BOUNDED COHOMOLOGY OF TRANSFORMATION GROUPS

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Abstract. Let M be a complete connected Riemannian manifold of finite volume. In this paper we present a new method of constructing classes in bounded cohomology of transformation groups such as $\mathrm{Homeo}_0(M,\mu)$, $\mathrm{Diff}_0(M,\mathrm{vol})$ and $\mathrm{Symp}_0(M,\omega)$ (in case M is symplectic). As an application we show that, under certain conditions on $\pi_1(M)$, the 3^{rd} bounded cohomology of these groups is infinite dimensional.

Let M be a complete connected Riemannian manifold of finite volume such that it has empty boundary. Suppose μ is a measure on M induced by the Riemannian metric. Let $\mathrm{Homeo}(M,\mu)$ be the group of all measure-preserving compactly supported homeomorphisms of M. We regard $\mathrm{Homeo}(M,\mu)$ as a topological group equipped with the C^0-topology. By $\mathrm{Homeo}_0(M,\mu)$ we denote the connected component of $\mathrm{Homeo}(M,\mu)$. In this paper we define and study a map:

$$\Gamma_b\colon \mathrm{H}_b^\bullet(\pi_1(M)) \to \mathrm{H}_b^\bullet(\mathrm{Homeo}_0(M,\mu)).$$

This map can be seen as a generalization of the construction given by Gambaudo-Ghys [14] and Polterovich [23], used and extended later to study different conjugacy invariant norms on transformation groups [5, 1, 3, 4, 6, 14]. The former constructions were restricted only to non-trivial homogeneous quasimorphisms, that is, to a certain subspace of the second bounded cohomology. Here we are able to deal with the bounded cohomology in all dimensions.

The map Γ_b can be defined also for the exact and reduced bounded cohomology. Moreover, and it is an another novelty of the present paper, Γ_b has a counterpart that works for the *ordinary cohomology* in all dimensions. However, the case of the ordinary cohomology seams to be less natural for our construction, and hence harder to work with.

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The group $\mathrm{Homeo}_0(M,\mu)$ and it subgroups preserve certain volume-like structure. This property is essential. It is known that Γ_b can not factor through larger groups as $\mathrm{Homeo}_0(M)$, see Remark 4.2.

We define and study the map Γ_b and give applications to bounded cohomology of certain transformation groups $\mathcal{T}_M < \mathrm{Homeo}_0(M,\mu)$. The groups we are interested in are the following:

- M is a complete Riemannian manifold of finite volume, μ is the measure induced by the Riemannian metric and we consider $\mathcal{T}_M = \mathrm{Homeo}_0(M,\mu)$.
- M is a complete Riemannian manifold of finite volume, vol is the volume form induced by the Riemannian metric and $\mathcal{T}_M = \mathrm{Diff}_0(M,\mathrm{vol})$, i.e., the connected component of the identity of the group of all volume-preserving compactly supported diffeomorphisms of M.
- (M,ω) is a symplectic manifold of finite volume which admits a complete Riemannian metric, and $\mathcal{T}_M = \mathrm{Symp}_0(M,\omega)$, i.e., the connected component of the identity of the group of all compactly supported symplectomorphisms of M. Here we regard $\mathcal{T}_M$ as a subgroup of $\mathrm{Homeo}_0(M,\mu)$, where μ is induced by the volume form $\omega^{\frac{1}{2}\dim M}$. We assume that μ is finite.

Let G be a group. By $\overline{\mathrm{EH}}_b^\bullet(G)$ we denote the reduced exact bounded cohomology of G (see Section 1), and by $\dim \overline{\mathrm{EH}}_b^\bullet(G)$ the real linear dimension of $\overline{\mathrm{EH}}_b^\bullet(G)$. We set π_M to be the quotient $\pi_1(M)/Z(\pi_1(M))$, where $Z(\pi_1(M))$ is the center of $\pi_1(M)$. By F_2 we denote the free group of rank 2.

Theorem A. *Suppose π_M surjects onto F_2. Then for every n there is an injective homomorphism*

$$\overline{\mathrm{EH}}_b^n(\mathrm{F}_2) \hookrightarrow \overline{\mathrm{EH}}_b^n(\mathcal{T}_M),$$

where $\mathcal{T}_M$ is $\mathrm{Homeo}_0(M,\mu)$ or $\mathrm{Diff}_0(M,\mathrm{vol})$ or $\mathrm{Symp}_0(M,\omega)$.

Theorem B. *Suppose $\mathrm{F}_2 \times K$ embeds hyperbolically into π_M, where K is a finite group. Then for every n we have*

$$\dim \overline{\mathrm{EH}}_b^n(\mathcal{T}_M) \geqslant \dim \overline{\mathrm{EH}}_b^n(\mathrm{F}_2),$$

where $\mathcal{T}_M$ is $\mathrm{Homeo}_0(M,\mu)$ or $\mathrm{Diff}_0(M,\mathrm{vol})$ or $\mathrm{Symp}_0(M,\omega)$.

The notion of hyperbolic embedding was defined in [11]. Note that Theorem B is applicable whenever $\pi_1(M)$ is a center-free acylindrically hyperbolic group, since such a group always admits a hyperbolically embedded $\mathrm{F}_2 \times K$, see [22, Theorem 1.2] and [11, Theorem 2.24]. Examples of acylindrically hyperbolic groups include:

(1) non-elementary hyperbolic groups and relatively hyperbolic groups,
(2) mapping class groups of hyperbolic surfaces and outer automorphism groups of non-abelian free groups,
(3) most 3-manifolds groups,
(4) right angled Artin groups that are not direct products.

The result of Soma [28] implies $\dim \overline{\mathrm{EH}}^3_b(\mathrm{F}_2) = 2^{\aleph_0}$. Thus we have:

Corollary. If assumptions of Theorem A or Theorem B hold, then

$$\dim \overline{\mathrm{EH}}^3_b(\mathcal{T}_M) \geqslant 2^{\aleph_0}.$$

Remark. There is a long standing open problem: Is bounded cohomology of non-abelian free group non-trivial beyond the dimension 3? Note that if the answer to the above problem is positive, then Theorem A implies that the corresponding bounded cohomology of $\mathrm{Homeo}_0(M, \mu)$ or $\mathrm{Diff}_0(M, \mathrm{vol})$ or $\mathrm{Symp}_0(M, \omega)$ is non-trivial in many cases, and in particular when M is a closed hyperbolic surface.

We would like to add that version of Theorem A and Theorem B for bounded cohomology hold in a more general setting, see Remark 4.1.

The non-trivial classes we construct in $\overline{\mathrm{EH}}^3_b(\mathcal{T}_M)$ come from non-trivial classes in $\overline{\mathrm{EH}}^3_b(\mathrm{F}_2)$. There is a nice family of elements in $\overline{\mathrm{EH}}^3_b(\mathrm{F}_2)$, whose representing cocycles are defined geometrically in terms of volumes of geodesic simplices in the hyperbolic 3-space $\mathbf{H}^3$. The elements of $\overline{\mathrm{EH}}^3_b(\mathcal{T}_M)$ that are constructed out of such classes in Theorem A retain similar geometrical description, see Remark 3.3.

