

BOUNDED COHOMOLOGY OF TRANSFORMATION GROUPS IN ALL DEGREES

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ABSTRACT. Let M be a compact connected oriented smooth manifold, of dimension $n \geq 2$. Let $\mathcal{T}_M$ be one of the following transformation groups: $\text{Homeo}_0(M, \mu)$, $\text{Diff}_0(M, \mu)$, $\text{Symp}_0(M, \omega)$ (in case M is symplectic) and $\text{Ham}(M, \omega)$ (in case M is symplectic). Denote by $\overline{H}_b^d(\mathcal{T}_M)$ reduced bounded cohomology of $\mathcal{T}_M$ in degree d , and by $\overline{EH}_b^d(\mathcal{T}_M)$ reduced exact bounded cohomology of $\mathcal{T}_M$ in degree d . In this paper we prove that if $n = 2$ and M is a closed surface Σ_g or a disc $\mathbb{D}$, then for every $d \geq 2$

$$\dim(\overline{H}_b^d(\mathcal{T}_M)) = \infty.$$

Moreover, $\dim \overline{EH}_b^d(\mathcal{T}_M) = \infty$ if $\mathcal{T}_M$ is either $\text{Diff}_0(\Sigma_g, \mu)$ or $\text{Ham}(\Sigma_g, \omega)$, or $\text{Ham}(\mathbb{D}, \omega)$; or $g > 1$. In case $n > 2$, under certain conditions on $\pi_1(M)$, we prove that for every degree $d \geq 2$

$$\dim \overline{EH}_b^d(\mathcal{T}_M) = \infty.$$

In particular, these results hold for $\mathcal{T}_M$ when M is a closed, orientable smooth manifold admitting a Riemannian metric of strictly negative sectional curvature of an arbitrary dimension.

1. INTRODUCTION

Gromov's work [26] established bounded cohomology as an important tool in geometry and topology. Its applications include simplicial volume, group actions, and symplectic geometry. Nevertheless, higher-degree bounded cohomology remains difficult to compute, even for free groups; see [21].

For transformation groups without a measure-preserving condition, Monod–Nariman [34] obtained complete computations and vanishing results. In particular, their Theorem 1.10 gives

$$H_b^q(\text{Homeo}_0(S^2); \mathbb{R}) = H_b^q(\text{Diff}_0(S^2); \mathbb{R}) = 0 \quad (q = 2, 3).$$

The area-preserving subgroups considered here exhibit a different behaviour: their reduced bounded cohomology is infinite-dimensional in every degree at least two.

In the measure-preserving setting, the constructions of Gambaudo–Ghys [24] and Polterovich [38] provide quasimorphisms from braid groups and

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fundamental groups. Such quasimorphisms have applications to conjugation-invariant norms and dynamics; see, for example, [9, 28]. Brandenbursky–Marcinkowski [10] constructed transfer maps in bounded cohomology and obtained infinite-dimensional third reduced exact bounded cohomology for several transformation groups, including those of hyperbolic surfaces. Kimura [29] established analogous configuration-space results for the disc and other surfaces of non-negative Euler characteristic. Nitsche [35, Theorems 4.2–4.3] obtained infinite-dimensionality in every positive even degree for volume-preserving homeomorphism groups of certain manifolds of dimension at least three and for area-preserving diffeomorphisms of the disc.

In this paper we obtain an all-degree conclusion for the area-preserving transformation groups of every closed oriented surface and of the disc. We also prove an all-degree result for higher-dimensional manifolds under the group-theoretic hypotheses stated below.

A principal difficulty in obtaining such results comes from the very limited understanding of the higher-degree bounded cohomology of free groups. Under suitable hypotheses on the fundamental group, Brandenbursky–Marcinkowski [10, Theorems A and B] proved that the reduced exact bounded cohomology of the transformation groups considered there contains a linear copy of $\overline{H}_b^d(F_2)$ for every $d \geq 2$. Thus, infinite-dimensionality of $\overline{H}_b^d(F_2)$ would immediately give the corresponding conclusion for those transformation groups in the same degree. However, understanding the bounded cohomology of F_2 in degrees greater than three remains a fundamental open problem; in particular, infinite-dimensionality of its reduced bounded cohomology in these degrees is not known. The degree-preserving transfer alone therefore does not yield the desired all-degree conclusion.

Our construction avoids this difficulty by using symmetric external products of degree-two and degree-three classes on products of free groups, rather than requiring higher-degree classes on F_2 itself. We transfer these classes by integration over products of manifolds and configuration spaces. Suitable restriction maps recover the external products and establish injectivity on the chosen spaces of classes. Since every integer $d \geq 2$ has the form $d = 2m + q$, with $m \geq 0$ and $q \in \{2, 3\}$, the known infinite-dimensionality in degrees two and three [12, 40] suffices to obtain our all-degree results.

Let M be a compact connected oriented smooth manifold of dimension $n \geq 2$ with possibly nonempty boundary. Let μ be a Borel probability measure induced by a smooth positive volume form on M . Let $\text{Homeo}_0(M, \mu)$ denote the identity component of the group of measure-preserving homeomorphisms of M which are the identity on a neighborhood of ∂M , and $\text{Diff}_0(M, \mu)$ denote the identity component of the group of volume-preserving diffeomorphisms M . If (M, ω) is symplectic, let $\text{Symp}_0(M, \omega)$ denote the identity component of the group of symplectomorphisms that are the identity near ∂M . Let $\text{Ham}(M, \omega)$ denote the group of Hamiltonian diffeomorphisms generated by Hamiltonian

isotopies; when the boundary is nonempty, the generating Hamiltonians have support in $\text{int } M$.

Let Σ_g denote the closed oriented surface of genus g , and fix $x_0 \in M$. We use the notation $\pi = \pi_1(M, x_0)$, $G = \pi_M = \pi/Z(\pi)$, $F_2 = \langle a, b \rangle$, where $Z(\pi)$ is the center of π and F_2 is the free group of rank two. As in [10], we use $\mathcal{T}_M$ for any one of the transformation groups under consideration. Theorem 1 concerns $n \geq 3$ under the hypotheses below. Theorem 2 concerns the disc and the sphere, and Theorem 3 concerns all closed oriented surfaces of positive genus.

Let K be a discrete group. We denote its bounded cohomology by $H_b^d(K)$ and its reduced bounded cohomology by

$$\overline{H}_b^d(K) = H_b^d(K) / \{u \in H_b^d(K) : \|u\| = 0\},$$

with the induced quotient norm. Let c_K^d denote the comparison map induced by inclusion of bounded cochains into ordinary cochains. Following [22], let $EH_b^d(K)$ denote its kernel, called the exact bounded cohomology of K . We write $\overline{EH}_b^d(K)$ for the reduced exact bounded cohomology:

$$\begin{aligned} EH_b^d(K) &= \ker(c_K^d : H_b^d(K) \longrightarrow H^d(K; \mathbb{R})), \\ \overline{EH}_b^d(K) &= \text{im}(EH_b^d(K) \longrightarrow \overline{H}_b^d(K)). \end{aligned}$$

Thus $\overline{EH}_b^d(K)$ is $EH_b^d(K)$ modulo its zero-seminorm subspace, with the norm induced from $\overline{H}_b^d(K)$. Consider the following conditions on $G = \pi/Z(\pi)$:

- (i) There are a group B , an amenable group A , and injective homomorphisms $A \rightarrow F_2$ and $A \rightarrow B$ such that $G \cong F_2 *_A B$.
- (ii) There is an epimorphism $r : G \twoheadrightarrow F_2$.
- (iii) The group G contains a hyperbolically embedded subgroup isomorphic to F_2 .

A surjection $\pi \twoheadrightarrow F_2$ sends $Z(\pi)$ to the identity, so (ii) is equivalent to Nitsche's epimorphism hypothesis on π itself.

Theorem 1 (Manifolds of dimension greater than two). *Let M be a compact, connected, oriented smooth manifold of dimension $n \geq 3$, with normalized smooth volume measure μ . Suppose that $G = \pi_1(M)/Z(\pi_1(M))$ satisfies at least one of (i), (ii), or (iii). Let $\mathcal{T}_M$ denote either of*

$$\text{Homeo}_0(M, \mu), \quad \text{Diff}_0(M, \mu).$$

If M has a symplectic form ω , one may instead take either of

$$\text{Symp}_0(M, \omega), \quad \text{Ham}(M, \omega),$$

with μ the normalized symplectic volume.

(a) **Infinite dimensionality.** For every integer $d \geq 2$,

$$(1) \quad \dim_{\mathbb{R}} \overline{EH}_b^d(\mathcal{T}_M) = \infty, \quad \dim_{\mathbb{R}} \overline{H}_b^d(\mathcal{T}_M) = \infty.$$

(b) **Injective maps.** There are a linear subspace $K_2 \subset \overline{H}_b^2(F_2)$ of codimension at most one and injective linear maps

$$(2) \quad \Phi_{0,2} : K_2 \longrightarrow \overline{EH}_b^2(\mathcal{T}_M),$$

$$\Phi_{0,3} : \overline{H}_b^3(F_2) \longrightarrow \overline{EH}_b^3(\mathcal{T}_M),$$

$$(3) \quad \Phi_{m,q} : \overline{H}_b^q(F_2) \longrightarrow \overline{EH}_b^{2m+q}(\mathcal{T}_M) \quad (m \geq 1, q \in \{2, 3\}).$$

Under (ii) or (iii), one may take $K_2 = \overline{H}_b^2(F_2)$.

Let M be a closed, connected, orientable smooth manifold of dimension $n \geq 3$ admitting a Riemannian metric of strictly negative sectional curvature. Then $\pi_1(M)$ is center-free and non-elementary torsion-free word-hyperbolic group, and hence it admits a hyperbolically embedded F_2 [15, Theorem 6.14(c)]. Now Theorem 1 implies the following

Corollary. Let M be a closed, connected, orientable smooth manifold of dimension $n \geq 3$ admitting a Riemannian metric of strictly negative sectional curvature. Let μ be the normalized measure induced by any smooth positive volume form. Then, for every $d \geq 2$,

$$\dim \overline{EH}_b^d(\text{Homeo}_0(M, \mu)) = \infty \quad \text{and} \quad \dim \overline{EH}_b^d(\text{Diff}_0(M, \mu)) = \infty.$$

In particular, their reduced bounded cohomology is infinite-dimensional in every degree at least two. If M also admits a symplectic form ω , the same reduced exact conclusions hold for

$$\text{Symp}_0(M, \omega) \quad \text{and} \quad \text{Ham}(M, \omega),$$

using the normalized symplectic volume.

The second theorem treats the closed two-disc $\mathbb{D}$ and the sphere S^2 , using three-point and four-point configurations, respectively.

Theorem 2 (The disc and the sphere). Let S be $\mathbb{D}$ or S^2 , equipped with an area form ω , and let μ be its induced measure. Let $d \geq 2$ be an integer. Then

$$\dim \overline{H}_b^d(\text{Homeo}_0(S, \mu)) = \infty, \quad \dim \overline{EH}_b^d(\text{Ham}(S, \omega)) = \infty.$$

For positive-genus surfaces we use a mixed transfer. Its distinguished factor is the compact one-point transfer when $g \geq 2$, and the two-point transfer when $g = 1$. For torus homeomorphisms, a family of two-point quasimorphisms supplies exactness in every even degree. For these homeomorphisms, in odd degrees we assert only the reduced bounded conclusion. Write $T = T^2 = \Sigma_1$.

For $n = 2, 3$, let $\text{Mod}(S, n)$ denote the mapping class group with n marked points, allowing their permutations, and set

$$K(S, n) = \ker(\text{Mod}(S, n) \longrightarrow \text{Mod}(S)).$$

The group $K(T, 2)$ in the next theorem is identified with $B_2(T)/Z(B_2(T))$ in the torus proof.

Theorem 3 (Closed surfaces of positive genus). *Let $S = \Sigma_g$ be a closed connected oriented surface of genus $g \geq 1$. Let ω be a smooth positive area form and μ its normalized area measure. Let $\mathcal{T}_S$ denote any one of*

$$\text{Homeo}_0(S, \mu), \quad \text{Symp}_0(S, \omega), \quad \text{Ham}(S, \omega).$$

(a) *For every integer $d \geq 2$,*

$$\dim_{\mathbb{R}} \overline{H}_b^d(\mathcal{T}_S) = \infty.$$

If $g \geq 2$, or if $\mathcal{T}_S$ is one of the two smooth groups, or if d is even, then also

$$\dim_{\mathbb{R}} \overline{EH}_b^d(\mathcal{T}_S) = \infty.$$

(b) *For every $m \geq 0$ and $q \in \{2, 3\}$, there are injective bounded linear maps, bounded below, as follows. For $g \geq 2$ they are*

$$\Phi_{m,q} : \overline{H}_b^q(F_2) \longrightarrow \overline{EH}_b^{2m+q}(\mathcal{T}_S).$$

For $g = 1$ they are

$$\Phi_{m,q}^T : \overline{H}_b^q(K(T, 2)) \longrightarrow \overline{H}_b^{2m+q}(\mathcal{T}_T).$$

The latter maps have reduced exact image for each of the smooth groups. For $\mathcal{T}_T = \text{Homeo}_0(T, \mu)$, there is a single infinite-dimensional subspace $\mathcal{E}_T \subset \overline{H}_b^2(K(T, 2))$ of codimension at most one such that every restriction $\Phi_{m,2}^T|_{\mathcal{E}_T}$ has reduced exact image.

Remark (Degree two). For a discrete group K , the map $f \mapsto [\delta f]$ identifies the space of real homogeneous quasimorphisms modulo homomorphisms $K \rightarrow \mathbb{R}$ with $EH_b^2(K)$; see [21, Sections 2.3–2.4]. Since $H_b^2(K)$ is normed, this is also $\overline{EH}_b^2(K)$ by [21, Theorem 6.5]. Thus the degree-two conclusions of Theorems 1 and 3, and the smooth conclusions of Theorem 2, give infinite-dimensional spaces of homogeneous quasimorphisms even after quotienting by homomorphisms. This is a bounded-cohomological proof of that consequence, not an assertion that the existence of such quasimorphisms is new. Relevant constructions and injectivity results already appear in [24, 38, 10, 9, 29]. For torus homeomorphisms, the required extension of two-point quasimorphisms follows from [7, Theorem 1.1]; see also [8].

Remark (Norm estimates). For each map in Theorems 1(b) and 3(b), there is a constant $c_{m,q} > 0$ such that

$$\|\Phi_{m,q}(v)\| \geq c_{m,q}\|v\| \quad \text{for every } v \text{ in the domain of } \Phi_{m,q}.$$

The same statement holds with $\Phi_{m,q}^T$ in the torus case. The constant may depend on M , $\mathcal{T}_M$, m , q , and the construction, but not on v . All the maps in Theorem 3 can be chosen bounded as well. In Theorem 1, these maps can be chosen bounded under (i) or (ii). Under (iii), the construction gives injective linear maps with the stated lower norm bounds; no upper norm bound is asserted.

We also record an ordinary-cohomology consequence of the classical mass-flow construction: a nonzero real character of $\pi_1(M)/Z(\pi_1(M))$ yields infinite-dimensional ordinary cohomology in every positive degree for the volume-preserving homeomorphism and diffeomorphism groups, see Theorem 29.

Symmetric external products. For a group K and $\sigma \in S_k$, we define a permutation of K^k by $\sigma(g_1, \dots, g_k) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(k)})$. Set

$$\text{Sym}_k \xi = \sum_{\sigma \in S_k} \sigma^* \xi.$$

A class on K^k is called *symmetric* if every σ^* fixes it. For every class ξ , the sum $\sum_{\sigma \in S_k} \sigma^* \xi$ is symmetric. If ξ is already symmetric, then $\text{Sym}_k \xi = k! \xi$. Thus the normalized operator $\frac{1}{k!} \text{Sym}_k$ acts as the identity on the symmetric subspace and is an idempotent linear projection onto that subspace. A transposition sends $u \times v$ to $(-1)^{\deg u \deg v} v \times u$; this need not equal $u \times v$. With degree-two fixed factors and at most one other factor, all permutation signs are positive.

Now we sketch the proof of Theorem 1, proofs of Theorems 2 and 3 use similar ideas, thus we do not sketch their proofs in the introduction.

Sketch of the proof of Theorem 1. We construct the maps in part (b) using only classes of degrees two and three on F_2 . Both $\overline{H}_b^2(F_2)$ and $\overline{H}_b^3(F_2)$ are infinite-dimensional, see, e.g., [12, 40]. For $m \geq 1$ and $q \in \{2, 3\}$, put $k = m + 1$ and $d = 2m + q$. Consider the composition

$$\overline{H}_b^q(F_2) \xrightarrow{\tilde{T}_m^q} \overline{H}_b^d(G^k) \xrightarrow{k!^{-1} \overline{\Gamma}_{b,k}^d} \overline{H}_b^d(\mathcal{T}_M).$$

The first map is the unnormalized sum of the extended external products, see Section 2. The second is the Gambaudo–Ghys map over M^k , divided by $k!$, see Section 3. Thus we transfer the normalized symmetric class, as in (26). We explain why their composition is injective and why its image lies in reduced exact bounded cohomology.

