcal{M}$ and $\mathcal{N}$ respectively. From [2] Lemma 1.3 and [7] Theorem 3.7, it follows that (i) $LK(\cdot, w) = \bar{\phi}(w)\tilde{K}(\cdot, w)$, (ii) $L^* f = \phi f$ and (iii) $K(z, w) = \phi(z)\bar{\phi}(w)$ for some holomorphic function $\phi$ on $\Omega \setminus \{\mathcal{M}\}$. As we have pointed out earlier, $\phi$ extends to a non-vanishing holomorphic function on $\Omega$.

Since $\mathcal{M}$ is in $\mathcal{B}_1(\Omega)$, it admits a decomposition as given in equation (3.1), with respect the generators $\tilde{g}_1, \ldots, \tilde{g}_t$ of $\mathcal{S}_{w_0}$. However, we may assume that $\tilde{g}_i = g_i$ for $1 \leq i \leq t$, because $\mathcal{S}_{w_0} = \mathcal{S}_{w_0}^{\mathcal{M}\mathcal{N}}$ for all $w_0 \in \Omega$. Thus for some $r > 0$, we have

$$\tilde{K}(\cdot, w) = \sum_{i=1}^t g_i(w)K^{(i)}(\cdot, w), \ w \in \Delta(w_0; r).$$

By applying the unitary $L$ to equation (3.1), we get

$$\bar{\phi}(w)\tilde{K}(\cdot, w) = LK(\cdot, w) = \sum_{i=1}^t g_i(w)LK^{(i)}(\cdot, w).$$

Since $\phi$ does not vanish on $\Omega$, we may choose

$$\tilde{K}^{(j)}(\cdot, w) = \frac{LK^{(j)}(\cdot, w)}{\phi(w)}, 1 \leq j \leq t, \ w \in \Delta(w_0; r).$$

From part (iii) of the decomposition Theorem ([2] Theorem 1.4]), the vectors $\tilde{K}^{(j)}(\cdot, w_0)$, $1 \leq j \leq t$, are uniquely determined by the generators $g_1, \ldots, g_t$. Therefore $\tilde{K}^{(j)}(\cdot, w_0) = \frac{LK^{(j)}(\cdot, w_0)}{\phi(w_0)}$. Now the
A decomposition for $\tilde{K}$ yields a holomorphic section $\tilde{s}_1(\theta) = \tilde{K}^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \tilde{K}^{(j)}(\cdot, w_0)$ for the holomorphic line bundle $\mathcal{L}_0(\mathcal{M})$ on the projective space $\mathbb{P}^{r}(w_0)$. Therefore

$$Ls_1(\theta) = LK^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \tilde{\theta}^{1}_j LK^{(j)}(\cdot, w_0)$$

$$= \tilde{\phi}(w_0)[\tilde{K}^{(1)}(\cdot, w_0) + \sum_{j=2}^{t} \tilde{\theta}^{1}_j \tilde{K}^{(j)}(\cdot, w_0)] = \tilde{\phi}(w_0)\tilde{s}_1(\theta).$$

From the unitarity of $L$, it follows that

$$\|s_1(\theta)\|^2 = \|Ls_1(\theta)\|^2 = \|\tilde{\phi}(w_0)\|^2 \|\tilde{s}_1(\theta)\|^2$$

and consequently the Hermitian holomorphic line bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\mathcal{\tilde{M}})$ on the projective space $\mathbb{P}^{r}(w_0)$ are equivalent. □