We finish the introduction by noting, that elements of $\mathrm{H}^n_b(\mathcal{T}_M)$ can be interpreted as bounded characteristic classes of foliated M-bundles with holonomy in $\mathcal{T}_M$. Such classes were studied in the case of ordinary cohomology for groups $\mathrm{Diff}(M, \mathrm{vol})$ and $\mathrm{Symp}(M, \omega)$, see ,e.g., [20, 19, 18, 21, 25, 24] and references therein. We want to stress out, that the classes constructed in the present paper are of entirely different nature.

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1. Preliminaries

1.A. **Bounded cohomology.** Bounded cohomology was defined in a seminal paper of Gromov [15]. Below we give basic definitions.

Let G be a group. A function $c\colon G^{n+1} \to \mathbb{R}$ is called homogeneous, if for every $h \in G$ and every $g_0, \ldots, g_n \in G$ we have

$$c(g_0 h, \ldots, g_n h) = c(g_0, \ldots, g_n).$$

Let us consider the space of bounded n-cochains:

$$\mathrm{C}_b^n(G) = \{c\colon G^{n+1} \to \mathbb{R} \mid c \text{ is homogeneous and bounded}\}.$$

The sequence $\{\mathrm{C}_b^n(G), d_n\}$ is a chain complex, where d is the ordinary coboundary operator $d_n\colon \mathrm{C}_b^n(G) \to \mathrm{C}_b^{n+1}(G)$. The **bounded cohomology** of the group G is the homology of this complex, i.e., $\mathrm{H}_b^n(G) = \ker(d_n)/\operatorname{im}(d_{n-1})$.

Note that $\mathrm{C}_b^n(G)$ is a subcomplex of the space of all homogeneous cochains, thus we have a canonical map $\mathrm{H}_n^b(G) \to \mathrm{H}^n(G, \mathbb{R})$, called the **comparison map**. The **exact bounded cohomology**, denoted $\mathrm{EH}_b^n(G)$, is defined to be the kernel of the comparison map. A class belongs to $\mathrm{EH}_b^n(G)$ if it is a coboundary of a cochain, but is never a coboundary of a bounded cochain.

On $\mathrm{C}_b^n(G)$ we have the supremum norm denoted by $\|\cdot\|_{sup}$. This norm induces a semi-norm on $\mathrm{H}_b^n(G)$, i.e., if $c \in \mathrm{H}_b^n(G)$, then

$$\|c\| = \min\{\|a\|_{sup} \mid [a] = c\}.$$

Let $\mathrm{N}^n(G) \leqslant \mathrm{H}_b^n(G)$ be the linear subspace consisting of classes of zero norm. The **reduced bounded cohomology** equals to the quotient

$$\overline{\mathrm{H}}_b^n(G) = \mathrm{H}_b^n(G)/\,\mathrm{N}^n(G).$$

Note that $\overline{\mathrm{H}}_b^n(G)$ with the induced norm, that we again denote by $\|\cdot\|$, is a Banach space. The **exact reduced bounded cohomology** is defined to be

$$\overline{\mathrm{EH}}_b^n(G) \;=\; \mathrm{EH}_b^n(G)/(\mathrm{EH}_b^n(G) \cap \mathrm{N}^n(G)).$$

Note that $\overline{\mathrm{EH}}_b^n(G) \leqslant \overline{\mathrm{H}}_b^n(G)$ and $\overline{\mathrm{EH}}_b^n(G)$ with the induced norm are Banach spaces.

1.B. **Measurable cocycles and induction on bounded cohomology.** Let G be a topological group, H a discrete group and (X, μ) a measurable space. Suppose that G acts on X by measure preserving homeomorphisms. A map $\gamma\colon G \times X \to H$ is called a **measurable cocycle**, if for all $g \in G$ the map $\gamma(g, \cdot)$ is measurable and for all $g_1, g_2 \in G$ we have

$$\gamma(g_1 g_2, x) = \gamma(g_1, g_2(x))\gamma(g_2, x)$$

for almost all $x \in X$. Now given a measurable cocycle γ, we produce a map

$$\mathrm{Ind}_b(\gamma)\colon\ \mathrm{H}_b^n(H) \to \mathrm{H}_b^n(G).$$

Let $c \in \mathrm{C}_b^n(H)$ and let γ be a measurable cocycle. First we define a map $\mathrm{Ind}_b'(\gamma)$ on the space of cocycles by the following formula:

$$\mathrm{Ind}_b'(\gamma)(c)(g_0, g_1, \ldots, g_n) = \int_M c(\gamma(g_0, x), \gamma(g_1, x), \ldots, \gamma(g_n, x))d\mu(x).$$

The next proposition shows that $\mathrm{Ind}_b'(\gamma)$ induces a map on the level of bounded cohomology which we call $\mathrm{Ind}_b(\gamma)$.

Proposition 1.1. *Let G be a group. Then the function*

$$x \to c(\gamma(g_0, x), \gamma(g_1, x), \ldots, \gamma(g_n, x))$$

is a μ-measurable function on X for every $g_0, g_1, \ldots, g_n \in G$, $\mathrm{Ind}_b'(\gamma)$ commutes with the coboundary d, and $\mathrm{Ind}_b'(\gamma)(c)$ is homogeneous.

Proof. The function $X \to H$ given by $x \to (\gamma(g_0, x), \gamma(g_1, x), \ldots, \gamma(g_n, x))$ is μ-measurable and the cochain c is continuous (H is discrete), thus their composition is μ-measurable.

The fact that $\mathrm{Ind}_b'(\gamma)$ commutes with the coboundary d follows directly from the definition of $\mathrm{Ind}_b'(\gamma)$.

For all $h, g_0, \ldots \in G$, we have:

$$\begin{aligned}
\mathrm{Ind}_b'(\gamma)(c)(g_0 h, \ldots) &= \int_X c(\gamma(g_0 h, x), \ldots)d\mu(x) \\
&= \int_X c(\gamma(g_0, h(x))\gamma(h, x), \ldots,)d\mu(x) \\
&= \int_X c(\gamma(g_0, h(x)), \ldots)d\mu(x) \\
&= \int_X c(\gamma(g_0, x), \ldots)dh^*\mu(x) = \mathrm{Ind}_b'(\gamma)(c)(g_0, \ldots).
\end{aligned}$$

In the above we first used the cocycle condition, then the homogeneity of c and that μ is h-invariant. □

We conclude this section by noting that similar maps were used in the study of ordinary continuous cohomology of Lie groups [16] and geometry of solvable and amenable groups [26, 27].

2. Definition of Γ_b

The goal of this section is to construct the homomorphism

$$\Gamma_b\colon \mathrm{H}_b^\bullet(\pi_1(M)) \to \mathrm{H}_b^\bullet(\mathrm{Homeo}_0(M,\mu)).$$

In order to do that, we first define a certain measurable cocycle γ and use it to define the above map.

2.A. **The cocycle.** Denote by $\mathcal{H}_M = \mathrm{Homeo}_0(M)$ the connected component of the identity of the group of all compactly supported homeomorphisms of M. We fix a basepoint $z \in M$. Let $\mathcal{H}_{M,z} < \mathcal{H}_M$ be the subgroup of all homeomorphisms in $\mathcal{H}_M$ that fix z. Consider the following fiber bundle

$$\mathcal{H}_{M,z} \to \mathcal{H}_M \xrightarrow{ev_z} M,$$

where ev_z is the evaluation map at the basepoint z, i.e., if $g \in \mathcal{H}_M$, then $ev_z(g) = g(z)$, and $\mathcal{H}_{M,z}$ is the fiber of ev_z. The group $\mathcal{H}_{M,z}$ has an interesting quotient. Namely, it is the group of connected components $\pi_0(\mathcal{H}_{M,z})$ which is closely related to $\pi_1(M)$. Recall that $\pi_M = \pi_1(M)/Z(\pi_1(M))$, where $Z(\pi_1(M))$ is the center of $\pi_1(M)$.