- Each of the three hypotheses provides an embedding $i : F_2 \hookrightarrow G$ and linear extension maps J_2, J_3 satisfying

$$i^* J_s = \text{id} \quad (s = 2, 3).$$

Lemmas 5 and 6 construct these maps and control their images under comparison. The image of J_3 lies in reduced exact bounded cohomology. The same holds for J_2 on a subspace $K_2 \subset \overline{H}_b^2(F_2)$ of codimension at most one; under (ii) or (iii), this is the whole space. Lemma 7 chooses in K_2 a nonzero finite linear combination θ of Brooks classes such that

$$\theta \smile u = 0 \quad (u \in \overline{H}_b^s(F_2), s \geq 1).$$

The vanishing is the theorem of Amontova–Bucher [1, Theorem A(b)]. External products with this fixed class increase the degree. Its cup-product vanishing property eliminates the terms with repeated coordinate assignments in the calculation.

- First consider two factors. For $v \in \overline{H}_b^q(F_2)$, take

$$\theta \times v + v \times \theta \in \overline{H}_b^{q+2}(F_2 \times F_2).$$

This depends injectively on v . To see it, choose a degree-two ℓ^1 -cycle z with $\langle \theta, z \rangle = 1$ (Lemma 7). The dual of the chain map $b \mapsto z \times b$ evaluates the first factor on z and leaves a reduced bounded-cohomology class on the second factor. Applied to the displayed sum, it gives

$$v \quad (q = 3), \quad v + \langle v, z \rangle \theta \quad (q = 2).$$

The second expression also determines v , since $\langle \theta, z \rangle = 1$ (see (21)).

For $k = m + 1$ factors, use the symmetric sum

$$T_m^q(v) = \text{Sym}_k(\theta^{\times m} \times v).$$

Here $\theta^{\times m}$ denotes m external factors. Proposition 9, using Lemma 8, proves that T_m^q has a bounded left inverse. Extend each factor separately and set

$$\tilde{T}_m^q(v) = \text{Sym}_k(J_2(\theta)^{\times m} \times J_q(v)).$$

For the product embedding $i^k(w_1, \dots, w_k) = (i(w_1), \dots, i(w_k))$, naturality gives

$$(i^k)^* \tilde{T}_m^q(v) = T_m^q(v).$$

Thus the extended classes remain independent. No compatibility of the maps J_s with cup products is required.

- Each point in M and an isotopy of a transformation, together with paths to a basepoint, produce an element of G in a natural way. For k points this gives an element of G^k . Integrating a bounded cocycle on these elements over M^k defines the second map in the composition. Lemma 10 proves that it is a bounded cochain construction and commutes with comparison.

For each fixed tuple of transformations the cocycle has finite image, so the same integral is defined even on ordinary cochains that are not bounded.

Every summand of $\tilde{T}_m^q(v)$ contains the factor $J_2(\theta) \in \overline{EH}_b^2(G)$, since $m \geq 1$. Its comparison image is therefore zero, and the transferred class belongs to reduced exact bounded cohomology as well. We denote this composition by $\Phi_{m,q}$.

- It remains to show that the transfer is injective on the subspace of external-product classes just constructed. Lemma 11 constructs homomorphisms $\rho_\varepsilon : F_2^k \rightarrow \mathcal{T}_M$ by finger-pushing isotopies in k disjoint figure-eight neighborhoods. On a subset $U_{j,\varepsilon}$ of the j th neighborhood, the associated homomorphism is $i \circ p_j$, up to conjugation, where $p_j : F_2^k \rightarrow F_2$ is the j th projection. The measure of $U_{j,\varepsilon}$ tends to a positive number λ_j . Other fixed subsets give cyclic or trivial associated homomorphisms; the remaining transition subset has measure tending to zero.

For two coordinates, the products $U_{1,\varepsilon} \times U_{2,\varepsilon}$ and $U_{2,\varepsilon} \times U_{1,\varepsilon}$ contribute the same class $\theta \times v + v \times \theta$. If both points lie in $U_{j,\varepsilon}$, both cocycle coordinates use the same projection p_j . The pullback then factors through the diagonal $\Delta(w) = (w, w)$, and

$$\Delta^*(\theta \times v) = \theta \smile v = 0.$$

The other summand vanishes for the same reason. Homomorphisms with cyclic or trivial image also give zero in positive-degree bounded cohomology. Finally, boundedness of the cocycle makes the contribution of the transition subset tend to zero.

For k coordinates, every repeated assignment puts at least two factors on the same free group. At least one of them is θ , since only one factor varies, so that internal cup product vanishes. The $k!$ bijective assignments give the remaining contributions and have the same product measure. Proposition 12 proves the resulting convergence

$$\rho_\varepsilon^* \Phi_{m,q}(v) \longrightarrow C_k T_m^q(v), \quad C_k = \prod_{j=1}^k \lambda_j > 0,$$

in reduced norm, first for two coordinates and then for general k .

As in the proof outline in [10, Section 3], the comparison between transfer and restriction can be summarized by the diagram

$$\begin{array}{ccccc}
 & & \Phi_{m,q} & & \\
 & \nearrow \tilde{T}_m^q & & \searrow k!^{-1} \bar{\Gamma}_{b,k}^d & \\
 \overline{H}_b^q(F_2) & \xrightarrow{\quad} & \overline{H}_b^d(G^k) & \xrightarrow{\quad} & \overline{H}_b^d(\mathcal{T}_M) \\
 \downarrow = & & \downarrow C_k(i^k)^* & \nearrow \rho_\varepsilon^* & \\
 \overline{H}_b^q(F_2) & \xrightarrow{\quad C_k T_m^q \quad} & \overline{H}_b^d(F_2^k) & &
 \end{array}$$

This diagram commutes in the limit $\varepsilon \rightarrow 0$ on the source space $\overline{H}_b^q(F_2)$: for each v , the two paths from the upper-left corner to the lower-right corner have values whose difference tends to zero in reduced norm. Indeed, the upper path gives $\rho_\varepsilon^* \Phi_{m,q}(v)$, whereas the lower path gives $C_k T_m^q(v)$ because $(i^k)^* \tilde{T}_m^q(v) = T_m^q(v)$.

We do not assert exact commutativity for fixed ε , or convergence on arbitrary classes in $\overline{H}_b^d(G^k)$. If $\Phi_{m,q}(v) = 0$, the left-hand side is zero for every ε . Thus $T_m^q(v)$ has zero reduced norm and is the zero class. Injectivity of T_m^q gives $v = 0$. This proves the required injectivity.

Degrees two and three use the same argument with one coordinate and no fixed external factor: apply the transfer to $J_2|_{K_2}$ and to J_3 , respectively. The degrees $2m + 2$ and $2m + 3$ exhaust all integers at least two. The injective maps therefore imply part (a). Lemma 11 gives the isotopies in the volume-preserving groups and, in the symplectic case and in Ham itself. The argument consequently applies to every group in the theorem. $\square$

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2. BOUNDED COHOMOLOGY OF PRODUCTS OF GROUPS AND TRANSFER MAPS

In this section establish the common algebraic and transfer lemmas.

Cup and external products. We first fix our cochain conventions. Let K be a discrete group, and let f and h be inhomogeneous cochains on K of degrees p and q , respectively. Their cup product is defined by

$$(4) \quad (f \smile h)(g_1, \dots, g_{p+q}) = f(g_1, \dots, g_p) h(g_{p+1}, \dots, g_{p+q}).$$

See [1, Section 2.2]. This product satisfies the Leibniz identity

$$\delta(f \smile h) = \delta f \smile h + (-1)^p f \smile \delta h.$$

If both cochains are bounded, then their cup product is bounded and

$$\|f \smile h\|_\infty \leq \|f\|_\infty \|h\|_\infty.$$

For discrete groups K and L , let $p_K : K \times L \rightarrow K$ and $p_L : K \times L \rightarrow L$ denote the coordinate projections. For cochains f on K and h on L , define their external product by $f \times h = p_K^* f \smile p_L^* h$. We use the same notation for the induced products on ordinary and bounded cohomology. For bounded-cohomology classes of the appropriate degrees, the canonical seminorms satisfy

$$(5) \quad \|u \smile v\| \leq \|u\| \|v\|, \quad \|u \times v\| \leq \|u\| \|v\|.$$

See [30, Section 2.3] and [35, Section 2]. Our convention does not include the factor $(-1)^{pq}$ appearing in Löh's definition. This difference has no effect when one of the degrees is even. The following lemma is proved in

Lemma 4 ([20]). *For every discrete group K , a class of zero seminorm in $H_b^d(K)$ has zero image in $H^d(K; \mathbb{R})$. Consequently the comparison map induces a linear map*

$$\bar{c}_K^d : \bar{H}_b^d(K) \longrightarrow H^d(K; \mathbb{R}), \quad \overline{EH}_b^d(K) = \ker \bar{c}_K^d.$$

External and cup products descend to reduced bounded cohomology and are compatible with these comparison maps.

The following lemma is a consequence of the isometric section theorem of Bucher–Burger–Frigerio–Iozzi–Pagliantini–Pozzetti [14]; see also [21, Theorem 9.7]. The assertions about comparison maps follow from the ordinary Mayer–Vietoris sequence of the amalgam.

Lemma 5. *Assume (i) and write*

$$(6) \quad G \cong F_2 *_A B.$$

Let $i : F_2 \hookrightarrow G$ and $j : B \hookrightarrow G$ denote the vertex inclusions. For every $q \geq 2$, there are isometric linear maps

$$\begin{aligned} J_b^q : H_b^q(F_2) &\longrightarrow H_b^q(G), \\ J_q : \bar{H}_b^q(F_2) &\longrightarrow \bar{H}_b^q(G) \end{aligned}$$

such that

$$(7) \quad \begin{aligned} i^* J_b^q &= \text{id}, & j^* J_b^q &= 0, \\ i^* J_q &= \text{id}, & j^* J_q &= 0. \end{aligned}$$

Moreover, $\bar{c}_G^q \circ J_q = 0$ ($q \geq 3$), $\text{rank}(\bar{c}_G^2 \circ J_2) \leq 1$. In particular, $K_2 = \ker(\bar{c}_G^2 \circ J_2)$ has codimension at most one in $\bar{H}_b^2(F_2)$.

Proof. Since G is countable and A is amenable, the isometric section theorem gives a linear map

$$\Theta^q : H_b^q(F_2) \oplus H_b^q(B) \longrightarrow H_b^q(G)$$

which is an isometric right inverse of the combined restriction map (i^*, j^*) . Here the direct sum carries the maximum seminorm. Define $J_b^q(u) = \Theta^q(u, 0)$. Then

$$i^* J_b^q(u) = u, \quad j^* J_b^q(u) = 0, \quad \|J_b^q(u)\| = \|u\|.$$

Thus J_b^q preserves zero-seminorm classes and induces an isometric map J_q on reduced bounded cohomology. This proves (7).

An amenable subgroup of a free group is either trivial or infinite cyclic. Consequently,

$$(8) \quad \begin{aligned} &A = 1 \text{ or } A \cong \mathbb{Z}, \\ &H^s(A; \mathbb{R}) = 0 \quad (s \geq 2), \quad \dim H^1(A; \mathbb{R}) \leq 1. \end{aligned}$$

We also have $H^q(F_2; \mathbb{R}) = 0$ for $q \geq 2$, see [27, Section 1.3].

Let $\iota_F : A \rightarrow F_2$ and $\iota_B : A \rightarrow B$ denote the edge inclusions. The ordinary Mayer–Vietoris sequence contains

$$\begin{aligned} H^{q-1}(F_2; \mathbb{R}) \oplus H^{q-1}(B; \mathbb{R}) &\xrightarrow{d^{q-1}} H^{q-1}(A; \mathbb{R}) \\ \xrightarrow{\partial_A^{q-1}} H^q(G; \mathbb{R}) &\xrightarrow{R^q} H^q(F_2; \mathbb{R}) \oplus H^q(B; \mathbb{R}), \end{aligned}$$

where

$$d^{q-1}(x, y) = \iota_F^* x - \iota_B^* y, \quad R^q(w) = (i^* w, j^* w).$$

Here ∂_A^{q-1} is the connecting homomorphism; see [13, Chapter II, Section 7].

Let $u \in \overline{H}_b^q(F_2)$ and put $w = \bar{c}_G^q(J_q(u))$. Naturality and (7) give

$$\begin{aligned} i^* w &= \bar{c}_{F_2}^q(i^* J_q(u)) = \bar{c}_{F_2}^q(u) = 0, \\ j^* w &= \bar{c}_B^q(j^* J_q(u)) = \bar{c}_B^q(0) = 0. \end{aligned}$$

Exactness of the Mayer–Vietoris sequence therefore gives

$$(9) \quad \text{im}(\bar{c}_G^q \circ J_q) \subseteq \ker R^q = \text{im } \partial_A^{q-1}.$$

For $q \geq 3$, the domain $H^{q-1}(A; \mathbb{R})$ of the connecting homomorphism is zero by (8). Hence (9) implies $\bar{c}_G^q \circ J_q = 0$.

For $q = 2$, the same inclusion gives $\text{rank}(\bar{c}_G^2 \circ J_2) \leq \dim H^1(A; \mathbb{R}) \leq 1$. More precisely, exactness identifies the possible obstruction space as

$$(10) \quad \ker R^2 \cong \frac{H^1(A; \mathbb{R})}{\text{im}(H^1(F_2; \mathbb{R}) \oplus H^1(B; \mathbb{R}) \xrightarrow{d^1} H^1(A; \mathbb{R}))}.$$

Thus the comparison image of J_2 lies in one fixed subspace of dimension at most one. Its kernel K_2 consequently has codimension at most one, as claimed. $\square$

Lemma 6. *Under (ii) or (iii) there are an embedding $i : F_2 \hookrightarrow G$ and linear maps*

$$J_q : \overline{H}_b^q(F_2) \longrightarrow \overline{EH}_b^q(G), \quad i^* J_q = \text{id} \quad (q = 2, 3).$$

Under (ii) these maps are isometric homomorphism pullbacks. Under (iii) they are algebraic linear right inverses of restriction.

Proof. Under (ii), freeness gives a section $i : F_2 \rightarrow G$ of r , so $r \circ i = \text{id}$. Put $J_q = r^*$. Naturality and the norm bound for pullback give

$$i^* J_q = \text{id}, \quad \|v\| \leq \|J_q(v)\| \leq \|v\|, \quad \bar{c}_G^q J_q = r^* \bar{c}_{F_2}^q = 0.$$

Here $H^q(F_2; \mathbb{R}) = 0$ for $q \geq 2$.

Under (iii), the exact extension theorem of Frigerio–Pozzetti–Sisto [22, Corollary 2 and Proposition 5.1] states that $i^* : EH_b^q(G) \rightarrow EH_b^q(F_2)$ is onto for $q \geq 2$. Since $EH_b^q(F_2) = H_b^q(F_2)$, lifting a representative of any reduced bounded-cohomology class and then passing to the quotient gives a surjection

$$i^* : \overline{EH}_b^q(G) \rightarrow \overline{H}_b^q(F_2).$$

Choose lifts of a Hamel basis and extend linearly to obtain J_q for $q = 2, 3$. Its image lies in reduced exact bounded cohomology and $i^* J_q = \text{id}$; compare [10, proof of Theorem B]. The inequality $\|J_q(v)\| \geq \|v\|$ follows from the norm bound for restriction. $\square$

For the remainder of the proof, fix i, J_2, J_3 as in Lemma 5 or Lemma 6. We use the following subspace for the degree-two assertion of Theorem 1:

$$(11) \quad \begin{aligned} i^* J_q &= \text{id} \quad (q = 2, 3), \quad J_3(\overline{H}_b^3(F_2)) \subset \overline{EH}_b^3(G), \\ K_2 &:= \ker(\bar{c}_G^2 \circ J_2) = \{v \in \overline{H}_b^2(F_2) : \bar{c}_G^2(J_2(v)) = 0\}. \end{aligned}$$

Thus K_2 consists precisely of the reduced bounded-cohomology classes whose chosen extension lies in $\overline{EH}_b^2(G)$. Lemma 5 gives $\dim(\overline{H}_b^2(F_2)/K_2) \leq 1$ under (i). Under (ii) or (iii), Lemma 6 gives $K_2 = \overline{H}_b^2(F_2)$. The approximation below will be applied to individual classes in the images of J_q .

Lemma 7. *There exist a nonzero class $\theta \in \overline{H}_b^2(F_2)$, a bounded representative given by the coboundary of a homogeneous quasimorphism, and an ℓ^1 bar cycle $z \in Z_2^{\ell^1}(F_2)$ such that*

$$J_2(\theta) \in \overline{EH}_b^2(G), \quad \langle \theta, z \rangle = 1, \quad \theta \smile v = 0 \quad (v \in \overline{H}_b^{>0}(F_2)).$$

The class θ may be chosen to be a finite linear combination of Brooks classes.