**Remark 3.5** (The case, where the dimension of the zero set $V(\mathscr{I})$ is $m - 1$). Let $\mathcal{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$. Assume that $\mathcal{M} = [\mathscr{I}, \mathcal{M}]$ for some polynomial ideal $\mathscr{I}$ and the dimension of the zero set of $\mathcal{M}$ is $m - 1$. Let the polynomials $p_1, \ldots, p_t$ be a minimal set of generators for $\mathcal{M}$. Let $q = \gcd(p_1, \ldots, p_t)$. Then the Beurling form (cf. [3]) of $\mathcal{M}$ is $q \mathcal{I}$, where $\mathcal{I}$ is generated by $\{p_1/q, \ldots, p_t/q\}$. From [3, Corollary 3.1.12], $\dim V(\mathscr{I}) \leq m - 2$ unless $\mathscr{I} = \mathbb{C}[z]$. The reproducing kernels $K$ of $\mathcal{M}$ is of the form $K(z, w) = q(z)\chi(z, w)q(w)$. Let $\mathcal{M}_1$ be the Hilbert module determined by the non-negative definite kernel $\chi$. The Hilbert module $\mathcal{M}$ is equivalent to $\mathcal{M}_1$. Now $\mathcal{M}_1 = [\mathcal{I}]$ and $V(\mathcal{I}_1) = V(\mathscr{I})$. If $V(\mathscr{I}) = \phi$, then the modules $\mathcal{M}_1$ belongs to Cowen-Douglas class of rank 1. Otherwise, $\dim V(\mathscr{I}) \leq m - 2$ and Theorem 3.3 determines its equivalence class.

The existence of the polynomials $q_1, \ldots, q_t$ such that $K^{(j)}(\cdot, w)|_{w=w_0} = q_j^{*}(\hat{D})K(\cdot, w)|_{w=w_0}, 1 \leq j \leq t$, is guaranteed by Lemma 2.1. From the decomposition Theorem (12, Theorem 1.4) and Lemma 2.1 that $\tilde{K}^{(j)}(\cdot, w)|_{w=w_0}$ is a linear combination of $q_j^{*}(\hat{D})K(\cdot, w)|_{w=w_0}, 1 \leq i \leq t$. But the following Lemma shows that

$$\tilde{K}^{(j)}(\cdot, w)|_{w=w_0} = q_j^{*}(\hat{D})\tilde{K}(\cdot, w)|_{w=w_0}, 1 \leq j \leq t,$$

which makes it possible to calculate the section for the line bundles $\mathcal{L}_0(\mathcal{M})$ and $\mathcal{L}_0(\mathcal{\tilde{M}})$ without any explicit reference to the generators of the stalks at $w_0$. In the following lemma, the decomposition of the reproducing kernels $K$ and $\tilde{K}$ are with respect to a common set of generators.

**Lemma 3.6.** Let $\mathcal{M}$ and $\mathcal{\tilde{M}}$ be two Hilbert modules both of which are completion of some polynomial ideal $\mathscr{I}$ with respect to two different inner products on the polynomial ring. Assume that they belong to the class $\mathfrak{B}_1(\Omega)$ and $\dim V(\mathscr{I}) \leq m - 2$. Let $K$ and $\tilde{K}$ be the corresponding reproducing kernels. Find polynomials $q_1, \ldots, q_t$, for which the vectors $K^{(j)}(\cdot, w) = q_j^{*}(\hat{D})K(\cdot, w)$ form a basis for the joint kernel at $w = w_0$. Then $\tilde{K}^{(j)}(\cdot, w) = q_j^{*}(\hat{D})\tilde{K}(\cdot, w)|_{w=w_0}$ is a basis for the joint kernel at $w_0$ in $\mathcal{\tilde{M}}$.

**Proof.** For $f \in \mathcal{M}$ and $1 \leq i \leq m$, we have

$$(f, \tilde{\partial}_iLK(\cdot, w)) = \hat{\partial}_i(f, LK(\cdot, w)) = \hat{\partial}_i(L^*f, K(\cdot, w)) = \langle L^*f, \hat{\partial}_iK(\cdot, w) \rangle = \langle f, L\hat{\partial}_iK(\cdot, w) \rangle,$$