Proposition 2.1. *There is an epimorphism $\delta\colon \mathcal{H}_{M,z} \to \pi_M$.*

Proof. Let us consider the long exact sequence of homotopy groups of the fibration

$$\mathcal{H}_{M,z} \to \mathcal{H}_M \xrightarrow{ev_z} M.$$

We have

$$\pi_1(\mathcal{H}_M) \xrightarrow{ev_z^*} \pi_1(M,z) \to \pi_0(\mathcal{H}_{M,z}) \to \pi_0(\mathcal{H}_M) = 1.$$

We shall show that $\mathrm{im}(ev_z^*) \subset Z(\pi_1(M,z))$. Indeed, let g_t, $t \in S^1$, be a loop in $\mathcal{H}_M$ based at the identity and $[g_t] \in \pi_1(\mathcal{H}_M)$. Then $ev_z^*([g_t])$ is a loop based at z represented by $g_t(z)$, $t \in S^1$. Let l_s, $s \in S^1$, be an arbitrary loop in M based at z. We have that the image of the map $S^1 \times S^1 \to M$ given by $(t,s) \to g_t(l_s)$, contains $g_t(z)$ and l_s, thus these loops commute in $\pi_1(M, z)$. We define δ to be the composition of the map

$$\mathcal{H}_{M,z} \to \pi_0(\mathcal{H}_{M,z}) \cong \pi_1(M, z)/\operatorname{im}(ev_z^*)$$

and the quotient map $\pi_1(M, z)/\operatorname{im}(ev_z) \to \pi_M$. □

Let $\mathcal{H}_{M,\mu} = \mathrm{Homeo}_0(M, \mu)$ and let $s\colon M \to \mathcal{H}_M$ be a measurable section of ev_z, i.e., $ev_z \circ s(x) = x$ for almost all $x \in M$. We define a measurable cocycle: $\gamma_s\colon \mathcal{H}_{M,\mu} \times M \to \pi_M$ by the formula

$$\gamma_s(g, x) = \delta(s_{g(x)}^{-1} \circ g \circ s_x).$$

It follows from the formula, that γ_s satisfies the cocycle condition.

2.B. **Example of a section s.** Let us consider the following set:

$$D = \operatorname{int}(\{\ x \in M \mid \text{there exists a unique geodesic between } z \text{ and } x\ \}).$$

The set $M \backslash D$ is called the cut locus of M. The Hausdorff dimension of $M \backslash D$ is at most $\dim(M) - 1$, see [17]. Thus $\mu(M \backslash D) = 0$, and $\mu(D) = \mu(M)$. We regard s as a measurable map, hence it is enough to define s on the full measure subset D. Let $x \in D$. We define s_x to be a point pushing map that transports z to x along the unique geodesic. It can be done such that s is continuous on D.

Let us now take a closer look at the cocycle γ_s defined using a section s as described in Subsection 2.A. Let $g \in \mathcal{H}_M$ and $x \in D \cap g^{-1}(D) \subset M$. The element $\gamma_s(g, x) \in \pi_M$ has a simple geometrical interpretation. It can be constructed as follows: let g_t, $t \in [0, 1]$, be any isotopy in $\mathcal{H}_M$ connecting the identity with g. Let α be the loop which is the concatenation of the unique geodesic from z to x, the path $g_t(x)$, $t \in [0, 1]$, and the unique geodesic from $g(x)$ to z. The loop α regarded as an element of π_M is well defined and equals $\gamma_s(g, x)$.

2.C. **The definition of Γ_b.** Let $\gamma_s\colon \mathcal{H}_{M,\mu} \times M \to \pi_M$ be a measurable cocycle given by a measurable section $s\colon M \to \mathcal{H}_M$. We define

$$\Gamma_b = \mathrm{Ind}_b(\gamma_s)\colon\ \mathrm{H}_b^\bullet(\pi_M) \to \mathrm{H}_b^\bullet(\mathcal{H}_{M,\mu}).$$

Note that the quotient map $\pi_1(M) \to \pi_M$ has an abelian kernel. It follows from the mapping theorem [15, Section 3.1] that a quotient map defines isometric isomorphisms $\mathrm{H}_b^\bullet(\pi_M) \to \mathrm{H}_b^\bullet(\pi_1(M))$. Thus we can regard Γ_b as a following map: $\Gamma_b\colon\ \mathrm{H}_b^\bullet(\pi_1(M)) \to \mathrm{H}_b^\bullet(\mathcal{H}_{M,\mu})$.

If $\mathcal{T}_M < \mathcal{H}_{M,\mu}$ is any subgroup, then we can compose the restriction map $\mathrm{H}_b^\bullet(\mathcal{H}_{M,\mu}) \to \mathrm{H}_b^\bullet(\mathcal{T}_M)$ with Γ_b and obtain the map

$$\Gamma_b(\mathcal{T}_M)\colon\ \mathrm{H}_b^\bullet(\pi_1(M)) \to \mathrm{H}_b^\bullet(\mathcal{T}_M).$$

Usually we abuse the notation and write Γ_b instead of $\Gamma_b(\mathcal{T}_M)$.

2.D. **Standard cohomology and exact bounded cohomology.** Let s be a section defined in Subsection 2.B and $\gamma = \gamma_s$. We show that for such γ, one can use the induction to define a map on the level of the ordinary cohomology. Note that induction in the case of the ordinary cohomology is a slightly more delicate operation then for bounded cohomology. Here the cocycles are not bounded and we need to show that the integral exists. In order to show integrability we need to choose carefully the section s. Let $c \in \mathrm{C}^n(\pi_M)$. We define

$$\mathrm{Ind}'(\gamma)(c)(g_0, g_1, \ldots, g_n) = \int_M c(\gamma(g_0, x), \gamma(g_1, x), \ldots, \gamma(g_n, x)) d\mu(x),$$

where $g_0, \cdots, g_n \in \mathcal{H}_{M,\mu}$.

Let us show, that $\mathrm{Ind}'(\gamma)$ defines a map on the level of ordinary cohomology which we call $\mathrm{Ind}'(\gamma)$. It follows from Proposition 1.1 that the function $x \to c(\gamma(g_0, x), \ldots, \gamma(g_n, x))$ is measurable. Integrability follows from Lemma 2.2, which shows that this function has essentially finite image. Every measurable function with essentially finite image is integrable. The fact that $d\,\mathrm{Ind}'(\gamma) = \mathrm{Ind}'(\gamma)d$ and that $\mathrm{Ind}'(\gamma)(c)$ is homogeneous follows immediately from Proposition 1.1. Thus for $\mathcal{T}_M \leqslant \mathcal{H}_{M,\mu}$ we have a well defined map

$$\Gamma(\mathcal{T}_M) := i^* \circ \mathrm{Ind}'(\gamma)\colon\ \mathrm{H}^\bullet(\pi_M) \to \mathrm{H}^\bullet(\mathcal{T}_M),$$

where $i^*\colon\ \mathrm{H}^\bullet(\mathcal{H}_{M,\mu}) \to \mathrm{H}^\bullet(\mathcal{T}_M)$ is induced by the inclusion.