Proof. The algebraic span of Brooks classes in $H_b^2(F_2)$ is infinite-dimensional by Brooks–Mitsumatsu [12, 33]; see also [21, Section 2.6]. By Matsumoto–Morita, the seminorm on $H_b^2(F_2)$ is a norm and the Kronecker map is an isomorphism

$$(12) \quad H_b^2(F_2) = \overline{H}_b^2(F_2) \xrightarrow{\cong} (\overline{H}_2^{\ell^1}(F_2))', \quad u \longmapsto \langle u, \cdot \rangle.$$

Here $\overline{H}_2^{\ell^1}(F_2)$ denotes reduced ℓ^1 -homology. We use [31, Theorem 2.3 and Corollary 2.7]; the needed statements are also given in [21, Theorem 6.5 and Corollary 6.7]. The same degree-two duality is used in [35, proof of Theorem 4.2].

By (11), the intersection of K_2 with the span of Brooks classes has codimension at most one. Choose a nonzero class θ in this intersection. It is a finite linear combination of Brooks classes and has a representative δq , where q is a homogeneous quasimorphism: replace each Brooks quasimorphism by its homogenization.

By (12), some ℓ^1 -cycle evaluates nontrivially on θ . Multiplying that cycle by a scalar gives $z \in Z_2^{\ell^1}(F_2)$ with $\langle \theta, z \rangle = 1$. The condition $\theta \in K_2$ gives exactness of $J_2(\theta)$. Finally, [1, Theorem A(b)] states that the cup product of a Brooks class with every bounded-cohomology class of positive degree is zero. Apply this to the finite linear combination defining θ and pass to reduced bounded cohomology using Lemma 4. $\square$

The following lemma is the bounded version of the classical slant product with a fixed homology class; see [41, Chapter 6, Sections 1 and 10]. For its bounded chain maps and homotopies, see [30, Lemma 2.6 and the proof of Proposition 2.5]. We retain the notation L_Z^* for the dual of cross product with a fixed cycle Z .

Let $C_*^{\ell^1}(K)$ denote the standard unnormalized ℓ^1 -chain complex of a group K with trivial real coefficients, and let $C_b^*(K)$ denote its continuous dual. Write $Z_p^{\ell^1}(K) = \ker \partial_p$. We use the standard shuffle cross product

$$\times : C_r^{\ell^1}(K) \otimes C_t^{\ell^1}(L) \longrightarrow C_{r+t}^{\ell^1}(K \times L).$$

For chains x and y of degrees r and t , respectively, it satisfies

$$(13) \quad \|x \times y\|_1 \leq \binom{r+t}{r} \|x\|_1 \|y\|_1$$

and

$$(14) \quad \partial(x \times y) = \partial x \times y + (-1)^r x \times \partial y.$$

See [30, Section 2.3]. The binomial coefficient in (13) is the number of (r, t) -shuffles.

Lemma 8. *Let K, L be discrete groups, let $p \geq 0$ be even, and let $Z \in Z_p^{\ell^1}(K)$. Cross product with Z defines*

$$L_Z : C_q^{\ell^1}(L) \longrightarrow C_{p+q}^{\ell^1}(K \times L), \quad L_Z(b) = Z \times b.$$

Its dual cochain map

$$L_Z^* : C_b^{p+q}(K \times L) \longrightarrow C_b^q(L), \quad (L_Z^* c)(b) = c(Z \times b),$$

induces a bounded linear map $L_Z^* : \overline{H}_b^{p+q}(K \times L) \longrightarrow \overline{H}_b^q(L)$ for every $q \geq 0$, with

$$(15) \quad \|L_Z^*\| \leq \binom{p+q}{p} \|Z\|_1.$$

For $u \in \overline{H}_b^s(K)$ and $v \in \overline{H}_b^t(L)$ with $s+t = p+q$, we have

$$(16) \quad L_Z^*(u \times v) = \begin{cases} \langle u, Z \rangle v, & s = p, \\ 0, & s \neq p. \end{cases}$$

Proof. Since $\partial Z = 0$ and p is even, (14) shows that L_Z commutes with the boundary. Its dual therefore commutes with the coboundary, and (13) gives (15). Being bounded, it preserves closures of coboundaries and hence induces the asserted map on reduced bounded cohomology.

The classical slant-product calculation gives (16); see [41, Chapter 6, Sections 1 and 10]. The same calculation applies to ℓ^1 -chains and bounded cochains because the Eilenberg–Zilber homotopies are bounded in every degree; see [30, Lemma 2.6 and the proof of Proposition 2.5]. In particular, evaluating the homotopy on the fixed cycle Z makes the cochain error a bounded coboundary. Thus the identity holds in bounded cohomology and passes to reduced bounded cohomology. $\square$

Proposition 9. *Let $\theta \in \overline{H}_b^2(F_2)$ and $z \in Z_2^{\ell^1}(F_2)$ satisfy $\langle \theta, z \rangle = 1$. For $m \geq 0$, $q \in \{2, 3\}$, put $k = m+1$ and $d = 2m+q$, and define*

$$(17) \quad \begin{aligned} T_m^q : \overline{H}_b^q(F_2) &\longrightarrow \overline{H}_b^d(F_2^k), \\ T_m^q(v) &= \text{Sym}_k(\theta^{\times m} \times v). \end{aligned}$$

Each T_m^q has a bounded left inverse. In particular it is bounded below, and its image is an infinite-dimensional space of symmetric classes, and each nonzero class in the image has positive norm.

Proof. Cross-product independence via Kronecker pairings is used in [30, Proposition 3.3], and a graded-symmetric version occurs in [35, Lemma 4.1(ii)]. We apply Lemma 8 to construct a bounded left inverse of T_m^q for each $q \in \{2, 3\}$. The construction uses only the fixed degree-two cycle z with $\langle \theta, z \rangle = 1$. In particular, for $q = 3$ it does not require a degree-three homology class detecting the class v .

Let σ act on F_2^k by permuting its coordinates. A sum over all such pullbacks is invariant under every coordinate permutation, so every class in the image of T_m^q is symmetric. It remains to prove injectivity with a lower norm estimate.

Put $Z_m = z^{\times m}$. Since an external product of cycles is a cycle, we have $Z_m \in Z_{2m}^{\ell^1}(F_2^m)$. Repeated application of (13) shows that it has finite norm;

more precisely the cross-product norm estimate gives

$$\|Z_m\|_1 = \|Z_{m-1} \times z\|_1 \leq \binom{2m}{2} \|Z_{m-1}\|_1 \|z\|_1,$$

because Z_{m-1} has degree $2m-2$ and z has degree two. Starting with $Z_1 = z$, we obtain

$$\|Z_m\|_1 \leq \left(\prod_{j=1}^m \binom{2j}{2} \right) \|z\|_1^m = \left(\prod_{j=1}^m \frac{(2j)!}{(2j-2)! 2!} \right) \|z\|_1^m = \frac{(2m)!}{(2!)^m} \|z\|_1^m.$$

In particular, Z_m has finite ℓ^1 -norm.

We next compute the pairing of $\theta^{\times m}$ with Z_m . By the definition of L_Z^* , for every class a and every cycle w of the appropriate degrees,

$$\langle a, Z \times w \rangle = \langle L_Z^* a, w \rangle.$$

For $m \geq 2$, apply Lemma 8 with $K = F_2^{m-1}$, $L = F_2$, and $Z = Z_{m-1}$. Since both $\theta^{\times(m-1)}$ and Z_{m-1} have degree $2m-2$, (16) gives

$$L_{Z_{m-1}}^*(\theta^{\times m}) = \langle \theta^{\times(m-1)}, Z_{m-1} \rangle \theta.$$

As $Z_m = Z_{m-1} \times z$, it follows that

$$\langle \theta^{\times m}, Z_m \rangle = \langle L_{Z_{m-1}}^*(\theta^{\times m}), z \rangle = \langle \theta^{\times(m-1)}, Z_{m-1} \rangle \langle \theta, z \rangle.$$

Starting with $Z_1 = z$ and $\langle \theta, z \rangle = 1$, induction therefore gives

$$(18) \quad \langle \theta^{\times m}, Z_m \rangle = \langle \theta, z \rangle^m = 1.$$

For $m = 0$, the same identity holds with the degree-zero unit $\theta^{\times 0} = 1$ and $Z_0 = []$.

Apply Lemma 8 with $K = F_2^m$, $L = F_2$, $p = 2m$ and $Z = Z_m$. It gives the map

$$L_{Z_m}^* : \overline{H}_b^{2m+q}(F_2^m \times F_2) \longrightarrow \overline{H}_b^q(F_2).$$

There are $m!$ permutations in (17) for each prescribed position of the variable class v : the other m factors all equal θ . Moving an even-degree factor past another factor produces no sign. Consequently these $m!$ copies of each term agree in reduced bounded cohomology. Graded commutativity is valid for bounded cohomology because the usual shuffle/flip homotopy is bounded in each degree; see [30, Section 2.3] and the bounded chain maps in Lemma 8.

Suppose $q = 3$. If v is in the last coordinate, the first m coordinates carry $\theta^{\times m}$, whose evaluation on Z_m is one by (18). Each of these $m!$ terms therefore gives v . If v is in one of the first m coordinates, the degree on those coordinates is

$$2(m-1) + 3 = 2m + 1.$$

Since Z_m has degree $2m$, the second line of (16) gives zero. We obtain

$$(19) \quad L_{Z_m}^* T_m^3(v) = m!v.$$

In particular, $m!^{-1} L_{Z_m}^*$ is a left inverse of T_m^3 .

Suppose $q = 2$. Evaluation on the fixed cycle z defines a bounded linear functional on $\overline{H}_b^2(F_2)$, since $|\langle v, z \rangle| \leq \|v\| \|z\|_1$. As before, the $m!$ terms with v last contribute $m!v$. If v is in the j th of the first m coordinates, the product pairing on these coordinates is $\langle v, z \rangle$: all the other evaluations are $\langle \theta, z \rangle = 1$. The last coordinate contains θ . There are $m!$ terms for each of these m choices, so

$$(20) \quad L_{Z_m}^* T_m^2(v) = m!(v + m\langle v, z \rangle \theta).$$

Define the following bounded linear maps on $\overline{H}_b^{2m+q}(F_2^{m+1})$, for $q = 3$ and $q = 2$, respectively:

$$(21) \quad Q_{m,3}(\eta) = \frac{1}{m!} L_{Z_m}^*(\eta), \quad Q_{m,2}(\eta) = \frac{1}{m!} \left(L_{Z_m}^*(\eta) - \frac{m}{m+1} \langle L_{Z_m}^*(\eta), z \rangle \theta \right).$$

Equation (19) gives $Q_{m,3} T_m^3 = \text{id}$. To check the second identity, pair (20) with z . Since $\langle \theta, z \rangle = 1$, this gives

$$\langle L_{Z_m}^* T_m^2(v), z \rangle = m!(m+1)\langle v, z \rangle.$$

Substituting this and (20) into (21) yields

$$Q_{m,2} T_m^2(v) = v + m\langle v, z \rangle \theta - m\langle v, z \rangle \theta = v.$$

Thus $Q_{m,q} T_m^q = \text{id}$. Both maps are bounded by Lemma 8 and the bound on the pairing with z . For $m = 0$ they are the identity maps. For example,

$$\|Q_{m,2}\| \leq \frac{1}{m!} \left(1 + \frac{m}{m+1} \|z\|_1 \|\theta\| \right) \binom{2m+2}{2m} \|Z_m\|_1,$$

and Lemma 8 gives the corresponding bound for $Q_{m,3}$.

It follows that $\|T_m^q(v)\| \geq \frac{1}{\|Q_{m,q}\|} \|v\|$. Thus T_m^q is injective and satisfies the asserted lower norm estimate.

The space $\overline{H}_b^2(F_2)$ is infinite-dimensional by [12, 33]. For $\overline{H}_b^3(F_2)$ we use [22, Corollary 3] (see [40] as well): the third reduced bounded cohomology of an acylindrically hyperbolic group has continuum dimension. It applies to F_2 , whose action on its Cayley tree is acylindrical and non-elementary. Another construction of positive-norm degree-three classes is given in [18, Sections 6–7]. Thus both spaces $\overline{H}_b^2(F_2)$ and $\overline{H}_b^3(F_2)$, and hence all their images under T_m^q , are infinite-dimensional. This proves the proposition. $\square$

3. PROOF OF THEOREM 1

Lemma 10. *For each $k \geq 1$ and each transformation group $\mathcal{T}_M$ in Theorem 1 or 3, the diagonal action on M^k defines compatible maps on reduced bounded cohomology and ordinary cohomology:*

$$\overline{\Gamma}_{b,k}^d : \overline{H}_b^d(G^k) \longrightarrow \overline{H}_b^d(\mathcal{T}_M), \quad \Gamma_k^d : H^d(G^k; \mathbb{R}) \longrightarrow H^d(\mathcal{T}_M; \mathbb{R}),$$

with $\|\bar{\Gamma}_{b,k}^d\| \leq 1$. The corresponding map on bounded cohomology is denoted by $\Gamma_{b,k}^d$. The maps commute with comparison. In particular, the reduced bounded-cohomology transfer preserves reduced exact bounded cohomology:

$$\bar{\Gamma}_{b,k}^d(\overline{EH}_b^d(G^k)) \subset \overline{EH}_b^d(\mathcal{T}_M).$$

Proof. This is the cocycle induction of [10, Sections 1.B and 2], applied to the diagonal action on M^k . Its covering-space formulation and compatibility with ordinary cohomology are given in [35, Section 3]; see also [38, 24]. We specify our conventions and check the finite-image hypothesis for the chosen section. Group multiplication is composition, $fg = f \circ g$.

Following [10, Section 1.A], we call a function $c : K^{d+1} \rightarrow \mathbb{R}$ *homogeneous* if

$$c(g_0h, \dots, g_dh) = c(g_0, \dots, g_d) \quad (g_0, \dots, g_d, h \in K).$$

Its coboundary is the alternating sum obtained by deleting one argument. This complex is isometrically identified with the inhomogeneous complex used in Lemma 8: the homogeneous cochain corresponding to an inhomogeneous cochain f is

$$c(g_0, \dots, g_d) = f(g_0g_1^{-1}, g_1g_2^{-1}, \dots, g_{d-1}g_d^{-1}).$$

Fix an auxiliary complete Riemannian metric on $\text{int } M$ and a basepoint $x_0 \in \text{int } M$. Outside the cut locus of x_0 , let $\alpha_{x_0,x}$ be the unique minimizing geodesic from x_0 to x . On the cut locus, which has μ -measure zero, choose the reference paths measurably. Put $\alpha_{x,x_0} = \alpha_{x_0,x}^{-1}$. This gives the measurable family of reference paths used in [10, Section 2.B].

For $f \in \mathcal{T}_M$, choose an isotopy $\{f_t\}_{t \in [0,1]}$ from the identity to f and define

$$(22) \quad \gamma(f, x) = [\alpha_{x_0,x} * \{f_t(x)\}_{t \in [0,1]} * \alpha_{f(x),x_0}]^{-1} \in G.$$

Here $*$ denotes path concatenation from left to right, and the brackets denote the homotopy class projected to $G = \pi_1(M, x_0)/Z(\pi_1(M, x_0))$. This class is independent of the chosen isotopy, since evaluation of a loop of isotopies has image in the center of $\pi_1(M, x_0)$; see [10, Proposition 2.1]. The resulting measurable cocycle satisfies

$$(23) \quad \gamma(fg, x) = \gamma(f, gx)\gamma(g, x)$$

for every $f, g \in \mathcal{T}_M$ and almost every $x \in M$.

By [10, Lemma 2.2], for each $f \in \mathcal{T}_M$ there is a finite set $S_f \subset G$ such that

$$(24) \quad \gamma(f, x) \in S_f \quad \text{for almost every } x \in M.$$

For $\mathbf{x} = (x_1, \dots, x_k)$ set

$$\gamma_k(f, \mathbf{x}) = (\gamma(f, x_1), \dots, \gamma(f, x_k)) \in G^k.$$

This is a cocycle for the diagonal action on M^k , by (23). Moreover,

$$\{\gamma_k(f, \mathbf{x}) : \mathbf{x} \in M^k\} \subset S_f^k,$$

so $\gamma_k(f, \cdot)$ has finite image for every fixed f .

For a homogeneous bounded d -cochain c on G^k , define $\Gamma'_{b,k}c$ by

$$(25) \quad (\Gamma'_{b,k}c)(f_0, \dots, f_d) = \int_{M^k} c(\gamma_k(f_0, \mathbf{x}), \dots, \gamma_k(f_d, \mathbf{x})) d\mu^k(\mathbf{x}).$$

The integrand is measurable and bounded. Since $\mu(M) = 1$, we have $\|\Gamma'_{b,k}c\|_\infty \leq \|c\|_\infty$. By the standard cocycle-induction calculation [10, Proposition 1.1 and Section 2.C], this is a homogeneous cochain map and

$$\delta\Gamma'_{b,k} = \Gamma'_{b,k}\delta.$$

Indeed, the cocycle identity and the change of variable $\mathbf{x} \mapsto h\mathbf{x}$ give right homogeneity, and the differential commutes with integration. The norm bound follows from $\mu^k(M^k) = 1$. Therefore the maps $\Gamma'_{b,k}$ and $\bar{\Gamma}_{b,k}^d$ are defined and have norm at most one.