implying $\tilde{\partial}_iLK(\cdot, w) = L\hat{\partial}_iK(\cdot, w)$. Thus

$$p(\hat{D})LK(\cdot, w) = Lp(\hat{D})K(\cdot, w) \text{ for any } p \in \mathbb{C}[z].$$
From equation (2.4), it follows that
\[
LK^{(j)}(\cdot, w_0) = L[q_j(\bar{D})K(\cdot, w)]_{w=w_0} = \{Lq_j(\bar{D})K(\cdot, w)\}_{w=w_0} = \{q_j(\bar{D})\phi(w)K(\cdot, w)\}_{w=w_0} = \sum_a a_a q_j(\bar{D})(\bar{w} - w_0)^a K(\cdot, w)]_{w=w_0} = \sum_a a_a \frac{\partial^a q_j}{\partial z^a}(\bar{D})K(\cdot, w)]_{w=w_0},
\]
where \( \phi(w) = \sum_a a_a (w - w_0)^a \), the power series expansion of \( \phi \) around \( w_0 \). Now for any \( p \in \mathcal{F} \) we have
\[
(p, \frac{\partial^a q_j}{\partial z^a}(\bar{D})K(\cdot, w)]_{w=w_0}) = (p, \frac{\partial^a q_j}{\partial z^a}(\bar{D})K(\cdot, w)]_{w=w_0}) = \frac{\partial^a q_j}{\partial z^a}(D)p(w)]_{w=w_0}.
\]
Since Lemma 2.1 ensures that \( \{[q^*_1], \ldots, [q^*_r]\} \) is a basis for \( \overline{V}_{w_0}(\mathcal{F})/\overline{V}_{w_0}(\mathcal{F}) \), it follows that
\[
(p, \frac{\partial^a q_j}{\partial z^a}(\bar{D})K(\cdot, w)]_{w=w_0}) = 0 \text{ for all } p \in \mathcal{F} \text{ and } |a| > 0.
\]
Therefore, we have \( \frac{\partial^a q_j}{\partial z^a}(\bar{D})K(\cdot, w)]_{w=w_0} = 0 \) for \( |a| > 0 \). Hence \( LK^{(j)}(\cdot, w_0) = a_0 q_j(\bar{D})K(\cdot, w)]_{w=w_0} = \phi(w_0)q_j(\bar{D})K(\cdot, w)]_{w=w_0} \) and consequently \( \tilde{K}^{(j)}(\cdot, w)]_{w=w_0} = q_j(\bar{D})K(\cdot, w)]_{w=w_0}, 1 \leq j \leq t. \)

4. Examples

We illustrate, by means of some examples, the nature of the invariants we obtain from the line bundle \( L_\theta \) that lives on the projective space. From Theorem 3.4 it follows that the curvature of the line bundle \( L_\theta \) is an invariant for the submodule. An example was given in [8] showing that the curvature is not a complete invariant. However the following lemma is useful for obtaining complete invariant in a large class of examples.

**Lemma 4.1.** Let \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) be Hilbert modules in \( \mathcal{B}_1(\Omega) \) for some bounded domain \( \Omega \subset C^m \). Suppose that \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) are in the Cowen-Douglas class \( \mathcal{B}_1(\Omega \setminus X) \), where \( \dim X \leq m - 2 \). Let \( \mathcal{M} \) (resp. \( \mathcal{M} \)) be a submodule of \( \mathcal{H} \) (resp. \( \tilde{\mathcal{H}} \)) such that

(i) \( V_w(\cdot, \mathcal{M}) = V_w(\cdot, \tilde{\mathcal{M}}) \) for all \( w \in \Omega \) and
(ii) \( \mathcal{M} = \cap_{w \in \Omega} \mathcal{M}_w \) and \( \mathcal{M} = \cap_{w \in \Omega} \tilde{\mathcal{M}}_w \), where

\( \mathcal{M}_w := \{ f \in \mathcal{H} : q(D)f|_w = 0 \text{ for all } q \in V_w(\cdot, \mathcal{M}) \}, \)

as before.

If \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) are equivalent, then \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) are equivalent.

**Proof.** Suppose \( U : \mathcal{H} \rightarrow \tilde{\mathcal{H}} \) is a unitary module map. Then \( U \) is is a multiplication operator induced by some holomorphic function, say \( \psi \), on \( \Omega \setminus X \) (cf. [7]). This function \( \psi \) extends non-vanishingly to all of \( \Omega \) by Hartog's Theorem. Let \( w_0 \in \Omega \) and \( q \in V_{w_0}(\cdot, \mathcal{M}) = V_{w_0}(\cdot, \mathcal{M}) \). Also let
$\psi(w) = \sum_a a_a(w - w_0)^a$ be the power series expansion around $w_0$. For $f \in \mathcal{M}$, we have
\[
q(D)(U f)|_{w = w_0} = q(D)(\psi f)|_{w = w_0} = q(D)\left(\sum_a a_a(w - w_0)^a f\right)|_{w = w_0}
\]
\[
= \sum_a a_a q(D)((w - w_0)^a f)|_{w = w_0}
\]
\[
= i \sum_{k \leq a} \binom{a}{k} (w - w_0)^{a-k} \frac{q(z)}{z^k} (D(f))|_{w = w_0}
\]
\[
= 0
\]
since $\frac{q(z)}{z^k} \in \mathcal{V}_{w_0}(\mathcal{M})$ for any multi index $k$ whenever $q \in \mathcal{V}_{w_0}(\mathcal{M})$. Therefore it follows that $U f \in \mathcal{M}$. A similar arguments shows that $U^* \mathcal{M} \subseteq \mathcal{M}$. The result follows from unitarity of $U$.