Any cocycle γ can be extended to a cocycle

$$\gamma'\colon\ \mathrm{Homeo}_0(M) \times M \to \pi_M,$$

using the same formula as for γ, i.e., $\gamma'(g, x) = \delta(s^{-1}_{g(x)} \circ g \circ s_x)$. Let X be a measurable space and let $\gamma\colon X \to Y$ be a measurable function. We say that γ has **essentially finite image**, if there exist a full measure set $Z \subset X$, such that γ has a finite image on Z.

Lemma 2.2. *Let $f \in \mathrm{Homeo}_0(M)$. Then $\gamma'(f, \cdot)\colon M \to \pi_M$ has essentially finite image.*

Proof. Let $f \in \mathrm{Homeo}_0(M)$ and let Let $\{f_t\}$ be the isotopy between the identity and f. The union of the supports $\bigcup_{t \in [0,1]} \mathrm{supp}(f_t)$ is a compact subset of M. Recall that M admits a complete Riemannian metric. Hence there exists $r > 0$ such that the geodesic ball $B_r(z)$ of radius r centered at z contains $\bigcup_{t \in [0,1]} \mathrm{supp}(f_t)$. Note that for each x in the full measure subset of $M \backslash B_r(z)$ the element $\gamma'(f, x)$ is trivial in

π_M. Hence it is enough to show that the set $\{\gamma'(f,x)\}$ where x belongs to the full measure subset of $B_r(z)$ is finite in $\pi_1(B_r(z),z)$. Here, since we have choosen the isotopy $\{f_t\}$, we can consider $\gamma'(f,x)$ as an element of $\pi_1(M,z)$.

The group $\mathrm{Homeo}_0(M)$ admits a fragmentation property with respect to any open cover of M, see [12, Corollary 1.3]. Hence the ball $B_r(z)$ can be covered with finite number of balls B_i with the following property: f can be written as a product of homeomorphisms h_i such that the support of h_i lies in B_i. Since M is a smooth manifold, for each i there exits a smooth ball B_i', such that it is ϵ-close to B_i and such that it is ϵ-homotopic to B_i, see smooth approximation theorem [7, Theorem 2.11.8]. Note that $\gamma'(f,x)$ satisfies a cocycle condition. It means that

$$\gamma'(f,x) = \gamma'(h_1 \circ \ldots \circ h_n, x) = \gamma'(h_1, h_2 \circ \ldots \circ h_n(x)) \ldots \gamma'(h_n, x).$$

Hence it is enough to prove that the set $\{\gamma'(h_i,x)\}$ where x belongs to the full measure subset of B_i is finite in $\pi_1(B_r(z),z)$.

The ball B_i' is smooth, thus it has finite diameter d_i. The group of homeomorphisms of a ball is connected. Every path inside B_i can be free ϵ-homotoped to a path in B_i' and hence to a path whose Riemannian length is less than the diameter d_i. Thus $\gamma'(h_i,x)$ can be represented by a path whose Riemannian length is less than $d_i + 2(r_i + \epsilon)$, where r_i is a radius of a geodesic ball $B_{r_i}(z)$ which contains B_i. By Milnor-Svarc lemma [8] the word length of $\gamma'(h_i,x)$ is bounded in $\pi_1(B_r(z),z)$ and we are done. □

We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet(\pi_M) & \xrightarrow{\Gamma} & \mathrm{H}^\bullet(\mathcal{T}_M) \\ \uparrow & & \uparrow \\ \mathrm{H}_b^\bullet(\pi_M) & \xrightarrow{\Gamma_b} & \mathrm{H}_b^\bullet(\mathcal{T}_M) \end{array}$$

It follows that Γ_b restricts to the exact part of bounded cohomology.

$$\mathrm{E}\Gamma_b(\mathcal{T}_M)\colon\ \mathrm{EH}_b^\bullet(\pi_M) \to \mathrm{EH}_b^\bullet(\mathcal{T}_M).$$

Remark 2.3. $\mathrm{EH}_b^2(G)$ is the space of non-trivial homogeneous quasi-morphisms on G [10, Chapter 2], and $\mathrm{E}\Gamma_b(\mathcal{T}_M)\colon\ \mathrm{EH}_b^2(\pi_M) \to \mathrm{EH}_b^2(\mathcal{T}_M)$ is the map defined by Polterovich [23].

2.E. **The reduced bounded cohomology.** It is straightforward to see, that $\Gamma_b\colon \mathrm{H}^\bullet_b(\pi_1(M)) \to \mathrm{H}^\bullet_b(\mathcal{H}_M)$ is a contraction map. Hence it defines a well defined map on the reduced bounded and the reduced exact bounded cohomology.

$$\overline{\Gamma}_b(\mathcal{T}_M)\colon\ \overline{\mathrm{H}}^\bullet_b(\pi_1(M)) \to \overline{\mathrm{H}}^\bullet_b(\mathcal{T}_M),$$

$$\overline{\mathrm{E}\Gamma}_b(\mathcal{T}_M)\colon\ \overline{\mathrm{EH}}^\bullet_b(\pi_M) \to \overline{\mathrm{EH}}^\bullet_b(\mathcal{T}_M).$$

3. Proofs

Recall that M is a complete Riemannian manifold of finite volume, and $\mathcal{T}_M$ is either $\mathrm{Homeo}_0(M,\mu)$ or $\mathrm{Diff}_0(M,\mathrm{vol})$. If, in addition M is a symplectic manifold, then $\mathcal{T}_M$ could be also $\mathrm{Symp}_0(M,\omega)$. Throughout this section we consider the map $\overline{\mathrm{E}\Gamma}_b$ induced by $\gamma = \gamma_s$, where s is a section defined in Subsection 2.B.

Before we proceed, let us give an outline of the proofs. Assumptions of both Theorem A and Theorem B imply that there is an embedding $i\colon \mathrm{F}_2 \to \pi_M$ such that in both cases $i^*\colon \overline{\mathrm{EH}}^\bullet_b(\pi_M) \to \overline{\mathrm{EH}}^\bullet_b(\mathrm{F}_2)$ is onto. Indeed, in Theorem A it is straightforward, and in Theorem B we use the result in [13], which in particular, implies that if $\mathrm{F}_2 \times K$ is hyperbolically embedded in π_M, then one can extend a class in $\overline{\mathrm{EH}}^\bullet_b(\mathrm{F}_2 \times K)$ to $\overline{\mathrm{EH}}^\bullet_b(\pi_M)$. That is why we require $\mathrm{F}_2 \times K$ to be hyperbolically embedded. Given an element $c \in \overline{\mathrm{EH}}^\bullet_b(\pi_M)$, we look at the restriction $\overline{\mathrm{E}\Gamma}_b(c)_{|\,\mathrm{F}_2}$ to a carefully embedded $\mathrm{F}_2 \to \mathcal{T}_M$. The construction of the embedding is made such that there exists a non-zero real number Λ so that $\Lambda i^*(c)$ and $\overline{\mathrm{E}\Gamma}_b(c)_{|\,\mathrm{F}_2}$ are close in the norm. This implies that $\overline{\mathrm{E}\Gamma}_b(c)$ is non-trivial provided $i^*(c)$ is non-trivial.

Before showing the main technical lemma, we need to introduce some notations. Let $i\colon \mathrm{F}_2 \to \pi_M$ be an embedding. Denote by a and b the generators of F_2. A loop α in M which is based at z, represents in a natural way an element in π_M. If $\dim(M) = 2$ we assume that $i(a)$ and $i(b)$ can be represented by simple loops based at z.