Changing the measurable reference paths does not change the induced map on bounded cohomology, by the bounded change-of-section homotopy; see [35, Lemma 3.3 and Remark 3.4]. This proves that the induced map is independent of the section.

We next define the map on ordinary cohomology and explain why finite image is needed. Fix $(f_0, \dots, f_d)$ and let $c : (G^k)^{d+1} \rightarrow \mathbb{R}$ be an ordinary homogeneous cochain, not necessarily bounded. By (24), the map

$$\mathbf{x} \mapsto (\gamma_k(f_0, \mathbf{x}), \dots, \gamma_k(f_d, \mathbf{x}))$$

has image in the finite set $\mathcal{A} = S_{f_0}^k \times \dots \times S_{f_d}^k$. For $\mathbf{a} = (a_0, \dots, a_d) \in \mathcal{A}$, let

$$E_{\mathbf{a}} = \{\mathbf{x} \in M^k : \gamma_k(f_i, \mathbf{x}) = a_i \text{ for } i = 0, \dots, d\}.$$

These are measurable sets and form a finite partition after empty sets are omitted. Thus the integrand in (25) has finite image in $\mathbb{R}$, and

$$(\Gamma'_k c)(f_0, \dots, f_d) = \sum_{\mathbf{a} \in \mathcal{A}} \mu^k(E_{\mathbf{a}}) c(a_0, \dots, a_d).$$

In particular, its absolute value is at most $\max_{\mathbf{a} \in \mathcal{A}} |c(\mathbf{a})| < \infty$, since $\mu^k(M^k) = 1$. This is an argument for each fixed tuple; it does not assert that $\Gamma'_k c$ is a bounded cochain. The same homogeneity and differential calculations are valid because all the integrals involved are finite. They give the ordinary cochain map Γ'_k and the induced map Γ_k^d . Since Γ'_k agrees with $\Gamma'_{b,k}$ on bounded cochains, we have

$$\bar{c}_{\mathcal{T}_M}^d \bar{\Gamma}_{b,k}^d = \Gamma_k^d \bar{c}_{G^k}^d.$$

In particular, if a bounded cocycle c has an ordinary primitive b , then

$$\Gamma'_{b,k}c = \delta(\Gamma'_k b).$$

The cochain $\Gamma'_k b$ is well-defined by the finite-image property. Therefore the induced map preserves reduced exact bounded cohomology. $\square$

Remark (Permutation invariance of product transfer). The product transfer in Lemma 10 is unchanged by permuting the group coordinates. More generally, let (X, ν) be a probability space with a measure-preserving action and a measurable K -valued cocycle. Apply that same cocycle to each of k independent points of X , and let $\Gamma_{X,k}^d$ denote the resulting transfer from $\overline{H}_b^d(K^k)$. Then

$$(26) \quad \Gamma_{X,k}^d(\sigma^* \xi) = \Gamma_{X,k}^d(\xi), \quad \Gamma_{X,k}^d(\xi) = \frac{1}{k!} \Gamma_{X,k}^d(\text{Sym}_k \xi).$$

This is an equality of cochain maps before passing to cohomology. Indeed, for a bounded homogeneous cochain c and a fixed transformation tuple $\mathbf{f} = (f_0, \dots, f_d)$, let $C_{\mathbf{f}}(\mathbf{x})$ denote its transfer integrand. The coordinatewise cocycle satisfies $\gamma_k(f, \sigma \mathbf{x}) = \sigma \gamma_k(f, \mathbf{x})$. Consequently,

$$(\Gamma'_{X,k}(\sigma^* c))(\mathbf{f}) = \int_{X^k} C_{\mathbf{f}}(\sigma \mathbf{x}) d\nu^k(\mathbf{x}) = \int_{X^k} C_{\mathbf{f}}(\mathbf{x}) d\nu^k(\mathbf{x}) = (\Gamma'_{X,k} c)(\mathbf{f}).$$

The second equality uses the measure-preserving permutation of the integration variables. Summing over S_k proves (26). The argument uses boundedness and finite measure, not compactness. It therefore applies to M^k , $X_3(\mathbb{D})^k$, $X_4(S^2)^k$, and $X_3(S)^k$ below.

We consistently transfer the normalized symmetric class $k!^{-1} \text{Sym}_k \xi$. This has exactly the same transfer as the ordered class ξ . For products of two different kinds of integration spaces, only equal factors are permuted. In particular, for $X_3(S)^m \times S$ for $g \geq 2$ and $X_3(T)^m \times X_2(T)$ for $g = 1$ in Theorem 3, the first m three-point factors are symmetrized. The last one-point or two-point factor is kept fixed; the normalization is $1/m!$.

Lemma 11. *Assume $n \geq 3$. Let $i : F_2 \hookrightarrow G$ be as above and fix $k \geq 1$. For every sufficiently small $\varepsilon > 0$ there is a homomorphism*

$$\rho_\varepsilon : F_2^k \longrightarrow \mathcal{T}_M$$

and a measurable $\rho_\varepsilon(F_2^k)$ -invariant subset $B_\varepsilon \subset M$ such that

$$\delta_\varepsilon = \mu(B_\varepsilon) \longrightarrow 0.$$

Every element of $\rho_\varepsilon(F_2^k)$ fixes $M \setminus B_\varepsilon$ pointwise. Up to sets of measure zero, this complement has a measurable partition into subsets $U_{j,\varepsilon}$, $V_{j,a,\varepsilon}$, $V_{j,b,\varepsilon}$, $1 \leq j \leq k$, and W . For each point in these subsets the homomorphism $w \mapsto \gamma(\rho_\varepsilon(w), x)$ is conjugate, respectively, to

$$i \circ p_j, \quad i \circ h_a \circ p_j, \quad i \circ h_b \circ p_j, \quad 1,$$

where $p_j : F_2^k \rightarrow F_2$ is the projection and

$$h_a(a) = a, \quad h_a(b) = 1, \quad h_b(a) = 1, \quad h_b(b) = b.$$

Moreover, $\lambda_{j,\varepsilon} = \mu(U_{j,\varepsilon}) \rightarrow \lambda_j > 0$.

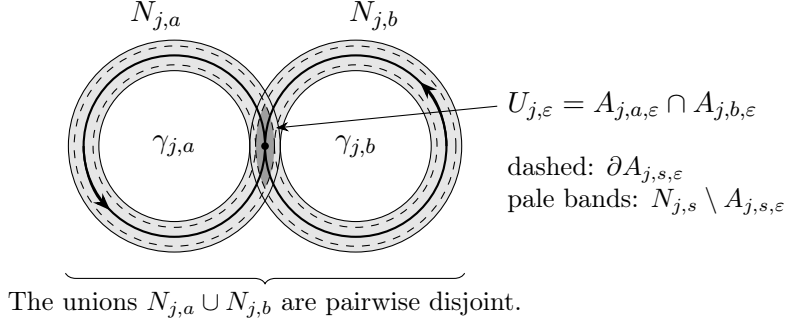


FIGURE 1. Tubular neighborhoods of a figure-eight embedding. Their contractible intersection contains the points whose trajectories record both generators, with the same conjugating element. The normal directions are omitted.

The homomorphisms can be chosen in $\text{Diff}_0(M, \mu)$. In the symplectic case they can be chosen in $\text{Ham}(M, \omega)$ itself, with Hamiltonian isotopies supported in the interior.

Proof. We use the finger-pushing construction of [10, Lemma 3.1], performed in k disjoint neighborhoods as in [35, proof of Theorem 4.2]. Since $n \geq 3$, general position gives k pairwise disjoint figure-eight embeddings in $\text{int } M$, whose based loops represent $i(a)^{-1}$ and $i(b)^{-1}$ in G . The inverses agree with the cocycle convention (22). Choose tubular neighborhoods $N_{j,a}, N_{j,b}$ with compact closures in $\text{int } M$, such that the unions $N_{j,a} \cup N_{j,b}$ are pairwise disjoint and each intersection $N_{j,a} \cap N_{j,b}$ is contractible and has positive measure. See Figure 1.

In product-volume coordinates $N = S^1 \times D^{n-1}$, with $S^1 = \mathbb{R}/\mathbb{Z}$, choose compact transverse cores K_ε exhausting D^{n-1} and smooth compactly supported cutoffs χ_ε equal to one near K_ε . Define

$$(27) \quad P_\varepsilon^t(\vartheta, y) = (\vartheta + t\chi_\varepsilon(y), y), \quad 0 \leq t \leq 1.$$

This isotopy preserves volume and extends by the identity outside N . On $A_\varepsilon = S^1 \times K_\varepsilon$, every trajectory makes one positive circuit, so the time-one map is the identity there. Moreover, $\mu(N \setminus A_\varepsilon) \rightarrow 0$.

For symplectic M of dimension 2ℓ , use the standard isotropic tubular model

$$N = S^1 \times (-r, r)_p \times D_z^{2\ell-2}, \quad \omega = d\vartheta \wedge dp + \omega_0;$$

see [42] and [6, Section 4.3]. Choose $\chi_\varepsilon(p, z)$ compactly supported in the transverse domain and equal to one near compact cores K_ε exhausting that domain. Set

$$(28) \quad H_\varepsilon(\vartheta, p, z) = -p\chi_\varepsilon(p, z), \quad \iota_{X_{H_\varepsilon}}\omega = -dH_\varepsilon.$$

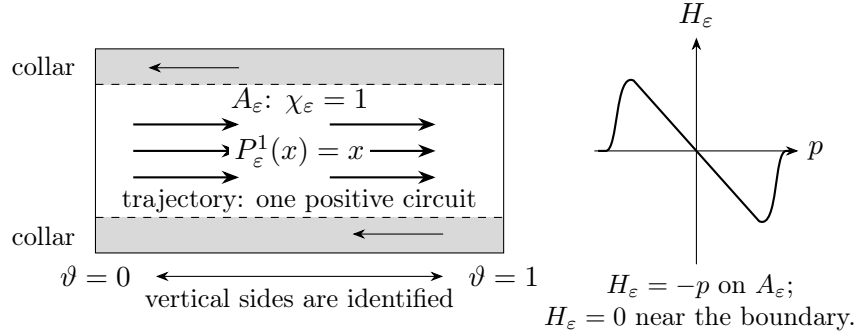


FIGURE 2. The finger-pushing isotopy on an annular slice and its Hamiltonian. The vertical sides are identified. Every point of A_ε returns at time one after one positive circuit. The transition region $N \setminus A_\varepsilon$ has measure tending to zero.

On $A_\varepsilon = S^1 \times K_\varepsilon$, the vector field is ∂_ϑ , so its flow has the same full-circuit property. Extension by zero gives a Hamiltonian isotopy on M . All cutoff effects lie in $N \setminus A_\varepsilon$, whose measure tends to zero. See Figure 2.

We apply the appropriate construction in every tube. Write $A_{j,s,\varepsilon}$ for its core and $f_{j,s,\varepsilon}$ for its time-one map, where $s \in \{a, b\}$. Now, let

$$\rho_{j,\varepsilon} : F_2 \rightarrow \mathcal{T}_M$$

send a, b to $f_{j,a,\varepsilon}, f_{j,b,\varepsilon}$. Disjoint supports imply that

$$\rho_\varepsilon(w_1, \dots, w_k) = \rho_{1,\varepsilon}(w_1) \cdots \rho_{k,\varepsilon}(w_k)$$

is a homomorphism. Set

$$B_\varepsilon = \bigcup_{j=1}^k \bigcup_{s \in \{a,b\}} (N_{j,s} \setminus A_{j,s,\varepsilon}).$$

Then $\mu(B_\varepsilon) \rightarrow 0$. Every generator fixes $M \setminus B_\varepsilon$ pointwise, so B_ε is invariant. Up to boundaries of measure zero, its complement is partitioned by

$$U_{j,\varepsilon} = A_{j,a,\varepsilon} \cap A_{j,b,\varepsilon}, \quad V_{j,a,\varepsilon} = A_{j,a,\varepsilon} \setminus N_{j,b}, \quad V_{j,b,\varepsilon} = A_{j,b,\varepsilon} \setminus N_{j,a},$$

$$W = M \setminus \bigcup_{j=1}^k (N_{j,a} \cup N_{j,b}).$$

For $x \in U_{j,\varepsilon}$, use a path c_x from the common vertex to x inside the contractible tube intersection to identify both generator trajectories. Hence the same conjugating element $u_x \in G$ works for both generators. Since all time-one maps fix x , the cocycle identity gives

$$\gamma(\rho_\varepsilon(w), x) = u_x i(w_j) u_x^{-1} \quad (w = (w_1, \dots, w_k) \in F_2^k).$$

On $V_{j,a,\varepsilon}$ only the a -isotopy moves x , giving a conjugate of $i \circ h_a \circ p_j$. Similarly, on $V_{j,b,\varepsilon}$ we obtain $i \circ h_b \circ p_j$, and on W the cocycle is trivial. The conjugators

can be chosen measurably by enumerating the countable group G and selecting the first element that gives the required identities on the generators.

Finally, exhaustion of the tubes by their cores gives

$$\mu(U_{j,\varepsilon}) \longrightarrow \mu(N_{j,a} \cap N_{j,b}) = \lambda_j > 0.$$

All isotopies have compact support in $\text{int } M$. The volume-preserving construction takes values in $\text{Diff}_0(M, \mu)$ and hence in $\text{Homeo}_0(M, \mu)$; the Hamiltonian construction takes values in $\text{Ham}(M, \omega)$ and hence in $\text{Symp}_0(M, \omega)$. $\square$

We recall the notation needed for the next proposition. Let $\mathcal{T}_M$ be one of the transformation groups in Theorem 1. Fix the embedding $i : F_2 \hookrightarrow G$ and the linear maps

$$J_s : \overline{H}_b^s(F_2) \longrightarrow \overline{H}_b^s(G), \quad i^* J_s = \text{id} \quad (s = 2, 3),$$

chosen in Lemma 5 or Lemma 6. Let $\theta \in \overline{H}_b^2(F_2)$ be the fixed nonzero finite linear combination of Brooks classes from Lemma 7. Recall that

$$J_2(\theta) \in \overline{EH}_b^2(G), \quad \theta \smile u = 0 \quad (u \in \overline{H}_b^s(F_2), \quad s \geq 1).$$

Fix integers $m \geq 0$ and $q \in \{2, 3\}$, and put $k = m + 1$ and $d = 2m + q$. Applying J_2 and J_q to the individual factors in Proposition 9 defines the linear map

$$\begin{aligned} \tilde{T}_m^q : \overline{H}_b^q(F_2) &\longrightarrow \overline{H}_b^d(G^k), \\ \tilde{T}_m^q(v) &= \text{Sym}_k(J_2(\theta)^{\times m} \times J_q(v)). \end{aligned}$$

Here $J_2(\theta)^{\times m}$ is the external product of m copies of $J_2(\theta)$. The operator Sym_k is the unnormalized sum of coordinate pullbacks fixed above. If $m = 0$, then $\tilde{T}_0^q = J_q$. Write $i^k : F_2^k \hookrightarrow G^k$ for the product embedding $i^k(w_1, \dots, w_k) = (i(w_1), \dots, i(w_k))$, so $i^2 = i \times i$. Naturality gives

$$(i^k)^* \tilde{T}_m^q(v) = \text{Sym}_k(\theta^{\times m} \times v) = T_m^q(v),$$

where T_m^q is the map of Proposition 9.

Fix the homomorphisms $\rho_\varepsilon : F_2^k \rightarrow \mathcal{T}_M$ and the subsets $B_\varepsilon, U_{j,\varepsilon}$ supplied by Lemma 11. Recall that $\delta_\varepsilon = \mu(B_\varepsilon) \rightarrow 0$ and $\lambda_{j,\varepsilon} = \mu(U_{j,\varepsilon}) \rightarrow \lambda_j > 0$. Put

$$(29) \quad c_{k,\varepsilon} = \prod_{j=1}^k \lambda_{j,\varepsilon}, \quad C_k = \prod_{j=1}^k \lambda_j > 0.$$

Thus $c_{k,\varepsilon} \rightarrow C_k$. We use the product transfer $\bar{\Gamma}_{b,k}^d : \overline{H}_b^d(G^k) \rightarrow \overline{H}_b^d(\mathcal{T}_M)$ of Lemma 10.

Proposition 12. *With the notation above, for every $\xi \in \text{im } \tilde{T}_m^q$ and every sufficiently small $\varepsilon > 0$,*

$$(30) \quad \left\| \rho_\varepsilon^* \bar{\Gamma}_{b,k}^d(\xi) - k! C_k (i^k)^* \xi \right\| \leq (k\delta_\varepsilon + k! |c_{k,\varepsilon} - C_k|) \|\xi\|,$$

where $c_{k,\varepsilon}$ and C_k are defined in (29). In particular, the left-hand side tends to zero for each fixed ξ .