4.1. The $(\alpha, \beta, \theta)$ examples: Weighted Bergman Modules in the unit ball. Let $\mathbb{B}^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ be the unit ball in $\mathbb{C}^2$. Let $L^2_{\alpha, \beta, \theta}(\mathbb{B}^2)$ be the Hilbert space of all (equivalence classes of) Borel measurable functions on $\mathbb{B}^2$ satisfying
\[
\|f\|^2_{\alpha, \beta, \theta} = \int_{\mathbb{B}^2} |f(z)|^2 d\mu(z_1, z_2) < +\infty,
\]
where the measure is
\[
d\mu(z_1, z_2) = (\alpha + \beta + \theta + 2)|z_1|^{2\theta}(1 - |z_1|^2)|z_2|^{2\theta}(1 - |z_2|^2) dA(z_1, z_2)
\]
for $(z_1, z_2) \in \mathbb{B}^2$, $-1 < \alpha, \beta, \theta < +\infty$ and $dA(z_1, z_2) = dA(z_1) dA(z_2)$. Here $dA$ denote the normalized area measure in the plane, that is $dA(z) = \frac{1}{\pi} dx dy$ for $z = x + iy$. The weighted Bergman space $\mathcal{A}^2_{\alpha, \beta, \theta}(\mathbb{B}^2)$ is the subspace of $L^2_{\alpha, \beta, \theta}(\mathbb{B}^2)$ consisting of the holomorphic functions on $\mathbb{B}^2$. The Hilbert space $\mathcal{A}^2_{\alpha, \beta, \theta}(\mathbb{B}^2)$ is non-trivial if we assume that the parameters $\alpha, \beta, \theta$ satisfy the additional condition:
\[
\alpha + \beta + \theta + 2 > 0.
\]
The reproducing kernel $K_{\alpha, \beta, \theta}(z, w) = \frac{1}{\alpha + \beta + \theta + 2} \left(1 - \frac{z_1 \bar{w}_1}{1 - z_1 \bar{w}_1}\right)^{\alpha+\beta+\theta+3} \left(1 - \frac{z_2 \bar{w}_2}{1 - z_2 \bar{w}_2}\right)^{\alpha+\beta+\theta+3} \left(\sum_{k=0}^{\infty} \frac{(\alpha + \beta + \theta + k + 2)(\alpha + \theta + 2)k}{(\theta + 1)_k} \left(\frac{z_2 \bar{w}_2}{1 - z_1 \bar{w}_1}\right)^k\right),
\]
where $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{B}^2$ and $(a)_k = a(a+1)...(a+k-1)$ is the Pochhammer symbol.