In the next lemma we construct a family of maps $\rho_\epsilon\colon \mathrm{F}_2 \to \mathcal{T}_M$ such that the diagram

$$\begin{array}{ccc} \overline{\mathrm{EH}}^\bullet_b(\pi_M) & \xrightarrow{\overline{\mathrm{E}\Gamma}_b} & \overline{\mathrm{EH}}^\bullet_b(\mathcal{T}_M) \\ \downarrow{\scriptstyle i^*} & \swarrow{\scriptstyle \rho_\epsilon^*} & \\ \overline{\mathrm{EH}}^\bullet_b(\mathrm{F}_2) & & \end{array}$$

is commutative up to scaling and a small error controlled by ϵ.

Lemma 3.1. *Assume that M, $\mathcal{T}_M$ and $i\colon \mathrm{F}_2 \to \pi_M$ are as above. Then there exists a family of representations $\rho_\epsilon\colon \mathrm{F}_2 \to \mathcal{T}_M$, indexed by $\epsilon \in (0,1)$, that satisfy the following property: there exists a non-zero real number Λ, such that for every class $c \in \overline{\mathrm{EH}}_b^\bullet(\pi_M)$*

$$\|\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b(c) - \Lambda i^*(c)\| \xrightarrow{\epsilon \to 0} 0.$$

Proof. Let $m = \dim(M)$ and let $B^{m-1} \subset \mathbb{R}^{m-1}$ be the $m-1$ dimensional closed unit ball and let $S^1 = \mathbb{R}/\mathbb{Z}$. We fix $\epsilon \in (0,1)$ and define an isotopy $P_\epsilon^t \in \mathrm{Diff}(S^1 \times B^{m-1})$, $t \in [0,1]$ on $S^1 \times B^{m-1}$ by the following formula

$$P_\epsilon^t(\psi, x) = (\psi + tf(\|x\|), x),$$

where f is a smooth function such that $f(t) = 1$ for $t \leqslant 1 - \epsilon$ and $f(1) = 0$. We call P_ϵ^t the finger-pushing isotopy and P_ϵ^1 the finger-pushing map. Note that $P_\epsilon^0 = Id$ and that P_ϵ^1 fixes point-wise the boundary of $S^1 \times B^{m-1}$ and fixes all points (ψ, x) for which $\|x\| \leqslant 1-\epsilon$.

Let g_0 be the Riemannian metric which is the product of the standard euclidean Riemannian metrics on B^{m-1} and S^1. Then g_0 is a Riemannian metric on $S^1 \times B^{m-1}$ that, by Fubini's theorem, is preserved by the maps P_ϵ^t for every $t \in [0,1]$ and every $\epsilon \in (0,1)$. In case $m = 2k$, we construct in a similar way finger-pushing isotopy $P_\epsilon^t \in \mathrm{Diff}(S^1 \times B^1 \times B^{2k-2})$, $t \in [0,1]$ which preserves the standard symplectic form $dx \wedge dy + \sum_{i=1}^{k-1} dp_i \wedge dq_i$ on $S^1 \times B^1 \times B^{2k-2}$. The precise construction is presented in [2, proof of Theorem 1.3].

Recall that a, b are generators of F_2. We represent $i(a)$ and $i(b)$ by loops α an β in M based at z that are embedded and intersect only at z. Note that if $m = 2$ we assumed that this is possible, and if $m > 2$ then any two elements of π_M can be represented in this way. Let $N(\alpha)$ be a closed tubular neighborhood of α and let $P_\epsilon^t(\alpha)$ be the isotopy defined by pulling-back P_ϵ^t via diffeomorphism $n_\alpha\colon N(\alpha) \to S^1 \times B^{m-1}$ and extending by the identity outside $N(\alpha)$. If $\mathcal{T}_M = \mathrm{Homeo}_0(M, \mu)$ or $\mathrm{Diff}_0(M, \mathrm{vol})$, then by Moser trick n_α is chosen such that $P_\epsilon^t(\alpha)$ preserves the vol form, and if $\mathcal{T}_M = \mathrm{Symp}_0(M, \omega)$, then by Moser trick n_α is chosen such that $P_\epsilon^t(\alpha)$ preserves the symplectic form ω. Let

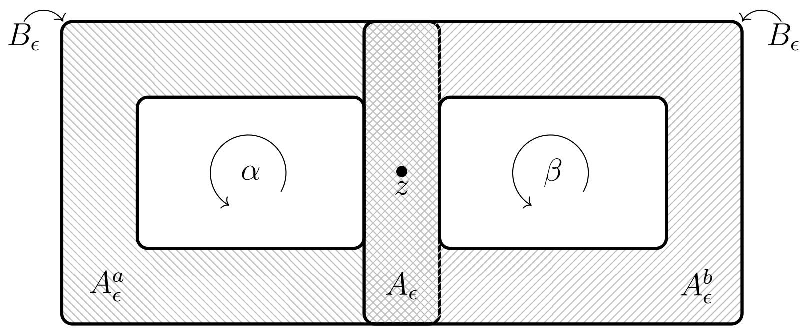
$$A_\epsilon(\alpha) = n_\alpha^{-1}(\{(\psi, x) \mid \|x\| \leqslant 1 - \epsilon\}), \qquad B_\epsilon(\alpha) = N(\alpha) - A_\epsilon(\alpha).$$

Note that $P_\epsilon^1(\alpha)$ fixes point-wise $A_\epsilon(\alpha)$. Similarly we define $P_\epsilon^t(\beta)$, $A_\epsilon(\beta)$, $B_\epsilon(\beta)$. The representation $\rho_\epsilon\colon \mathrm{F}_2 \to \mathcal{T}_M$ is given by:

$$\rho_\epsilon(a) = P_\epsilon^1(\alpha), \qquad \rho_\epsilon(b) = P_\epsilon^1(\beta).$$

Now we show that $\|\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b([c]) - \Lambda i^*([c])\| \xrightarrow{\epsilon\to 0} 0$ for certain non-zero real number Λ and every $[c] \in \overline{\mathrm{EH}}_b^\bullet(\pi_M)$. To simplify the notation, we identify F_2 with its image $i(\mathrm{F}_2)$.

First we investigate the values of γ on elements of the form $\rho_\epsilon(w)$, where $w \in \mathrm{F}_2$. Let $h_a\colon \mathrm{F}_2 \to \langle a\rangle$ be the retraction onto the subgroup generated by a that sends b to the trivial element. We define analogously $h_b\colon \mathrm{F}_2 \to \langle b\rangle$.