Proof. We compute the pullback directly from the transfer formula (25), using the partition and the associated homomorphisms of Lemma 11. We first treat two coordinates, identify the nonzero contributions, and estimate the integral over the exceptional set. We then give the argument for arbitrary k .

The case $k = 2$. Here $m = 1$, $d = 2 + q$, and $q \in \{2, 3\}$. Let

$$v \in \overline{H}_b^q(F_2), \quad \xi = \tilde{T}_1^q(v) = J_2(\theta) \times J_q(v) + J_q(v) \times J_2(\theta).$$

Recall that restriction along $i \times i$ is the map

$$(i \times i)^* : \overline{H}_b^d(G \times G) \longrightarrow \overline{H}_b^d(F_2 \times F_2).$$

The maps J_2, J_3 chosen in Lemma 5 or Lemma 6 are right inverses to restriction along i :

$$i^* J_s = \text{id} \quad (s = 2, 3),$$

as recorded in (11). In particular, $i^* J_2(\theta) = \theta$ and $i^* J_q(v) = v$. Naturality of the external product therefore gives

$$(i \times i)^* \xi = (i^* J_2(\theta)) \times (i^* J_q(v)) + (i^* J_q(v)) \times (i^* J_2(\theta)) = \theta \times v + v \times \theta.$$

Let $\tau : F_2 \times F_2 \rightarrow F_2 \times F_2$ exchange the factors. Then $\tau^*((i \times i)^* \xi) = (i \times i)^* \xi$; there is no sign when the factors are exchanged, since the degree of θ is two. For $k = 2$, the homomorphism ρ_ε of Lemma 11 is

$$\rho_\varepsilon : F_2 \times F_2 \longrightarrow \mathcal{T}_M, \quad \rho_\varepsilon(g, h) = \rho_{1,\varepsilon}(g) \rho_{2,\varepsilon}(h).$$

Thus g determines the pushes in $N_{1,a} \cup N_{1,b}$, and h determines the pushes in the disjoint union $N_{2,a} \cup N_{2,b}$. Set

$$\mathcal{E}_\varepsilon = (B_\varepsilon \times M) \cup (M \times B_\varepsilon).$$

The set $\mathcal{E}_\varepsilon$ is invariant under the diagonal action of $\rho_\varepsilon(F_2^2)$, since B_ε is invariant. As $\mu(M) = 1$,

$$\mu^2(\mathcal{E}_\varepsilon) = 2\delta_\varepsilon - \delta_\varepsilon^2 \leq 2\delta_\varepsilon.$$

For a homogeneous bounded d -cochain c on G^2 , denote the integral over this subset by

$$(\mathcal{R}_\varepsilon c)(w_0, \dots, w_d) = \int_{\mathcal{E}_\varepsilon} c(\gamma_2(\rho_\varepsilon(w_0), \mathbf{x}), \dots, \gamma_2(\rho_\varepsilon(w_d), \mathbf{x})) d\mu^2(\mathbf{x}).$$

Here $w_0, \dots, w_d \in F_2^2$. The cocycle identity (23), followed by the measure-preserving change of variable $\mathbf{x} \mapsto \rho_\varepsilon(w)\mathbf{x}$ for fixed $w \in F_2^2$, proves right invariance of this cochain. The change of variable preserves $\mathcal{E}_\varepsilon$. The differential commutes with the integral, so $\mathcal{R}_\varepsilon$ is a cochain map. Moreover,

$$\|\mathcal{R}_\varepsilon c\|_\infty \leq 2\delta_\varepsilon \|c\|_\infty.$$

It therefore induces a map, denoted by the same symbol, on reduced bounded cohomology, and

$$\|\mathcal{R}_\varepsilon(\xi)\| \leq 2\delta_\varepsilon \|\xi\|.$$

Thus all pairs with a coordinate in B_ε are accounted for by this error term.

Up to a null set, the complement of $\mathcal{E}_\varepsilon$ is partitioned into products $C = X \times Y$, where X, Y belong to $\{U_{1,\varepsilon}, U_{2,\varepsilon}, V_{1,a,\varepsilon}, V_{1,b,\varepsilon}, V_{2,a,\varepsilon}, V_{2,b,\varepsilon}, W\}$. Every time-one map fixes every point in C . It follows that for $(x, y) \in C$, the map

$$w \mapsto \gamma_2(\rho_\varepsilon(w), (x, y))$$

is a homomorphism. By Lemma 11, it has the form

$$w \mapsto u_{x,y} \varphi_C(w) u_{x,y}^{-1},$$

where $\varphi_C : F_2^2 \rightarrow G^2$ is independent of $(x, y) \in C$, and $u_{x,y}$ is measurable.

Inner automorphisms induce the identity on bounded cohomology, through a cochain homotopy bounded independently of the conjugating element; see [30, Lemma A.2]. Integrating this homotopy shows that the contribution of C is $\mu^2(C) \varphi_C^*$, also on reduced bounded cohomology. We therefore omit the conjugations in the following calculations.

If X or Y is one of the subsets $V_{j,a,\varepsilon}$, $V_{j,b,\varepsilon}$, or W , the corresponding coordinate of φ_C factors through a cyclic or trivial group. For any external product of positive-degree classes,

$$\varphi_C^*(\tilde{\alpha} \times \tilde{\beta}) = \varphi_{C,1}^* \tilde{\alpha} \smile \varphi_{C,2}^* \tilde{\beta},$$

where $\varphi_{C,1}, \varphi_{C,2} : F_2^2 \rightarrow G$ are its coordinate homomorphisms. The factor corresponding to that cyclic or trivial coordinate is zero, since the positive-degree bounded cohomology of an amenable group vanishes [21, Corollary 3.7]. Both summands of ξ are such external products. Hence all these subsets contribute zero to the pullback of ξ . The vanishing is in reduced bounded cohomology.

It remains to compute the four products

$$U_{1,\varepsilon} \times U_{2,\varepsilon}, \quad U_{2,\varepsilon} \times U_{1,\varepsilon}, \quad U_{1,\varepsilon} \times U_{1,\varepsilon}, \quad U_{2,\varepsilon} \times U_{2,\varepsilon}.$$

For this calculation let $\tilde{\alpha} \in \overline{H}_b^r(G)$ and $\tilde{\beta} \in \overline{H}_b^s(G)$, with $r, s > 0$ and $r+s = d$, and put

$$\alpha = i^* \tilde{\alpha}, \quad \beta = i^* \tilde{\beta}.$$

If $x \in U_{1,\varepsilon}$ and $y \in U_{2,\varepsilon}$, the two associated homomorphisms depend on g and h separately. The associated homomorphism is

$$\varphi_{12} : F_2 \times F_2 \longrightarrow G \times G, \quad \varphi_{12}(g, h) = (i(g), i(h)),$$

and therefore

$$\varphi_{12}^*(\tilde{\alpha} \times \tilde{\beta}) = (i \times i)^*(\tilde{\alpha} \times \tilde{\beta}) = \alpha \times \beta.$$

For $x \in U_{2,\varepsilon}$ and $y \in U_{1,\varepsilon}$, we have $\varphi_{21}(g, h) = (i(h), i(g)) = (i \times i)\tau(g, h)$, and so $\varphi_{21}^*(\tilde{\alpha} \times \tilde{\beta}) = \tau^*(\alpha \times \beta)$.

Now suppose that $x, y \in U_{1,\varepsilon}$. Both associated homomorphisms depend on g , while the pushes determined by h are supported in the disjoint second neighborhood. Thus

$$\varphi_{11} : F_2 \times F_2 \longrightarrow G \times G, \quad \varphi_{11}(g, h) = (i(g), i(g)).$$

Let $p_1, p_2 : F_2 \times F_2 \rightarrow F_2$ be the projections, and let $\Delta : F_2 \rightarrow F_2 \times F_2$ be given by $\Delta(g) = (g, g)$. Then $F_2 \times F_2 \xrightarrow{p_1} F_2 \xrightarrow{\Delta} F_2 \times F_2 \xrightarrow{i \times i} G \times G$ is a factorization of φ_{11} . Since $\alpha \times \beta = p_1^* \alpha \smile p_2^* \beta$ and $p_1 \Delta = p_2 \Delta = \text{id}_{F_2}$, naturality gives $\Delta^*(\alpha \times \beta) = (p_1 \Delta)^* \alpha \smile (p_2 \Delta)^* \beta = \alpha \smile \beta$. Consequently,

$$\varphi_{11}^*(\tilde{\alpha} \times \tilde{\beta}) = p_1^* \Delta^*(i \times i)^*(\tilde{\alpha} \times \tilde{\beta}) = p_1^* \Delta^*(\alpha \times \beta) = p_1^*(\alpha \smile \beta).$$

The same calculation on $U_{2,\varepsilon} \times U_{2,\varepsilon}$ gives $p_2^*(\alpha \smile \beta)$.

The measure of each of the first two products is $c_{2,\varepsilon} = \lambda_{1,\varepsilon} \lambda_{2,\varepsilon}$, while the measures of the last two are $\lambda_{1,\varepsilon}^2$ and $\lambda_{2,\varepsilon}^2$. Including the error term, the calculation gives

$$\begin{aligned} \rho_\varepsilon^* \bar{\Gamma}_{b,2}^d(\tilde{\alpha} \times \tilde{\beta}) &= c_{2,\varepsilon}(\alpha \times \beta + \tau^*(\alpha \times \beta)) \\ &\quad + \lambda_{1,\varepsilon}^2 p_1^*(\alpha \smile \beta) + \lambda_{2,\varepsilon}^2 p_2^*(\alpha \smile \beta) + \mathcal{R}_\varepsilon(\tilde{\alpha} \times \tilde{\beta}). \end{aligned}$$

Figure 3 shows the four products and their pullbacks.

$x_2 \in U_2$	$(i(g_1), i(g_2))$ external product	$(i(g_2), i(g_2))$ $p_2^*(\alpha \smile \beta) = 0$	Each off-diagonal product has measure $\lambda_{1,\varepsilon} \lambda_{2,\varepsilon}$. Their sum gives symmetrization.
$x_2 \in U_1$	$(i(g_1), i(g_1))$ $p_1^*(\alpha \smile \beta) = 0$	$(i(g_2), i(g_1))$ $\tau^*(\alpha \times \beta)$	
	$x_1 \in U_1$	$x_1 \in U_2$	

FIGURE 3. The four products $U_{j,\varepsilon} \times U_{l,\varepsilon}$ for $k = 2$. On the shaded products the coordinate homomorphism factors through the diagonal, giving $p_j^*(\alpha \smile \beta)$. This class is zero when one factor is θ . The other two products give $\alpha \times \beta$ and $\tau^*(\alpha \times \beta)$, where τ exchanges the factors. Tuples with a coordinate in B_ε belong to $\mathcal{E}_\varepsilon$.

Apply this identity to the two summands of ξ . Their restrictions are $\theta \times v$ and $v \times \theta$. By Lemma 7 and graded commutativity,

$$\theta \smile v = 0, \quad v \smile \theta = (-1)^{2q} \theta \smile v = 0.$$

Hence the two products in which both points belong to the same $U_{j,\varepsilon}$ contribute zero. The remaining two products give

$$\rho_\varepsilon^* \bar{\Gamma}_{b,2}^d(\xi) = c_{2,\varepsilon}((i \times i)^* \xi + \tau^*((i \times i)^* \xi)) + \mathcal{R}_\varepsilon(\xi) = 2c_{2,\varepsilon}(i \times i)^* \xi + \mathcal{R}_\varepsilon(\xi).$$

Subtract $2C_2(i \times i)^*\xi$, where $C_2 = \lambda_1\lambda_2 > 0$. Since restriction is norm non-increasing,

$$\begin{aligned} & \|\rho_\varepsilon^* \bar{\Gamma}_{b,2}^d(\xi) - 2C_2(i \times i)^*\xi\| \\ & \leq 2|c_{2,\varepsilon} - C_2| \|(i \times i)^*\xi\| + \|\mathcal{R}_\varepsilon(\xi)\| \\ & \leq (2|c_{2,\varepsilon} - C_2| + 2\delta_\varepsilon) \|\xi\|. \end{aligned}$$

As $\varepsilon \rightarrow 0$, the right-hand side tends to zero by Lemma 11. This proves (30) for $k = 2$.

The same calculation also explains the injectivity argument. If $\bar{\Gamma}_{b,2}^d(\xi) = 0$, then the left-hand side of the last estimate equals $2C_2\|(i \times i)^*\xi\|$ for every ε . Thus $(i \times i)^*\xi = 0$ in reduced bounded cohomology. But $(i \times i)^*\xi = T_1^q(v)$, and Proposition 9 says that T_1^q is injective. Hence $v = 0$ and $\xi = 0$. Thus $\bar{\Gamma}_{b,2}^{q+2}$ is injective on $\text{im } \tilde{T}_1^q$ for $q = 2, 3$.

The general case. Let $k = m + 1$ be arbitrary and $\xi \in \text{im } \tilde{T}_m^q$. Take ρ_ε , B_ε , and $U_{j,\varepsilon}$ from Lemma 11, as fixed before the proposition. Set

$$\mathcal{E}_\varepsilon = \{(x_1, \dots, x_k) \in M^k : x_\ell \in B_\varepsilon \text{ for some } \ell\}.$$

This set is invariant, and the union bound gives

$$\mu^k(\mathcal{E}_\varepsilon) \leq \sum_{\ell=1}^k \mu^k(M^{\ell-1} \times B_\varepsilon \times M^{k-\ell}) = k\delta_\varepsilon.$$

Define

$$(\mathcal{R}_\varepsilon c)(w_0, \dots, w_d) = \int_{\mathcal{E}_\varepsilon} c(\gamma_k(\rho_\varepsilon(w_0), \mathbf{x}), \dots, \gamma_k(\rho_\varepsilon(w_d), \mathbf{x})) d\mu^k(\mathbf{x}).$$

The cochain-map argument above uses only invariance of $\mathcal{E}_\varepsilon$ and the cocycle identity, so it applies without change. The norm of this cochain map is at most $k\delta_\varepsilon$, and its induced map on reduced bounded cohomology satisfies

$$(31) \quad \|\mathcal{R}_\varepsilon(\xi)\| \leq k\delta_\varepsilon \|\xi\|.$$

Outside $\mathcal{E}_\varepsilon$ we partition M^k into products C of the sets in Lemma 11. On each such product the associated homomorphism has the form $w \mapsto u_{\mathbf{x}} \varphi_C(w) u_{\mathbf{x}}^{-1}$, where $\varphi_C : F_2^k \rightarrow G^k$ is independent of $\mathbf{x} \in C$. This product contributes $\mu^k(C) \varphi_C^* \xi$. If one coordinate lies in $V_{j,a,\varepsilon}$, $V_{j,b,\varepsilon}$, or W , the corresponding positive-degree factor restricts through an amenable group and is zero. These contributions therefore vanish exactly as for two coordinates.

It remains to consider

$$C_\tau = U_{\tau(1),\varepsilon} \times \dots \times U_{\tau(k),\varepsilon}, \quad \tau : \{1, \dots, k\} \rightarrow \{1, \dots, k\}.$$

The associated homomorphism is $\varphi_{C_\tau} = (i^k) \circ D_\tau$, where

$$D_\tau(w_1, \dots, w_k) = (w_{\tau(1)}, \dots, w_{\tau(k)}).$$

Thus its contribution is $\mu^k(C_\tau)D_\tau^*((i^k)^*\xi)$. For positive-degree classes $u_1, \dots, u_k$ on F_2 , the definition of the external product and naturality give (32)

$$D_\tau^*(u_1 \times \dots \times u_k) = D_\tau^*(p_1^*u_1 \smile \dots \smile p_k^*u_k) = p_{\tau(1)}^*u_1 \smile \dots \smile p_{\tau(k)}^*u_k.$$

This is the general form of the diagonal calculation above: all factors assigned to one $U_{j,\varepsilon}$ become an internal cup product on the j th free-group factor.

We claim that $D_\tau^*((i^k)^*\xi) = 0$ whenever τ is not injective. Each summand of $(i^k)^*\xi$ contains m copies of θ and one class $v \in \overline{H}_b^q(F_2)$. Group the factors with the same value of $\tau(\ell)$ in (32). Graded commutativity permits this regrouping without a sign, since all degrees are even except possibly for the single degree-three factor. A group of indices containing at least two elements contains a θ , because at most one factor differs from θ . After reordering, its internal product is $\theta \smile \zeta$ for a positive-degree class ζ , and hence is zero by Lemma 7. Every summand therefore vanishes, which proves the claim. As in the two-coordinate case, this argument uses cup-product vanishing on F_2 .