Let $\mathcal{F}$ be an ideal in $\mathcal{C}[z_1, z_2]$ such that $V(\mathcal{F}) = \{P\} \subset \mathbb{B}^2$. We have
\[
\dim \ker(M_{j} - w) = \begin{cases} 
1 & \text{for } w \in \mathbb{B}^2 \setminus \{P\}; \\
\dim \mathcal{F}/m_{\mathcal{F}}(\mathcal{F})(>1) & \text{for } w = P.
\end{cases}
\]
Hence $[\mathcal{F}]_{\alpha, \beta, \theta}(\mathbb{B}^2)$ (the completion of $\mathcal{F}$ in $\mathcal{A}^2_{\alpha, \beta, \theta}(\mathbb{B}^2)$) is not equivalent to $[\mathcal{F}]_{\alpha, \beta, \theta}(\mathbb{B}^2)$ (the completion of $\mathcal{F}$ in $\mathcal{A}^2_{\alpha, \beta, \theta}(\mathbb{B}^2)$) if $P \neq \mathcal{F}$. Now let us determine when two modules in the set
\[
\{[\mathcal{F}]_{\alpha, \beta, \theta}(\mathbb{B}^2) : -1 < \alpha, \beta, \theta < +\infty \text{ and } \alpha + \beta + \theta + 2 > 0\}
\]
are equivalent. In the following proposition, without loss of generality, we assume that $P = 0$. 
Proposition 4.2. Suppose $\mathcal{I}$ is an ideal in $\mathbb{C}[z_1, z_2]$ with $V(\mathcal{I}) = \{0\}$. Then the Hilbert modules $[\mathcal{I}, \alpha^2, \beta, \theta, r, s](\mathbb{B}^2)$ and $[\mathcal{I}, \alpha', \beta', \theta', r', s'](\mathbb{B}^2)$ are unitarily equivalent if and only if $\alpha = \alpha', \beta = \beta'$ and $\theta = \theta'$.

Proof. From the Hilbert Nullstellensatz, it follows that there exist an natural number $N$ such that $m_N \subset \mathcal{I}$. Let $\mathcal{I}_{m,n}$ be the polynomial ideal generated by $z_1^m$ and $z_2^n$. Combining (2.1) with Lemma 4.1, we see, in particular, that the submodules $[\mathcal{I}_{m,n}, \alpha^2, \beta, \theta, r, s](\mathbb{B}^2)$ and $[\mathcal{I}_{m,n}, \alpha', \beta', \theta', r', s'](\mathbb{B}^2)$ are unitarily equivalent for $m, n \geq N$. Let $K_{m,n}$ be the reproducing kernel for $[\mathcal{I}_{m,n}, \alpha^2, \beta, \theta, r, s](\mathbb{B}^2)$. We write $K_{m,n}(z, w) = \sum_{i,j \geq 0} b_{ij} z_1^i z_2^j$ where

$$b_{ij} = \frac{\alpha + \beta + \theta + j + 2}{\alpha + \beta + \theta + 2} \cdot \frac{(\alpha + \theta + 2)_i}{(\theta + 1)_j \cdot i!}.$$  

Let $I_{m,n} := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i, j \geq 0, i \geq m \text{ or } j \geq n\}$. We note that

$$K_{m,n}(z, w) = \sum_{(i, j) \in I_{m,n}} b_{ij} z_1^i z_2^j.$$  

One easily see that the set $\{z_1^m, z_2^n\}$ forms a minimal set of generators for the sheaf corresponding to $[\mathcal{I}_{m,n}, \alpha^2, \beta, \theta, 2](\mathbb{B}^2)$. The reproducing kernel then can be decomposed as

$$K_{m,n}(z, w) = \tilde{w}_1^m K_1^{m,n}(z, w) + \tilde{w}_2^n K_2^{m,n}(z, w)$$

for some $r > 0$ and $w \in \Delta(0; r)$.

Successive differentiation, using Leibnitz rule, gives

$$K_1^{m,n}(z, w)|_{w=0} = \frac{1}{m!} \tilde{z}_1^m K_{m,n}(z, w)|_{w=(0,0)} = b_{m0} z_1^m$$

and

$$K_2^{m,n}(z, w)|_{w=0} = \frac{1}{n!} \tilde{z}_2^n K_{m,n}(z, w)|_{w=(0,0)} = b_{0n} z_2^n.$$  

Therefore

$$s_1(\theta_1) = b_{m0} z_1^m + \theta_1 b_{0n} z_2^n,$$

where $\theta_1$ denotes co-ordinate for the corresponding open chart in $\mathbb{P}$. Thus

$$||s_1(\theta_1)||^2 = b_{m0}^2 ||z_1^m||^2 + b_{0n}^2 ||z_2^n||^2 ||\theta_1||^2 = b_{m0}^2 + b_{0n}^2 ||\theta_1||^2.$$  