From the description of γ in Subsection 2.B, we see that if x belongs to the set $A_\epsilon := A_\epsilon(\alpha)\cap A_\epsilon(\beta)$, then $\gamma(\rho_\epsilon(w), x)$ is conjugated to w. In the same way, if $x \in A_\epsilon^a := A_\epsilon(\alpha) - N(\beta)$, then $\gamma(\rho_\epsilon(w), x)$ is conjugated to $h_a(w)$ and similarly, if $x \in A_\epsilon^b := A_\epsilon(\beta) - N(\alpha)$, then $\gamma(\rho_\epsilon(w), x)$ is conjugated to $h_b(w)$. If $x \in B_\epsilon := B_\epsilon(\alpha) \cup B_\epsilon(\beta)$, then we do not have much control over the loops we get, but this case is negligible if ϵ is small. To sum up, we have:

$$\gamma(\rho_\epsilon(w), x) = \begin{cases} e & x \in M - (N(\alpha)\cup N(\beta)),\\ u_x w u_x^{-1} & x \in A_\epsilon = A_\epsilon(\alpha)\cap A_\epsilon(\beta),\\ u_{a,x} h_a(w) u_{a,x}^{-1} & x \in A_\epsilon^a = A_\epsilon(\alpha) - N(\beta),\\ u_{b,x} h_b(w) u_{b,x}^{-1} & x \in A_\epsilon^b = A_\epsilon(\beta) - N(\alpha),\\ ? & x \in B_\epsilon = B_\epsilon(\alpha)\cup B_\epsilon(\beta),\end{cases}$$

for some $u_x, u_{a,x}, u_{b,x} \in \pi_M$. Let n be a natural number and $[c] \in \overline{\mathrm{EH}}_b^n(\pi_M)$. Without loss of generality, we can assume that $c(e,\dots,e) = 0$. Denote by $\bar g = (g_0, g_1, \dots) \in \mathcal{T}_M^n$. We set $\gamma(\bar g, x) = (\gamma(g_0,x), \gamma(g_1,x),\dots)$. Let $\overline{w} \in \mathrm{F}_2^n$. We have:

$$\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b(c)(\overline{w}) = \overline{\mathrm{E}\Gamma}_b(c)(\rho_\epsilon(\overline{w})) = \int_M c(\gamma(\rho_\epsilon(\overline{w}), x))d\mu(x).$$

Denote by $u.c(\overline{w}) = c(u\overline{w}u^{-1})$, where $u \in \pi_M$. We compute separately the integral on subsets A_ϵ, A_ϵ^a, A_ϵ^b and B_ϵ:

$$\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b(c)(\overline{w}) = \int_{A_\epsilon} u_x.c(\overline{w}) d\mu(x) + \int_{A_\epsilon^a} u_{a,x}.c(h_a(\overline{w}))d\mu(x) + \int_{A_\epsilon^b} u_{b,x}.c(h_b(\overline{w}))d\mu(x) + \int_{B_\epsilon} c(\gamma(\rho_\epsilon(\overline{w}), x))d\mu(x).$$

Recall that conjugation acts trivially on the cohomology. Thus we have $[u.c] = [c]$. Both $\mathrm{Z}_b^n(G) = \ker(d_n)$ and $\overline{\mathrm{H}}_b^n(G)$ are Banach spaces and $[\cdot]\colon \mathrm{Z}_b^n(G) \to \overline{\mathrm{H}}_b^n(G)$ is a continuous linear map. Thus we have:

$$\begin{aligned}[\int_{A_\epsilon} u_x.c(\overline{w})d\mu(x)] &= [\sum_{u \in \pi_M} \mu(\{x \in A_\epsilon \mid u_x = u\}) u.c(\overline{w})] \\ &= \sum_{u\in\pi_M} \mu(\{x \in A_\epsilon \mid u_x = u\}) i^*[u.c] = \mu(A_\epsilon) i^*([c]).\end{aligned}$$

Let $u.c_{|a}$ be the restriction of $u.c$ to the subgroup generated by the generator a. The function $\overline{w} \to c(uh_a(\overline{w})u^{-1})$ is equal to the pull-back of the cocycle $u.c_{|a}$, namely:

$$c(uh_a(\overline{w})u^{-1}) = h_a^*(u.c_{|a})(\overline{w}).$$

Moreover, since $\overline{\mathrm{EH}}_b^n(\mathbb{Z})$ is trivial, then $\overline{w} \to c(uh_a(\overline{w})u^{-1})$ defines the trivial class in $\overline{\mathrm{EH}}_b^n(\mathrm{F}_2)$. It follows, that

$$[\int_{A_\epsilon^a} u_{a,x}.c(h_a(\overline{w}))d\mu(x)] = \sum_{u\in\pi_M} \mu(\{x \in A_\epsilon \mid u_{a,x} = u\}) h_a^*([u.c_{|a}]) = 0.$$

The same holds for the integral over A_ϵ^b. We denote

$$c_{res}^\epsilon(\overline{w}) = \int_{B_\epsilon} c(\gamma(\rho_\epsilon(\overline{w}), x))d\mu(x).$$

Note that c_{res}^ϵ is a cocycle on F_2. Now we can write:

$$\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b([c]) = \mu(A_\epsilon) i^*([c]) + [c_{res}^\epsilon].$$

We have that

$$\|[c_{res}^\epsilon]\| \leqslant \mu(B_\epsilon) \|c\|_{sup}.$$

Moreover, $\mu(A_\epsilon) \xrightarrow{\epsilon \to 0} \mu(N_a \cap N_b) \neq 0$ and $\mu(B_\epsilon) \xrightarrow{\epsilon\to 0} 0$. It follows that:

$$\|\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b([c]) - \mu(N_a \cap N_b) i^*([c])\| \leqslant [\mu(A_\epsilon) - \mu(N_a \cap N_b)] \|i^*([c])\| + \mu(B_\epsilon)\|c\|_{sup}.$$

Hence $\|\rho_\epsilon^* \overline{\mathrm{E}\Gamma}_b([c]) - \mu(N_a \cap N_b) i^*([c])\| \xrightarrow{\epsilon\to 0} 0$. $\square$

Remark 3.2. In what follows we apply Lemma 3.1 for injective i. However, Lemma 3.1 holds also for i which is not an embedding. Injectivity of i was used only to simplify the notation, when we identified F_2 with its image $i(\mathrm{F}_2)$.

Now we prove Theorem A and Theorem B.

Theorem A. *Suppose that* $p\colon \pi_M \to \mathrm{F}_2$ *is a surjective homomorphism. Then there is an injective homomorphism*

$$\overline{\mathrm{EH}}^\bullet_b(\mathrm{F}_2) \hookrightarrow \overline{\mathrm{EH}}^\bullet_b(\mathcal{T}_M).$$

Proof. Let $m = \dim(M)$. If $m > 3$, then we take $i\colon \mathrm{F}_2 \to \pi_M$ to be any section of p. If $m = 2$, then it is easy to find two embedded loops based at z and intersecting only at z, such that they generate F_2 and there is a retraction $\pi_1(M) \to \mathrm{F}_2$. Instead of the original p, we use this retraction which from now on we call p. Let i be a section of this new p. We show that $\overline{\mathrm{E}\Gamma}_b \circ p^*$ is an embedding.