If τ is injective, it is a permutation. Moreover,

$$\mu^k(C_\tau) = \lambda_{\tau(1),\varepsilon} \cdots \lambda_{\tau(k),\varepsilon} = c_{k,\varepsilon},$$

independently of the permutation. Summing these $k!$ contributions and adding the error term gives

$$(33) \quad \rho_\varepsilon^* \overline{\Gamma}_{b,k}^d(\xi) = c_{k,\varepsilon} \text{Sym}_k((i^k)^*\xi) + \mathcal{R}_\varepsilon(\xi) \quad \text{in } \overline{H}_b^d(F_2^k).$$

By construction $(i^k)^*\xi$ is symmetric, so $\text{Sym}_k((i^k)^*\xi) = k!(i^k)^*\xi$. Subtract $k!C_k(i^k)^*\xi$ from (33). Using (31) and that restriction is norm non-increasing, we obtain

$$\begin{aligned} \|\rho_\varepsilon^* \overline{\Gamma}_{b,k}^d(\xi) - k!C_k(i^k)^*\xi\| &\leq k!|c_{k,\varepsilon} - C_k| \|(i^k)^*\xi\| + \|\mathcal{R}_\varepsilon(\xi)\| \\ &\leq (k!|c_{k,\varepsilon} - C_k| + k\delta_\varepsilon) \|\xi\|. \end{aligned}$$

Finally, Lemma 11 gives $\delta_\varepsilon \rightarrow 0$ and $c_{k,\varepsilon} = \prod_j \lambda_{j,\varepsilon} \rightarrow \prod_j \lambda_j = C_k > 0$. This proves the proposition. $\square$

Proof of Theorem 1. We use the restriction-and-limit argument of [10, proofs of Theorems A and B]; see also [35, Lemma 4.1(i)]. Choose i, J_2, J_3 from Lemma 5 or Lemma 6, according to the hypothesis being used, and put $K_2 = \ker(\bar{c}_G^2 \circ J_2)$ as in (11). Choose θ, z from Lemma 7.

We first define the maps asserted in part (b). For $m \geq 0$ and $q \in \{2, 3\}$, put $d = 2m + q$ and $k = m + 1$, and set

$$\Phi_{m,q}(v) := \frac{1}{k!} \overline{\Gamma}_{b,k}^d(\tilde{T}_m^q(v)).$$

The domain is K_2 when $(m, q) = (0, 2)$ and $\overline{H}_b^q(F_2)$ in every other case. In particular, the two maps in the initial degrees are

$$\Phi_{0,2} = (\overline{\Gamma}_{b,1}^2 \circ J_2)|_{K_2}, \quad \Phi_{0,3} = \overline{\Gamma}_{b,1}^3 \circ J_3.$$

For $m \geq 1$, the definition is equivalently

$$\Phi_{m,q}(v) = \frac{1}{(m+1)!} \overline{\Gamma}_{b,m+1}^{2m+q} \left(\text{Sym}_{m+1}(J_2(\theta)^{\times m} \times J_q(v)) \right).$$

By (26), this is also the transfer of the ordered product $J_2(\theta)^{\times m} \times J_q(v)$. Values of these maps lie in $\overline{H}_b^{2m+q}(\mathcal{T}_M)$; we will verify that their images are exact. Using $i^* J_q = \text{id}$ from (11) and naturality of the external product, we obtain

$$(i^k)^* (\tilde{T}_m^q(v)) = \text{Sym}_k(\theta^{\times m} \times v) = T_m^q(v).$$

Since T_m^q is injective, $\tilde{T}_m^q$ is injective. Moreover, restriction $(i^k)^*$ is injective on $\text{im } \tilde{T}_m^q$.

We first prove injectivity of the transfer on that image. Let $\xi \in \text{im } \tilde{T}_m^q$ and suppose $\overline{\Gamma}_{b,k}^d(\xi) = 0$. Then $\rho_\varepsilon^* \overline{\Gamma}_{b,k}^d(\xi) = 0$ for every ε . By Proposition 12,

$$k! C_k \| (i^k)^* \xi \| = \| \rho_\varepsilon^* \overline{\Gamma}_{b,k}^d(\xi) - k! C_k (i^k)^* \xi \| \longrightarrow 0.$$

The number on the left is independent of ε , and $C_k > 0$. Thus $(i^k)^* \xi$ has norm zero. Since $(i^k)^* \xi$ is a class in reduced bounded cohomology, it is zero. Injectivity of $(i^k)^*$ on $\text{im } \tilde{T}_m^q$ gives $\xi = 0$. It follows that $\Phi_{m,q}$ is injective.

Pullback by ρ_ε is norm non-increasing, so

$$\| \overline{\Gamma}_{b,k}^d(\xi) \| \geq \lim_{\varepsilon \rightarrow 0} \| \rho_\varepsilon^* \overline{\Gamma}_{b,k}^d(\xi) \| = k! C_k \| (i^k)^* \xi \|.$$

Apply this to $\xi = \tilde{T}_m^q(v)$, divide by $k!$, and use the bounded left inverse $Q_{m,q}$ from (21). We obtain

$$\| \Phi_{m,q}(v) \| \geq C_k \| T_m^q(v) \| \geq \frac{C_k}{\| Q_{m,q} \|} \| v \|.$$

We next show that the images specified in the theorem lie in reduced exact bounded cohomology. If $m \geq 1$, each summand of $\tilde{T}_m^q(v)$ contains the factor $J_2(\theta)$, which has zero comparison image. By compatibility of comparison with external products, the comparison image of this summand is zero. Permutations are homomorphism pullbacks and preserve exactness; summing gives an ordinary primitive for the symmetric class. Thus

$$\tilde{T}_m^q(v) \in \overline{EH}_b^{2m+q}(G^k) \quad (m \geq 1, v \in \overline{H}_b^q(F_2)).$$

Its transfer belongs to reduced exact bounded cohomology by Lemma 10.

For $m = 0, q = 3$, the section has $J_3(\overline{H}_b^3(F_2)) \subset \overline{EH}_b^3(G)$, so the same transfer argument proves exactness. For $m = 0, q = 2$, restrict to $K_2 = \ker(\bar{c}_G^2 J_2)$. Under (i) this has codimension at most one in the infinite-dimensional space

$\overline{H}_b^2(F_2)$ and is therefore infinite-dimensional; under (ii) or (iii) it is all of $\overline{H}_b^2(F_2)$. This proves part (b), with the domains and codomains stated in (2)–(3). The two degree patterns $2m + 2$ and $2m + 3$, $m \geq 0$, cover every integer $d \geq 2$. Part (a) now follows from injectivity and the infinite-dimensionality of $\overline{H}_b^2(F_2)$ and $\overline{H}_b^3(F_2)$.

Finally, for $\text{Diff}_0(M, \mu)$ we use the volume-preserving isotopies (27). These are also isotopies in $\text{Homeo}_0(M, \mu)$. For $\text{Ham}(M, \omega)$ we use the Hamiltonians (28); their flows are also isotopies in $\text{Symp}_0(M, \omega)$. Each group preserves the measure, so Lemma 10 defines its transfer. The time-one maps define homomorphisms for which Proposition 12 gives the required pullback estimate. This completes the proof. $\square$

4. PROOF OF THEOREM 2

4.1. Disc case. We first prove the disc case of Theorem 2; the sphere proof follows it. The positive-genus surfaces are treated afterward in Theorem 3. In the disc proof, let $\mathcal{T}_{\mathbb{D}}$ denote any one of its four transformation groups. Multiplying the area form by a positive constant changes none of these groups, so assume $\int_{\mathbb{D}} \omega = 1$ and write μ for this area measure. All configuration-space measures below are restrictions of product measures.

Write $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$ and let $P_3 \triangleleft B_3$ be the pure braid group. Put $\Delta^2 = (\sigma_1 \sigma_2)^3$. The standard decomposition is

$$P_3 = \langle \sigma_1^2, \sigma_2^2 \rangle \times \langle \Delta^2 \rangle \cong F_2 \times \mathbb{Z}.$$

Let $j : F_2 \hookrightarrow B_3$ be the inclusion given by $j(a) = \sigma_1^2$ and $j(b) = \sigma_2^2$. These braid-group facts are recalled in [29, Sections 2 and 4.1].

Lemma 13. *For $q = 2, 3$, restriction gives an isometric embedding*

$$j^* : \overline{H}_b^q(B_3) \longrightarrow \overline{H}_b^q(F_2).$$

The domain $\overline{H}_b^q(B_3)$ is infinite-dimensional and equals $\overline{EH}_b^q(B_3)$. There are a fixed nonzero class $\alpha \in \overline{H}_b^2(B_3)$ and an ℓ^1 -cycle $z \in Z_2^{\ell^1}(F_2)$ with

$$(34) \quad \langle j^* \alpha, z \rangle = 1.$$

Proof. We first prove infinite-dimensionality independently of the restriction map. Let $p : B_3 \rightarrow \text{PSL}(2, \mathbb{Z})$ denote the central quotient homomorphism. Its kernel is $Z(B_3) = \langle \Delta^2 \rangle \cong \mathbb{Z}$, and $\text{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ is a non-elementary hyperbolic group. Since the kernel is amenable, we obtain an isometric isomorphism

$$p^* : \overline{H}_b^q(\text{PSL}(2, \mathbb{Z})) \longrightarrow \overline{H}_b^q(B_3) \quad (q \geq 2);$$

see [29, Theorem 2.2] and [21, Chapter 3]. The second bounded cohomology of this quotient is infinite-dimensional by [16]; in degree two it coincides with

reduced bounded cohomology. In degree three, [22, Corollary 3] gives

$$\dim \overline{EH}_b^3(\mathrm{PSL}(2, \mathbb{Z})) = \infty.$$

Pullback preserves exactness by naturality of the comparison map. The isometry of p^* therefore implies that $\overline{EH}_b^3(B_3)$ is infinite-dimensional. In particular, $\overline{H}_b^q(B_3)$ is infinite-dimensional for $q = 2, 3$. See also [29, proof of Corollary 1.2] for the degree-three argument.

For exactness, recall Arnol'd's computation of the ordinary cohomology of braid groups [2]. In particular, $H^r(B_3; \mathbb{R}) = 0$ ($r > 1$). For B_3 , this also follows from the one-relator computation in [13, Chapter II, Section 4, Example 3]. Consequently, the comparison map $c_{B_3}^q$ is zero for every $q \geq 2$, hence

$$EH_b^q(B_3) = H_b^q(B_3), \quad \overline{EH}_b^q(B_3) = \overline{H}_b^q(B_3) \quad (q \geq 2).$$

We now prove the assertion about restriction. Restriction from B_3 to its finite-index subgroup P_3 is isometric onto its image: averaging over the cosets is a norm non-increasing left inverse to restriction. The reduced bounded cohomology of $P_3 = F_2 \times \mathbb{Z}$ is identified isometrically with that of F_2 , with the inverse given by restriction to the free factor. Composing proves the assertion about j^* ; see [29, Theorems 2.1–2.2] and [21, Chapter 3]. Restriction is not asserted to be onto the entire space $\overline{H}_b^q(F_2)$. Its image is infinite-dimensional by the first part of the proof.

Finally, choose $0 \neq \alpha \in \overline{H}_b^2(B_3)$. Isometry makes $j^*\alpha$ nonzero. The degree-two duality used in Lemma 7 gives an ℓ^1 -cycle with nonzero evaluation. Rescale it to obtain (34). $\square$

Let $X_3(\mathbb{D})$ denote the space of ordered configurations of three points in $\mathrm{int} \mathbb{D}$:

$$X_3(\mathbb{D}) = \{(x_1, x_2, x_3) \in (\mathrm{int} \mathbb{D})^3 : x_i \neq x_j \text{ for } i \neq j\}.$$

It has measure one. A surface transformation acts on it diagonally, by $f(x_1, x_2, x_3) = (f(x_1), f(x_2), f(x_3))$. In the product $X_3(\mathbb{D})^k$, points belonging to different triples are not required to be distinct. In particular, we do not identify this space with $X_{3k}(\mathbb{D})$.

Lemma 14. *For every $k \geq 1$ and $d \geq 0$, the braid cocycle for the diagonal action on $X_3(\mathbb{D})^k$ induces the product Gambaudo–Ghys map*

$$\overline{\Gamma}_{b, \mathbb{D}, k}^d : \overline{H}_b^d(B_3^k) \longrightarrow \overline{H}_b^d(\mathcal{T}_{\mathbb{D}}), \quad \|\overline{\Gamma}_{b, \mathbb{D}, k}^d\| \leq 1.$$

For $k = 1$ we write $\overline{\Gamma}_{b, \mathbb{D}}^d$ for this map. The product maps satisfy the permutation identity (26) with $X = X_3(\mathbb{D})$. For each of the smooth groups in Theorem 2, these maps preserve reduced exact bounded cohomology. If $d_1, \dots, d_k \geq 0$ and $u_j \in \overline{H}_b^{d_j}(B_3)$, then

$$(35) \quad \overline{\Gamma}_{b, \mathbb{D}, k}^{d_1 + \dots + d_k}(u_1 \times \dots \times u_k) = \overline{\Gamma}_{b, \mathbb{D}}^{d_1}(u_1) \smile \dots \smile \overline{\Gamma}_{b, \mathbb{D}}^{d_k}(u_k).$$

Proof. We use the construction of [11, Section 2]. Fix a base configuration $\mathbf{z} = (z_1, z_2, z_3) \in X_3(\mathbb{D})$. For almost every $\mathbf{x} = (x_1, x_2, x_3) \in X_3(\mathbb{D})$, the straight-line path

$$\alpha_{\mathbf{z}, \mathbf{x}}(t) = ((1-t)z_1 + tx_1, (1-t)z_2 + tx_2, (1-t)z_3 + tx_3), \quad 0 \leq t \leq 1,$$

lies in $X_3(\mathbb{D})$ and joins $\mathbf{z}$ to $\mathbf{x}$. On the exceptional set of measure zero, choose reference paths measurably. Let $\alpha_{\mathbf{x}, \mathbf{z}} = \alpha_{\mathbf{z}, \mathbf{x}}^{-1}$ denote the reverse path.

For $f \in \mathcal{T}_{\mathbb{D}}$, choose an isotopy $\{f_t\}_{t \in [0,1]}$ from the identity to f and write

$$f_t(\mathbf{x}) = (f_t(x_1), f_t(x_2), f_t(x_3)).$$

Define

$$\gamma_{\mathbb{D}}(f, \mathbf{x}) = [\alpha_{\mathbf{z}, \mathbf{x}} * \{f_t(\mathbf{x})\}_{t \in [0,1]} * \alpha_{f(\mathbf{x}), \mathbf{z}}]^{-1} \in \pi_1(X_3(\mathbb{D}), \mathbf{z}) = P_3 \subset B_3.$$

The homotopy class is independent of the chosen isotopy, since the group of homeomorphisms of the disc fixing its boundary is contractible. This defines a measurable cocycle, with values in B_3 , satisfying

$$\gamma_{\mathbb{D}}(fh, \mathbf{x}) = \gamma_{\mathbb{D}}(f, h(\mathbf{x}))\gamma_{\mathbb{D}}(h, \mathbf{x})$$

for every $f, h \in \mathcal{T}_{\mathbb{D}}$ and almost every $\mathbf{x} \in X_3(\mathbb{D})$.