Let $a_{m,n} = b_{0n}/b_{m0}$. Let $\mathcal{H}_{m,n}$ denote the curvature corresponding to the bundle $\mathcal{L}_{0,m,n}$ which is determined on the projective space $\mathbb{P}^1$ by the module $[\mathcal{I}_{m,n}, \alpha^2, \beta, \theta, 2](\mathbb{B}^2)$. Thus we have

$$\mathcal{H}_{m,n}(\theta_1) = \frac{2 \partial \bar{\partial} \ln ||s_1(\theta_1)||^2}{a_{m,n} \theta_1^2} = \frac{2 \partial \bar{\partial} \ln (1 + a_{m,n} ||\theta_1||^2)}{a_{m,n} (1 + a_{m,n} ||\theta_1||^2)^2}.$$  

Let $\mathcal{H}_{m,n}'$ denote the curvature corresponding to the bundle $\mathcal{L}_{0,m,n}'$ which is determined on the projective space $\mathbb{P}^1$ by the module $[\mathcal{I}_{m,n}, \alpha', \beta', \theta', 2](\mathbb{B}^2)$. As above, from Lemma 3.6, we have

$$\mathcal{H}_{m,n}'(\theta_1) = \frac{a_{m,n}'}{(1 + a_{m,n}' ||\theta_1||^2)^2}.$$  

Since the submodules $[\mathcal{I}_{m,n}, \alpha^2, \beta, \theta, 2](\mathbb{B}^2)$ and $[\mathcal{I}_{m,n}, \alpha', \beta', \theta', 2](\mathbb{B}^2)$ are unitarily equivalent, from Theorem 3.4, it follows that $\mathcal{H}_{m,n}(\theta_1) = \mathcal{H}_{m,n}'(\theta_1)$ for $\theta_1$ in an open chart $\mathbb{P}^1$ and $m, n \geq N$. Thus

$$\frac{a_{m,n}}{(1 + a_{m,n} ||\theta_1||^2)^2} = \frac{a_{m,n}'}{(1 + a_{m,n}' ||\theta_1||^2)^2}.$$
This shows that \((a_{m,n} - a'_{m,n})(1 + a_{m,n}a'_{m,n}|\theta|^2) = 0\). So \(a_{m,n} = a'_{m,n}\) and hence

\[
\frac{b_{mN}}{b'_{mN}} = \frac{b'_{mN}}{b_{mN}}
\]

(4.2)

for all \(m, n \geq N\). This also follows directly from the equation (3.2). It is enough to consider the cases \((m, n) = (N, N), (N, N+1), (N, N+2)\) and \((N+1, N)\) to prove the Proposition. From equation (4.2), we have

\[
\frac{b_{(N+1)0}}{b_{0N}} = \frac{b'_{(N+1)0}}{b'_{0N}}, \quad \frac{b_{0(N+1)}}{b_{0N}} = \frac{b'_{0(N+1)}}{b'_{0N}} \quad \text{and} \quad \frac{b_{0(N+2)}}{b_{0(N+1)}} = \frac{b'_{0(N+2)}}{b'_{0(N+1)}}.
\]

(4.3)

Let \(A = \alpha + \beta + \theta, B = \alpha + \theta\) and \(C = \theta\). From equation (4.1), we have

\[
\frac{b_{0(N+1)}}{b_{0N}} = \frac{A + N + 3}{N + 1}, \quad \frac{b_{0(N+2)}}{b_{0(N+1)}} = \frac{A + N + 4}{A + N + 3} \cdot \frac{B + N + 3}{C + N + 2}.
\]

From (4.3), it follows that \(A = A'\) and

\[
B'C' + B(N+1) + C'(N+2) = B'C + B'(N+1) + C(N+2), \quad (4.4)
\]

\[
B'C' + B(N+2) + C'(N+3) = B'C + B'(N+2) + C(N+3). \quad (4.5)
\]

Subtracting (4.5) from (4.4), we get \(B - C = B' - C'\) and thus \(\theta = \theta'\). Therefore \(\frac{b_{0(N+1)}}{b_{0N}} = \frac{b'_{0(N+1)}}{b'_{0N}}\), implying \(B = B'\) and hence \(\alpha = \alpha'\). Lastly \(A = A'\) and in consequence \(\beta = \beta'\).

\[\square\]

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