The section i satisfies the assumptions of Lemma 3.1. Let $\{\rho_\epsilon\}$ be the family of representations from Lemma 3.1. We have a diagram

$$\begin{array}{ccc} \overline{\mathrm{EH}}^\bullet_b(\pi_M) & \xrightarrow{\overline{\mathrm{E}\Gamma}_b} & \overline{\mathrm{EH}}^\bullet_b(\mathcal{T}_M) \\ {\scriptstyle p^*}\uparrow\downarrow{\scriptstyle i^*} & \swarrow{\scriptstyle \rho^*_\epsilon} & \\ \overline{\mathrm{EH}}^\bullet_b(\mathrm{F}_2) & & \end{array}$$

Note that $i^* \circ p^* = id$. Suppose that $d \in \overline{\mathrm{EH}}^\bullet_b(\mathrm{F}_2)$ is a non-trivial class. In the reduced cohomology it means that $\|d\| > 0$. Let $c = p^*(d)$. Then we have that $\|i^*(c)\| = \|d\| > 0$. Since

$$\|\rho^*_\epsilon \overline{\mathrm{E}\Gamma}_b(c) - \Lambda i^*(c)\| \xrightarrow{\epsilon \to 0} 0,$$

then for some small ϵ we have $\|\rho^*_\epsilon \overline{\mathrm{E}\Gamma}_b(c)\| > 0$. It follows that

$$\overline{\mathrm{E}\Gamma}_b(c) = \overline{\mathrm{E}\Gamma}_b(p^*(d)) \neq 0.$$

Thus $\overline{\mathrm{E}\Gamma}_b \circ p^*$ is an embedding. □

Remark 3.3. A typical class in $\overline{\mathrm{EH}}^3_b(\mathrm{F}_2)$ is defined by choosing an isometric action ρ of F_2 on the 3-dimensional hyperbolic space $\mathbf{H}^3$ and defining a cocycle $\mathrm{vol}_\rho(a_1, \ldots, a_4)$ to be the signed volume of the geodesic simplex $\Delta(\rho(a_1)x, \ldots, \rho(a_4)x)$, where $x \in \mathbf{H}^3$. For certain ρ, classes defined by vol_ρ have a positive norm, see [28]. The classes in $\overline{\mathrm{EH}}^3_b(\mathcal{T}_M)$ which are constructed in Theorem A have similar geometrical interpretation. Namely, the value of $\overline{\mathrm{E}\Gamma}_b(p^*(\mathrm{vol}_\rho))(f_1,\ldots,f_4)$ is the average value of signed volumes of $\Delta(p\gamma(f_1,x),\ldots,p\gamma(f_4,x))$ over M. Note that since every $\gamma(f_i,x)$ takes essentially finitely many values, then this average is actually a finite sum of weighted signed volumes of certain simplices in $\mathbf{H}^3$. Here the signed volume equals $\mathrm{vol}(\Delta)$ if Δ has the same orientation as $\mathbf{H}^3$ and $-\mathrm{vol}(\Delta)$ otherwise.

Theorem B. *Suppose $j\colon \mathrm{F}_2\times K\to\pi_M$ is a hyperbolic embedding, where K is finite. Then*

$$\dim\overline{\mathrm{EH}}_b^\bullet(\mathcal{T}_M)\geqslant\dim\overline{\mathrm{EH}}_b^\bullet(\mathrm{F}_2).$$

Proof. It is easy to see, that if F_2 embeds in π_M when $\dim(M)=2$, then one can find a retraction $\pi_M\to\mathrm{F}_2$. Thus if $\dim(M)=2$, the statement follows from Theorem A.

Let us assume, that $\dim(M)>2$. Let $i\colon\mathrm{F}_2\to\pi_M$ be j restricted to $\mathrm{F}_2\times\{e\}$. Since $\dim(M)>2$, $i(a)$ and $i(b)$ can be represented by based loops that are disjoint everywhere except the base-point z. Let $\{\rho_\epsilon\}_\epsilon$ be a family of maps $\rho_\epsilon\colon\mathrm{F}_2\to\mathcal{T}_M$ constructed in Lemma 3.1. We have

$$\begin{array}{ccc}
\overline{\mathrm{EH}}_b^\bullet(\pi_M) & \xrightarrow{\overline{\mathrm{E}\Gamma}_b} & \overline{\mathrm{EH}}_b^\bullet(\mathcal{T}_M)\\
\big\downarrow{\scriptstyle i^*} & \swarrow{\scriptstyle \rho_\epsilon^*} & \\
\overline{\mathrm{EH}}^\bullet(\mathrm{F}_2) & &
\end{array}$$

First we show that $\ker(\overline{\mathrm{E}\Gamma}_b)\subset\ker(i^*)$. There is a non-zero real number Λ such that for every $c\in\overline{\mathrm{EH}}_b^\bullet(\pi_1(M))$ we have

$$\|\rho_\epsilon^*\overline{\mathrm{E}\Gamma}_b(c)-\Lambda i^*(c)\|\xrightarrow{\epsilon\to0}0.$$

Let $c\in\overline{\mathrm{EH}}_b^\bullet(\pi_1(M))$ be such that $\overline{\mathrm{E}\Gamma}_b(c)=0$. Then

$$\|\Lambda i^*(c)\|=\|\rho_\epsilon^*\overline{\mathrm{E}\Gamma}_b(c)-\Lambda i^*(c)\|\xrightarrow{\epsilon\to0}0.$$

Hence $\|i^*(c)\|=0$ and $c\in\ker(i^*)$. Thus $\ker(\overline{\mathrm{E}\Gamma}_b)\subset\ker(i^*)$. It follows that

$$\dim\overline{\mathrm{EH}}_b^\bullet(\mathcal{T}_M)\geqslant\dim(\overline{\mathrm{EH}}_b^\bullet(\pi_1(M))/\ker(\overline{\mathrm{E}\Gamma}_b))\geqslant\dim(\overline{\mathrm{EH}}_b^\bullet(\pi_1(M))/\ker(i^*)).$$

In [13] it is shown, that if $j\colon\mathrm{F}_2\times K\to\pi_M$ is a hyperbolic embedding, then the map $j^*\colon\overline{\mathrm{EH}}_b^\bullet(\pi_M)\to\overline{\mathrm{EH}}_b^\bullet(\mathrm{F}_2\times K)$ is onto. Using identification $\overline{\mathrm{EH}}_b^\bullet(\mathrm{F}_2\times K)=\overline{\mathrm{EH}}_b^\bullet(\mathrm{F}_2)$, we can write that $i^*=j^*$. Thus i^* is onto and $\overline{\mathrm{EH}}_b^\bullet(\pi_1(M))/\ker(i^*)=\overline{\mathrm{EH}}_b^\bullet(\mathrm{F}_2)$. □

4. Questions and final remarks

Remark 4.1. The version of Theorem A and Theorem B where $\overline{\mathrm{EH}}^\bullet_b$ is substituted with $\overline{\mathrm{H}}^\bullet_b$ holds in a more general setting. Namely, it holds for a topological manifold M equipped with a regular finite Borel measure μ which is positive on open sets and zero on nowhere dense sets, and $\mathcal{T}_M$ is the group of isotopic to the identity, not necessary compactly supported measure-preserving homeomorphisms of M. Similarly, $\mathcal{T}_M$ can be the identity component of the group of volume-preserving diffeomorphisms or symplectomorphisms.

Remark 4.2. Note that $\mathrm{E}\Gamma_b\colon \mathrm{EH}^\bullet_b(\pi_1(M)) \to \mathrm{EH}^\bullet_b(\mathrm{Diff}_0(M, \mathrm{vol}))$ does not factor through $\mathrm{Diff}_0(M)$ (similar argument works for homeomorphisms). The reason is that if it is factored, then one can construct non-trivial homogeneous quasimorphisms on $\mathrm{Diff}_0(M)$, and it is known, that for many M, the group $\mathrm{Diff}_0(M)$ does not admit such quasimorphisms. More precisely, let M be such manifold, for example a closed connected hyperbolic 3-dimensional manifold. Recall that $\mathrm{EH}^2_b(G)$ is the space of non-trivial homogeneous quasimorphisms on G. It follows from [9, Theorem 1.11] that $\mathrm{EH}^2_b(\mathrm{Diff}_0(M)) = 0$. Since $\mathrm{EH}^2_b(\pi_1(M)) \neq 0$, it is easy to see, that $\mathrm{E}\Gamma_b\colon \mathrm{EH}^2_b(\pi_1(M)) \to \mathrm{EH}^2_b(\mathrm{Diff}_0(M, \mu))$ is an embedding (see ,e.g., the proof for surfaces [5, Theorem 2.5]), and hence cannot factor through the trivial group.