We now define the product transfer on cochains. A point of $X_3(\mathbb{D})^k$ is a tuple $\mathbf{X} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)})$, with each $\mathbf{x}^{(j)}$ an ordered triple. The action is

$$f\mathbf{X} = (f\mathbf{x}^{(1)}, \dots, f\mathbf{x}^{(k)}), \quad f\mathbf{x}^{(j)} = (f(x_1^{(j)}), f(x_2^{(j)}), f(x_3^{(j)})).$$

This is an action of the group $\mathcal{T}_{\mathbb{D}}$. It preserves the probability measure

$$\nu_k = (\mu^3|_{X_3(\mathbb{D})})^{\otimes k}.$$

Using these paths in each factor, define the product cocycle by

$$\begin{aligned} \gamma_{\mathbb{D}, k} : \mathcal{T}_{\mathbb{D}} \times X_3(\mathbb{D})^k &\longrightarrow P_3^k \subset B_3^k, \\ \gamma_{\mathbb{D}, k}(f, \mathbf{X}) &= (\gamma_{\mathbb{D}}(f, \mathbf{x}^{(1)}), \dots, \gamma_{\mathbb{D}}(f, \mathbf{x}^{(k)})). \end{aligned}$$

The preceding cocycle identity in each coordinate gives

$$\gamma_{\mathbb{D}, k}(fh, \mathbf{X}) = \gamma_{\mathbb{D}, k}(f, h\mathbf{X})\gamma_{\mathbb{D}, k}(h, \mathbf{X}).$$

Let $c : (B_3^k)^{d+1} \rightarrow \mathbb{R}$ be a bounded cochain invariant under right multiplication. Define

$$(\Gamma'_{b, \mathbb{D}, k} c)(f_0, \dots, f_d) = \int_{X_3(\mathbb{D})^k} c(\gamma_{\mathbb{D}, k}(f_0, \mathbf{X}), \dots, \gamma_{\mathbb{D}, k}(f_d, \mathbf{X})) d\nu_k(\mathbf{X}).$$

In particular, when $k = 1$ this is

$$(\Gamma'_{b, \mathbb{D}} c)(f_0, \dots, f_d) = \int_{X_3(\mathbb{D})} c(\gamma_{\mathbb{D}}(f_0, \mathbf{x}), \dots, \gamma_{\mathbb{D}}(f_d, \mathbf{x})) d\mu^3(\mathbf{x}).$$

The product integral is absolutely convergent. Since $\nu_k(X_3(\mathbb{D})^k) = 1$,

$$\|\Gamma'_{b, \mathbb{D}, k} c\|_{\infty} \leq \|c\|_{\infty}.$$

We check that it is a right-homogeneous cochain. For $h \in \mathcal{T}_{\mathbb{D}}$ the cocycle identity gives

$$\begin{aligned} (\Gamma'_{b,\mathbb{D},k}c)(f_0h, \dots, f_dh) &= \\ \int_{X_3(\mathbb{D})^k} c(\gamma_{\mathbb{D},k}(f_0, h\mathbf{X}), \gamma_{\mathbb{D},k}(h, \mathbf{X}), \dots, \gamma_{\mathbb{D},k}(f_d, h\mathbf{X}), \gamma_{\mathbb{D},k}(h, \mathbf{X})) d\nu_k(\mathbf{X}) &= \\ \int_{X_3(\mathbb{D})^k} c(\gamma_{\mathbb{D},k}(f_0, h\mathbf{X}), \dots, \gamma_{\mathbb{D},k}(f_d, h\mathbf{X})) d\nu_k(\mathbf{X}) &= (\Gamma'_{b,\mathbb{D},k}c)(f_0, \dots, f_d). \end{aligned}$$

The second equality uses right invariance of c , with the same multiplier $\gamma_{\mathbb{D},k}(h, \mathbf{X}) \in B_3^k$ in all $d+1$ arguments. The last uses the measure-preserving change of variable $\mathbf{Y} = h\mathbf{X}$.

We also check the differential. The homogeneous differential is the alternating sum obtained by deleting one argument, so

$$\begin{aligned} (\delta\Gamma'_{b,\mathbb{D},k}c)(f_0, \dots, f_{d+1}) &= \\ \sum_{\ell=0}^{d+1} (-1)^\ell \int_{X_3(\mathbb{D})^k} c(\gamma_{\mathbb{D},k}(f_0, \mathbf{X}), \dots, \widehat{\gamma_{\mathbb{D},k}(f_\ell, \mathbf{X})}, \dots, \gamma_{\mathbb{D},k}(f_{d+1}, \mathbf{X})) d\nu_k(\mathbf{X}) &= \\ \int_{X_3(\mathbb{D})^k} (\delta c)(\gamma_{\mathbb{D},k}(f_0, \mathbf{X}), \dots, \gamma_{\mathbb{D},k}(f_{d+1}, \mathbf{X})) d\nu_k(\mathbf{X}) &= \\ = (\Gamma'_{b,\mathbb{D},k}\delta c)(f_0, \dots, f_{d+1}). \end{aligned}$$

The hat means that the indicated argument is omitted. Thus $\delta\Gamma'_{b,\mathbb{D},k} = \Gamma'_{b,\mathbb{D},k}\delta$. We obtain a map on bounded cohomology, and its norm bound sends zero-seminorm classes to zero-seminorm classes. Its induced map on reduced bounded cohomology is $\bar{\Gamma}_{b,\mathbb{D},k}^d$. The construction applies to area-preserving homeomorphisms as well as smooth transformations.

For the smooth transformation groups, the finite-image property follows from [9, Lemma 2.1]: with the reference paths chosen there, for every fixed f there is a finite set $S_f \subset P_3$ such that

$$\gamma_{\mathbb{D}}(f, \mathbf{x}) \in S_f \quad \text{for almost every } \mathbf{x} \in X_3(\mathbb{D}).$$

We may use these reference paths, since the induced bounded transfer is independent of their choice. Consequently, $\gamma_{\mathbb{D},k}(f, \mathbf{X}) \in S_f^k$ for almost every $\mathbf{X} \in X_3(\mathbb{D})^k$.

For each fixed tuple $(f_0, \dots, f_d)$, the integrand in the transfer formula therefore has finite image almost everywhere, even when the cochain is unbounded. Since ν_k is finite, the same integral defines a map $\Gamma'_{\mathbb{D},k}$ on ordinary right-homogeneous cochains. The preceding cocycle calculation shows that this is a cochain map, and it agrees with $\Gamma'_{b,\mathbb{D},k}$ on bounded cochains.

In particular, if a bounded cocycle c has an ordinary primitive A , then

$$\Gamma'_{b,\mathbb{D},k}c = \Gamma'_{\mathbb{D},k}(\delta A) = \delta(\Gamma'_{\mathbb{D},k}A).$$

Thus the transfer preserves exact bounded cohomology and, by its norm bound, reduced exact bounded cohomology.

We finish by proving (35) on cochains, first for two factors. In the right-homogeneous convention, the cup product of a p -cochain c and a q -cochain e on the same group is

$$(c \smile e)(g_0, \dots, g_{p+q}) = c(g_0, \dots, g_p)e(g_p, \dots, g_{p+q}).$$

This is (4) under the homogeneous identification in Lemma 10. If c and e are cochains on B_3 , their external product is the cochain on $B_3 \times B_3$ given by

$$(c \times e)((g_0, h_0), \dots, (g_{p+q}, h_{p+q})) = c(g_0, \dots, g_p)e(h_p, \dots, h_{p+q}).$$

Let c, e be bounded; they need not be cocycles for the following identity. For $(\mathbf{x}, \mathbf{y}) \in X_3(\mathbb{D})^2$ we have

$$\gamma_{\mathbb{D},2}(f, (\mathbf{x}, \mathbf{y})) = (\gamma_{\mathbb{D}}(f, \mathbf{x}), \gamma_{\mathbb{D}}(f, \mathbf{y})).$$

For this computation, fix $\mathbf{f} = (f_0, \dots, f_{p+q})$ and write the two integrands as

$$\begin{aligned} c_{\mathbf{f}}(\mathbf{x}) &= c(\gamma_{\mathbb{D}}(f_0, \mathbf{x}), \dots, \gamma_{\mathbb{D}}(f_p, \mathbf{x})), \\ e_{\mathbf{f}}(\mathbf{y}) &= e(\gamma_{\mathbb{D}}(f_p, \mathbf{y}), \dots, \gamma_{\mathbb{D}}(f_{p+q}, \mathbf{y})). \end{aligned}$$

Substitution into the product transfer gives

$$\begin{aligned} (\Gamma'_{b,\mathbb{D},2}(c \times e))(f_0, \dots, f_{p+q}) &= \int_{X_3(\mathbb{D})^2} c_{\mathbf{f}}(\mathbf{x}) e_{\mathbf{f}}(\mathbf{y}) d\nu_2(\mathbf{x}, \mathbf{y}) \\ &= \left[\int_{X_3(\mathbb{D})} c_{\mathbf{f}}(\mathbf{x}) d\mu^3(\mathbf{x}) \right] \left[\int_{X_3(\mathbb{D})} e_{\mathbf{f}}(\mathbf{y}) d\mu^3(\mathbf{y}) \right] \\ &= (\Gamma'_{b,\mathbb{D}}c)(f_0, \dots, f_p)(\Gamma'_{b,\mathbb{D}}e)(f_p, \dots, f_{p+q}) \\ &= ((\Gamma'_{b,\mathbb{D}}c) \smile (\Gamma'_{b,\mathbb{D}}e))(f_0, \dots, f_{p+q}). \end{aligned}$$

Thus we have the actual cochain identity

$$\Gamma'_{b,\mathbb{D},2}(c \times e) = (\Gamma'_{b,\mathbb{D}}c) \smile (\Gamma'_{b,\mathbb{D}}e).$$

For completeness, let c_j be a bounded right-homogeneous d_j -cochain on B_3 , put $D_0 = 0$ and $D_j = d_1 + \dots + d_j$, and write $d = D_k$. For $\mathbf{h}_\ell = (h_\ell^{(1)}, \dots, h_\ell^{(k)}) \in B_3^k$, our external-product convention gives

$$(c_1 \times \dots \times c_k)(\mathbf{h}_0, \dots, \mathbf{h}_d) = \prod_{j=1}^k c_j(h_{D_{j-1}}^{(j)}, \dots, h_{D_j}^{(j)}).$$

For the fixed tuple $\mathbf{f} = (f_0, \dots, f_d)$, put

$$c_{j,\mathbf{f}}(\mathbf{x}) = c_j(\gamma_{\mathbb{D}}(f_{D_{j-1}}, \mathbf{x}), \dots, \gamma_{\mathbb{D}}(f_{D_j}, \mathbf{x})).$$

The same substitution and Fubini's theorem give

$$\begin{aligned}
(\Gamma'_{b,\mathbb{D},k}(c_1 \times \cdots \times c_k))(f_0, \dots, f_d) &= \int_{X_3(\mathbb{D})^k} \prod_{j=1}^k c_{j,\mathbf{f}}(\mathbf{x}^{(j)}) d\nu_k(\mathbf{X}) \\
&= \prod_{j=1}^k \left[\int_{X_3(\mathbb{D})} c_{j,\mathbf{f}}(\mathbf{x}) d\mu^3(\mathbf{x}) \right] = \prod_{j=1}^k (\Gamma'_{b,\mathbb{D}} c_j)(f_{D_{j-1}}, \dots, f_{D_j}) \\
&= ((\Gamma'_{b,\mathbb{D}} c_1) \smile \cdots \smile (\Gamma'_{b,\mathbb{D}} c_k))(f_0, \dots, f_d).
\end{aligned}$$

All integrands are bounded in absolute value by $\prod_j \|c_j\|_\infty$. Taking cocycles representing u_j proves the product identity on bounded cohomology. The transfer and product norm bounds, including (5), show that zero-seminorm classes give zero-seminorm terms on both sides, so the identity passes to reduced bounded cohomology and gives (35). $\square$

We next construct the simultaneous rotations used to compute the pullback of this transfer. The number of copies k is fixed in each construction. Choose k pairwise disjoint open discs $E_j \subseteq \text{int } \mathbb{D}$. Inside each E_j choose two overlapping round discs $N_{j,1}, N_{j,2}$ in area coordinates. The pairs are congruent, and their closures lie in E_j . Each of the three sets

$$N_{j,1} \setminus \overline{N_{j,2}}, \quad N_{j,1} \cap N_{j,2}, \quad N_{j,2} \setminus \overline{N_{j,1}}$$

has positive area. For any fixed k , this is possible in one area-coordinate chart by taking the radii sufficiently small. Figure 4 illustrates the case $k = 2$.

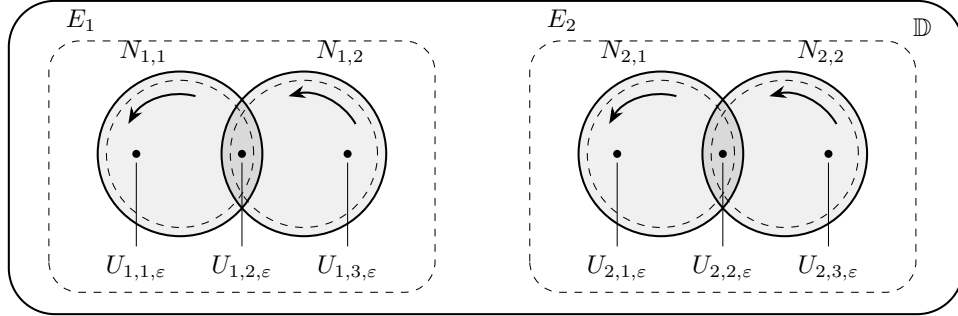


FIGURE 4. Two disjoint copies of the disc rotation construction. The dashed inner circles bound the regions rotated through one complete turn; the thin annuli between the circles are transition regions. A triple with one point in each of the three indicated subsets in a single E_j gives the free braid homomorphism. A triple cannot have two points in each of two different E_j .

For $s = 1, 2$, let $N_{j,s}^\varepsilon$ be the concentric disc of radius $R - \varepsilon$ inside $N_{j,s}$, for $\varepsilon > 0$ small. Choose a smooth radial function χ_ε , equal to one on $[0, R - \varepsilon]$,

equal to zero near R , and with $0 \leq \chi_\varepsilon \leq 1$. The isotopy

$$(r, \varphi) \mapsto (r, \varphi + 2\pi t \chi_\varepsilon(r)), \quad 0 \leq t \leq 1,$$

extended by the identity outside $N_{j,s}$, is Hamiltonian. Let $f_{j,s,\varepsilon}$ be its time-one map. It fixes $N_{j,s}^\varepsilon$ pointwise, but its isotopy makes each point in that disc travel once around the center.

Define $\rho_{j,\varepsilon} : F_2 \rightarrow \text{Ham}(\mathbb{D}, \omega)$ by $\rho_{j,\varepsilon}(a) = f_{j,1,\varepsilon}$, $\rho_{j,\varepsilon}(b) = f_{j,2,\varepsilon}$. Images belonging to different j have disjoint supports and commute. Hence

$$(36) \quad \rho_\varepsilon : F_2^k \longrightarrow \text{Ham}(\mathbb{D}, \omega), \quad \rho_\varepsilon(w_1, \dots, w_k) = \prod_{j=1}^k \rho_{j,\varepsilon}(w_j)$$

is a homomorphism, also viewed as taking values in any $\mathcal{T}_\mathbb{D}$. Put

$$B_\varepsilon = \bigcup_{j,s} (N_{j,s} \setminus N_{j,s}^\varepsilon), \quad b_\varepsilon = \mu(B_\varepsilon) \longrightarrow 0.$$

Each time-one map fixes $\mathbb{D} \setminus B_\varepsilon$ pointwise, so B_ε is invariant under the generated group. Define

$$\begin{aligned} U_{j,1,\varepsilon} &= (N_{j,1} \setminus \overline{N_{j,2}}) \setminus B_\varepsilon, \\ U_{j,2,\varepsilon} &= (N_{j,1} \cap N_{j,2}) \setminus B_\varepsilon, \\ U_{j,3,\varepsilon} &= (N_{j,2} \setminus \overline{N_{j,1}}) \setminus B_\varepsilon. \end{aligned}$$

Their areas are independent of j . Set

$$(37) \quad \lambda_\varepsilon = 6 \prod_{s=1}^3 \mu(U_{j,s,\varepsilon}), \quad \lambda = \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon > 0.$$

Let $p_j : F_2^k \rightarrow F_2$ denote projection onto the j th factor.

Lemma 15. *For every $d \geq 2$ and $u \in \overline{H}_b^d(B_3)$,*

$$(38) \quad \left\| \rho_\varepsilon^* \overline{\Gamma}_{b,\mathbb{D}}^d(u) - \lambda \sum_{j=1}^k p_j^* j^* u \right\| \leq (3b_\varepsilon + k|\lambda_\varepsilon - \lambda|) \|u\|.$$

In particular, the expression on the left tends to zero.

Proof. We give the complete decomposition of a single three-point configuration; the same decomposition applies simultaneously to all k copies. This is the local computation of [29, Lemma 4.2 and Section 4.3], in the disjoint setting of [35, proof of Theorem 4.3].

Let $\mathcal{E}_\varepsilon \subset X_3(\mathbb{D})$ consist of triples with at least one coordinate in B_ε . It is invariant under the diagonal action of the time-one maps, and

$$\mu^3(\mathcal{E}_\varepsilon) = 1 - (1 - b_\varepsilon)^3 \leq 3b_\varepsilon.$$

For a bounded cocycle c representing u , integration over $\mathcal{E}_\varepsilon$ is a bounded cochain map: invariance gives the measure-preserving change of variables

needed for right homogeneity, and the differential commutes with integration. Its value has norm at most $3b_\varepsilon\|c\|_\infty$.

Every point of a triple outside $\mathcal{E}_\varepsilon$ is fixed by each time-one map. Consequently $w \mapsto \gamma_{\mathbb{D}}(\rho_\varepsilon(w), \mathbf{x})$ is a group homomorphism. A rotation supported in an E_j containing at most one point of the triple gives the trivial braid: that point traces a loop in the simply connected disc E_j , which contains neither of the other points. Since the E_j are disjoint and a triple has only three points, at most one E_j contains two or more points. Thus the associated homomorphism is either trivial or factors through one projection p_j . This also deals with configurations spread among different copies; they cannot produce two independent nonabelian braid contributions.

Fix the possibly nontrivial copy j . The braid made by one complete rotation of $N_{j,s}$ is trivial if it contains zero or one marked point, a full twist of the enclosed pair if it contains two, and the central full twist Δ^2 if it contains all three. If one of the two discs contains zero or one point, one generator is trivial. If it contains all three, its generator is central. In both cases the two generators have abelian image. If both discs contain exactly two points and those pairs are equal, the corresponding twists are equal up to the same conjugacy; both occur inside E_j away from the third point. This again gives cyclic image. The only remaining case is two different pairs of marked points. They share one point, and the triple then has exactly one point in each of $U_{j,1,\varepsilon}, U_{j,2,\varepsilon}, U_{j,3,\varepsilon}$.