Remark 4.3. The assumption that homeomorphisms we deal with are isotopic to the identity can be dropped for the price of substituting the group $\pi_1(M)$ with a mapping class group (such approach was used in [5] for surfaces). Indeed, let $\delta^{ext}\colon \mathrm{Homeo}(M, z) \to \mathrm{MCG}(M, z)$ be the quotient map, where $\mathrm{MCG}(M, z) = \pi_0(\mathrm{Homeo}(M, z))$ and $\mathrm{Homeo}(M, z)$ is the group of homeomorphisms of M fixing z. Consider a cocycle $\gamma^{ext}\colon \mathrm{Homeo}(M, \mu) \times M \to \mathrm{MCG}(M, z)$, given by the following formula $\gamma^{ext}(g, x) = \delta^{ext}(s^{-1}_{g(x)} \circ g \circ s_x)$. This cocycle induces the following map $\Gamma^{ext}_b = \mathrm{Ind}_b(\gamma^{ext})\colon \mathrm{H}^\bullet_b(\mathrm{MCG}(M, z)) \to \mathrm{H}^\bullet_b(\mathrm{Homeo}(M, \mu))$. The problem with this approach is that almost nothing is known about bounded cohomology of mapping class groups of manifolds of dimension greater than 2.

One source of non-trivial classes in $\mathrm{H}^n_b(\pi_1(M))$ is the following construction. Let M^n be a compact Riemannian manifold with negative sectional curvature and let $\pi_1(M)$ act by deck-transformations on $\widetilde{M}$. It is known, that there is a common bound for volumes of geodesic simplices in $\widetilde{M}$. In the same spirit as in Remark 3.3, one can define a non-trivial class $[\mathrm{vol}_M] \in \mathrm{H}_b^n(\pi_1(M))$.

Question 4.4. Is the class $\Gamma_b([\mathrm{vol}_M]) \in \mathrm{H}_b^n(\mathrm{Homeo}_0(M,\mu))$ non-trivial?

A similiar, but in a sense more general question is for the ordinary cohomology. Let M^n be a closed Riemannian manifold such that $\pi_1(M)$ has a trivial center. Then $\pi_1(M) = \pi_M$ and we have a map $\Gamma\colon \mathrm{H}^n(\pi_1(M)) \to \mathrm{H}^n(\mathrm{Homeo}_0(M,\mu))$. Let $f\colon M \to B\pi_1(M)$ be a map classifying the universal cover, i.e., a map that induces an isomorphism $\pi_1(f)\colon \pi_1(M) \to \pi_1(B\pi_1(M))$. Let $[M] \in \mathrm{H}^n(M)$ be the fundamental class, then $f^*[M] \in \mathrm{H}^n(B\pi_1(M)) = \mathrm{H}^n(\pi_1(M))$.

Question 4.5. Is the class $\Gamma(f^*[M]) \in \mathrm{H}^n(\mathrm{Homeo}_0(M,\mu))$ non-trivial?

References

[1] Michael Brandenbursky. Bi-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces. *Internat. J. Math.*, 26(9):1550066, 29 pages, 2015.

[2] Michael Brandenbursky and Jarek Kędra. Fragmentation norm and relative quasimorphisms. *Arxiv:1804.06592*.

[3] Michael Brandenbursky and Jarek Kędra. On the autonomous metric on the group of area-preserving diffeomorphisms of the 2-disc. *Algebr. Geom. Topol.*, 13(2):795–816, 2013.

[4] Michael Brandenbursky, Jarek Kędra, and Egor Shelukhin. On the autonomous norm on the group of Hamiltonian diffeomorphisms of the torus. *Commun. Contemp. Math.*, 20(2):1750042, 27, 2018.

[5] Michael Brandenbursky and Michal Marcinkowski. Entropy and quasimorphisms. *arXiv:1707.06020*.

[6] Michael Brandenbursky and Egor Shelukhin. On the L^p-geometry of autonomous Hamiltonian diffeomorphisms of surfaces. *Math. Res. Lett.*, 22(5):1275–1294, 2015.

[7] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[8] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[9] Dmitri Burago, Sergei Ivanov, and Leonid Polterovich. Conjugation-invariant norms on groups of geometric origin. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 221–250. Math. Soc. Japan, Tokyo, 2008.

[10] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[11] F. Dahmani, V. Guirardel, and D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.*, 245(1156):v+152, 2017.

[12] Robert D. Edwards and Robion C. Kirby. Deformations of spaces of imbeddings. *Ann. Math. (2)*, 93:63–88, 1971.

[13] R. Frigerio, M. B. Pozzetti, and A. Sisto. Extending higher-dimensional quasi-cocycles. *J. Topol.*, 8(4):1123–1155, 2015.

[14] Jean-Marc Gambaudo and Étienne Ghys. Commutators and diffeomorphisms of surfaces. *Ergodic Theory Dynam. Systems*, 24(5):1591–1617, 2004.

[15] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[16] A. Guichardet. *Cohomologie des groupes topologiques et des algèbres de Lie*, volume 2 of *Textes Mathématiques [Mathematical Texts]*. CEDIC, Paris, 1980.

[17] Jin-ichi Itoh and Minoru Tanaka. The dimension of a cut locus on a smooth Riemannian manifold. *Tohoku Math. J. (2)*, 50(4):571–575, 1998.

[18] Maxim Kontsevich. Rozansky-Witten invariants via formal geometry. *Compositio Math.*, 115(1):115–127, 1999.

[19] D. Kotschick and S. Morita. Signatures of foliated surface bundles and the symplectomorphism groups of surfaces. *Topology*, 44(1):131–149, 2005.

[20] D. Kotschick and S. Morita. Characteristic classes of foliated surface bundles with area-preserving holonomy. *J. Differential Geom.*, 75(2):273–302, 2007.

[21] Sam Nariman. On the moduli space of flat symplectic surface bundles. *to appear in Journal of Differential Geometry.*

[22] D. Osin. Acylindrically hyperbolic groups. *Trans. Amer. Math. Soc.*, 368(2):851–888, 2016.

[23] Leonid Polterovich. Floer homology, dynamics and groups. In *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 417–438. Springer, Dordrecht, 2006.

[24] Alexander Reznikov. Continuous cohomology of the group of volume-preserving and symplectic diffeomorphisms, measurable transfer and higher asymptotic cycles. *Selecta Math. (N.S.)*, 5(1):181–198, 1999.

[25] Alexander G. Reznikov. Characteristic classes in symplectic topology. *Selecta Math. (N.S.)*, 3(4):601–642, 1997. Appendix D by Ludmil Katzarkov.

[26] R. Sauer. Homological invariants and quasi-isometry. *Geom. Funct. Anal.*, 16(2):476–515, 2006.

[27] Yehuda Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.*, 192(2):119–185, 2004.

[28] T. Soma. Bounded cohomology and topologically tame Kleinian groups. *Duke Math. J.*, 88(2):357–370, 1997.

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