Order those three regions from the first exclusive part through the intersection to the second exclusive part. In that ordering the two rotations make the braids σ_1^2, σ_2^2 . For an arbitrary labeling and arbitrary closing paths, the simultaneous associated homomorphism is

$$w \mapsto \beta_{\mathbf{x}} j(w_j) \beta_{\mathbf{x}}^{-1}, \quad \beta_{\mathbf{x}} \in B_3.$$

The same conjugator works on both generators, and hence on every word: an orientation-preserving identification of the two overlapping marked discs with the model identifies both twists simultaneously. This is precisely the common-conjugator assertion of [29, Lemma 4.2]. The conjugator does not depend on w .

The six possible assignments of labels give six product subsets, each of measure $\prod_s \mu(U_{j,s,\varepsilon})$. Because u is a class on B_3 , inner conjugation by $\beta_{\mathbf{x}}$ acts trivially on it. This is not true for an arbitrary class on P_3 . The contribution of these six subsets is therefore $\lambda_\varepsilon p_j^* j^* u$. All other subsets have amenable image and their positive-degree bounded-cohomology contribution is zero. Partitioning by the countably many associated homomorphisms gives the same conclusion for their union: finite partial sums are zero in reduced bounded cohomology, and the remaining integral is bounded by its measure times $\|c\|_\infty$.

Adding the k contributions and the exceptional contribution proves

$$\left\| \rho_\varepsilon^* \bar{\Gamma}_{b, \mathbb{D}}^d(u) - \lambda_\varepsilon \sum_{j=1}^k p_j^* j^* u \right\| \leq 3b_\varepsilon \|u\|.$$

Here we take the infimum over bounded representatives of u . Since each p_j^* and j^* has norm at most one, replacing λ_ε by λ gives (38). $\square$

Remark (The fixed Brooks class for the disc). For the disc proof, we now choose the fixed class α so that

$$j^* \alpha = [\delta \phi_{ab}] + [\delta \phi_{ba}],$$

where ϕ_w is the Brooks quasimorphism counting occurrences of w minus occurrences of w^{-1} in reduced words. For this choice, [1, Theorem A(b)] and bilinearity imply

$$j^* \alpha \smile j^* \beta = 0 \quad (\beta \in \bar{H}_b^q(B_3), \quad q \geq 1).$$

This will make every repeated-coordinate term vanish in the disc calculation, exactly as for the fixed class in Theorem 1. We verify both nonvanishing and extension to B_3 before making this choice in the proof.

We verify that this choice extends to B_3 . Put $e = [\delta \phi_{ab}]$ and $\eta = e + [\delta \phi_{ba}]$. In $P_3 / \langle \Delta^2 \rangle = F_2$, conjugation by $\sigma_1 \sigma_2 \sigma_1$ and $\sigma_1 \sigma_2$ induces, respectively,

$$S(a) = b, \quad S(b) = a, \quad T(a) = b, \quad T(b) = b^{-1} a^{-1}.$$

These formulas follow from the braid relation and the central quotient; see the presentation preceding Lemma 13. The two conjugating braids generate B_3 . Clearly $S^* e = [\delta \phi_{ba}]$ and $S^* \eta = \eta$. Let $\varepsilon_b : F_2 \rightarrow \mathbb{R}$ be the exponent-sum homomorphism with $\varepsilon_b(a) = 0$ and $\varepsilon_b(b) = 1$. Direct counting gives

$$\phi_{ab}(T(g)) = \phi_{ab}(g) - \varepsilon_b(g) \quad (g \in F_2).$$

Indeed, the substitution is $a \mapsto b$, $a^{-1} \mapsto b^{-1}$, $b \mapsto b^{-1} a^{-1}$, $b^{-1} \mapsto ab$. The cancellations at the adjacent pairs ab and $b^{-1} a^{-1}$ remove, respectively, one negative and one positive occurrence counted by ϕ_{ab} ; no other cancellation occurs for the reduced word to which the substitution is applied. Before these cancellations the signed count is $-\varepsilon_b(g)$, which proves the displayed identity. Taking coboundaries gives $T^* e = e$. Checking a and b also gives

$$S \circ T \circ S = \text{Inn}(a^{-1}) \circ T^{-1}.$$

Since inner automorphisms act trivially on bounded cohomology, this identity implies $T^* S^* e = S^* e$. Hence $T^* \eta = \eta$, so η is invariant under the full braid-group action on the central quotient. Let Q be the homogenization of $\phi_{ab} + \phi_{ba}$. Since $Q(a) = Q(b) = 0$ and $Q(ab) = 2$, the class $\eta = [\delta Q]$ is nonzero in bounded cohomology, and hence in reduced bounded cohomology.

We first replace Q by an invariant representative of the same bounded-cohomology class. Let $\varepsilon : F_2 \rightarrow \mathbb{R}$ be the homomorphism satisfying

$\varepsilon(a) = \varepsilon(b) = 1$, and put $Q_0 = Q - \frac{2}{3}\varepsilon$. Since $T^*[\delta Q] = [\delta Q]$ and Q is homogeneous, $Q \circ T - Q$ is a homomorphism. Evaluating on a, b gives

$$Q \circ T - Q = -2\varepsilon_b, \quad \varepsilon \circ T - \varepsilon = -3\varepsilon_b.$$

Hence $Q_0 \circ T = Q_0$. Both Q and ε are invariant under S , so $Q_0 \circ S = Q_0$ as well. Thus Q_0 is invariant under the induced B_3 -action, and $\delta Q_0 = \delta Q$.

Let $p : P_3 \rightarrow F_2$ be the projection. The homogeneous quasimorphism $Q_0 \circ p$ is invariant under conjugation by B_3 . Since P_3 is normal of index 6, it extends to a homogeneous quasimorphism $\overline{Q} : B_3 \rightarrow \mathbb{R}$. Explicitly,

$$\overline{Q}(\beta) = \frac{1}{6}Q_0(p(\beta^6)), \quad \beta \in B_3.$$

The formula is defined because $\beta^6 \in P_3$. For $h \in P_3$, homogeneity gives $\overline{Q}(h) = Q_0(p(h))$. Define $\alpha = [\delta \overline{Q}] \in \overline{H}_b^2(B_3)$. Since $p \circ j = \text{id}$, we obtain

$$j^*\alpha = [\delta Q_0] = [\delta Q] = \eta \neq 0.$$

Choose z as in Lemma 13, so that $\langle j^*\alpha, z \rangle = 1$.

Proof of Theorem 2 for the disc. Choose α, z by the preceding Brooks-class construction, and put $\theta = j^*\alpha$. Thus $\langle \theta, z \rangle = 1$ and $\theta \smile u = 0$ for every positive-degree bounded class u on F_2 . In this proof, T_m^q denotes the map of Proposition 9 for this fixed θ, z . Let $m \geq 0$, $q \in \{2, 3\}$, $k = m + 1$, and $d = 2m + q$. For $\beta \in \overline{H}_b^q(B_3)$ define

$$\xi_\beta = \text{Sym}_k(\alpha^{\times m} \times \beta) \in \overline{H}_b^d(B_3^k).$$

This class is symmetric. Naturality gives $(j^k)^*\xi_\beta = T_m^q(j^*\beta)$, so $\beta \mapsto \xi_\beta$ is injective by Proposition 9 and Lemma 13. Define

$$(39) \quad \begin{aligned} \Psi_{m,q} : \overline{H}_b^q(B_3) &\longrightarrow \overline{H}_b^d(\mathcal{T}_{\mathbb{D}}), \\ \Psi_{m,q}(\beta) &= \frac{1}{k!} \overline{\Gamma}_{b,\mathbb{D},k}^d(\xi_\beta). \end{aligned}$$

By (26) and (35), this is equivalently

$$\Psi_{m,q}(\beta) = (\overline{\Gamma}_{b,\mathbb{D}}^2(\alpha))^{\smile m} \smile \overline{\Gamma}_{b,\mathbb{D}}^q(\beta).$$

Thus the map is the same cup product of transferred classes, now written as a transfer of a symmetric external product.

Two factors: $m = 1$, $k = 2$. Fix $q = 2$ or 3 and write $b = j^*\beta$. Use the two disjoint rotation constructions preceding Lemma 15. They give one homomorphism $\rho_\varepsilon : F_2^2 \rightarrow \text{Ham}(\mathbb{D}, \omega)$, independent of β . Apply that lemma first to α in degree two and then to β in degree q . The constant $\lambda > 0$ is defined in (37). With the same ρ_ε and λ in both applications, we obtain

$$\rho_\varepsilon^* \overline{\Gamma}_{b,\mathbb{D}}^2(\alpha) \longrightarrow \lambda(p_1^*\theta + p_2^*\theta), \quad \rho_\varepsilon^* \overline{\Gamma}_{b,\mathbb{D}}^q(\beta) \longrightarrow \lambda(p_1^*b + p_2^*b).$$

These are norm limits. The product estimate (5) allows us to multiply them. More explicitly, put $u_\varepsilon = \rho_\varepsilon^* \bar{\Gamma}_{b, \mathbb{D}}^2(\alpha)$, $w_\varepsilon = \rho_\varepsilon^* \bar{\Gamma}_{b, \mathbb{D}}^q(\beta)$, and denote their displayed limits by u_0, w_0 . Then

$$\|u_\varepsilon \smile w_\varepsilon - u_0 \smile w_0\| \leq \|u_\varepsilon - u_0\| \|w_\varepsilon\| + \|u_0\| \|w_\varepsilon - w_0\| \longrightarrow 0.$$

The norms $\|w_\varepsilon\|$ are bounded by $\|\beta\|$. Expanding the limiting product gives

$$(40) \quad \rho_\varepsilon^* \Psi_{1,q}(\beta) \longrightarrow \lambda^2 [p_1^*(\theta \smile b) + \theta \times b + b \times \theta + p_2^*(\theta \smile b)].$$

The first and fourth terms vanish by the choice of θ . The two mixed terms have the same positive sign, since $(-1)^{2q} = 1$. Hence (40) reduces to

$$(41) \quad \rho_\varepsilon^* \Psi_{1,q}(\beta) \longrightarrow \lambda^2 (\theta \times b + b \times \theta) = \lambda^2 T_1^q(b).$$

This is the same symmetric-product limit as in Theorem 1. If $\Psi_{1,q}(\beta) = 0$, then (41) gives $T_1^q(b) = 0$. Proposition 9 implies $b = 0$, and injectivity of j^* gives $\beta = 0$.

An arbitrary number of factors. For $m \geq 2$, use $k = m + 1$ disjoint copies and apply Lemma 15 to α and β , with the same homomorphism. Replacing one factor at a time in the cup product, using the preceding estimate, gives

$$\rho_\varepsilon^* \Psi_{m,q}(\beta) \longrightarrow \lambda^k \left(\sum_{j=1}^k p_j^* \theta \right)^{\smile m} \smile \left(\sum_{j=1}^k p_j^* b \right).$$

There are k labelled factors of positive degree. In a nonbijective assignment to the k coordinates, a repeated coordinate contains a fixed θ and another positive-degree class. Their internal cup product is zero. Only bijective assignments remain, and their sum is $\text{Sym}_k(\theta^{\times m} \times b) = T_m^q(b)$. Therefore

$$(42) \quad \rho_\varepsilon^* \Psi_{m,q}(\beta) \longrightarrow \lambda^k T_m^q(j^* \beta).$$

The same formula holds for $m = 1$ by (41) and for $m = 0$ by Lemma 15, with $T_0^q = \text{id}$. Since $\lambda > 0$, (42), Proposition 9, and injectivity of j^* prove that every $\Psi_{m,q}$ is injective. In particular, a nonzero class gives an image of positive reduced norm. For example, if $Q_{m,q}$ is the bounded left inverse from (21), then

$$\|\Psi_{m,q}(\beta)\| \geq \lambda^k \|T_m^q(j^* \beta)\| \geq \frac{\lambda^k}{\|Q_{m,q}\|} \|\beta\|.$$

We used that j^* is isometric; the constants may depend on k .

Both $\bar{H}_b^2(B_3)$ and $\bar{H}_b^3(B_3)$ are infinite-dimensional by Lemma 13, which proves the reduced bounded-cohomology assertion for the disc. For smooth transformations, α and β are reduced exact by that lemma, and Lemma 14 preserves exactness. Each summand of ξ_β is exact, so its normalized transfer in (39) is reduced exact. For homeomorphisms we only use bounded cochains on the finite-measure configuration space and do not assert an ordinary transfer. All testing homomorphisms take values in $\text{Ham}(\mathbb{D}, \omega)$, so the same proof applies to each of the four transformation groups. $\square$

4.2. Sphere case. We now prove the sphere case of Theorem 2. Write $\mathcal{T}_{S^2}$ for any of the four area-preserving transformation groups in that theorem, and normalize the area form so that $\mu(S^2) = 1$. The domain of the transfer is the reduced bounded cohomology of the full mapping class group of the sphere with four marked points, not that of its pure subgroup. We first explain this choice and its relation to the three-strand disc braid group.

Fix a disc in S^2 and a fourth point outside it. Adding the constant fourth strand to a three-strand pure disc braid defines a homomorphism

$$\kappa : P_3(\mathbb{D}) \longrightarrow P_4(S^2).$$

For $r = 3, 4$, write Δ_r^2 for the full twist on r strands, in the disc or sphere as appropriate. Thus $\Delta_3^2 = (\sigma_1\sigma_2)^3$, and Δ_4^2 has order two in $P_4(S^2)$. Denote by δ_1, δ_2 the elementary spherical braids supported in the chosen three-punctured disc. In particular, $\kappa(\sigma_s^2) = \delta_s^2$ for $s = 1, 2$. We recall the following standard braid and mapping class group identifications; see [17] and [29, Theorem 2.4 and Section 5.2]. The homomorphism κ is onto and

$$(43) \quad \begin{aligned} \ker \kappa &= \langle (\Delta_3^2)^2 \rangle, & \kappa(\Delta_3^2) &= \Delta_4^2, \\ P_4(S^2) &= \langle \delta_1^2, \delta_2^2 \rangle \times \langle \Delta_4^2 \rangle \cong F_2 \times \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Let $\text{Mod}(S^2, 4)$ denote the orientation-preserving mapping class group of the four-punctured sphere. We identify

$$\text{Mod}(S^2, 4) = B_4(S^2)/\langle \Delta_4^2 \rangle.$$

The images of δ_1^2, δ_2^2 define an embedding $j_{S^2} : F_2 \hookrightarrow \text{Mod}(S^2, 4)$ whose image is the pure mapping class group and has index 24.

Lemma 16. *For $q = 2, 3$, restriction is an isometric embedding*

$$(44) \quad j_{S^2}^* : \overline{H}_b^q(\text{Mod}(S^2, 4)) \longrightarrow \overline{H}_b^q(F_2).$$

Its domain is infinite-dimensional and equals $\overline{EH}_b^q(\text{Mod}(S^2, 4))$. There are a fixed nonzero class $\alpha \in \overline{H}_b^2(\text{Mod}(S^2, 4))$ and a cycle $z \in Z_2^{\ell^1}(F_2)$ such that

$$(45) \quad \langle j_{S^2}^* \alpha, z \rangle = 1.$$

Proof. Since the pure subgroup has finite index, restriction is isometric in bounded cohomology; see [29, Theorem 2.1]. It therefore induces the isometric embedding (44) on reduced bounded cohomology.

The group $\text{Mod}(S^2, 4)$ is non-elementary hyperbolic, since it contains F_2 with finite index. Its reduced bounded cohomology is infinite-dimensional in degrees two and three by [16] and [22, Corollary 3], respectively; see also [29, Section 5.4] for the degree-three conclusion.

Ordinary restriction to a finite-index subgroup is injective. Since $H^q(F_2) = 0$ for $q \geq 2$, it follows that

$$H^q(\text{Mod}(S^2, 4)) = 0 \quad (q \geq 2).$$

Thus the comparison map is zero in these degrees, and reduced bounded cohomology equals reduced exact bounded cohomology.

Choose $0 \neq \alpha \in \overline{H}_b^2(\text{Mod}(S^2, 4))$. Its restriction is nonzero by isometry. The degree-two duality recalled in Lemma 7 gives an ℓ^1 -cycle with nonzero pairing with $j_{S^2}^* \alpha$; rescaling that cycle gives (45). $\square$

In particular, using $\text{Mod}(S^2, 4)$ instead of $B_4(S^2)$ does not discard bounded-cohomology classes: pullback by the quotient with finite central kernel is isometric in bounded and reduced bounded cohomology. It is also an isomorphism on ordinary real cohomology. It does not identify the domain of the transfer with all of $\overline{H}_b^q(F_2)$; only the image in (44) is used.

Let $X_4(S^2)$ denote the ordered configuration space

$$X_4(S^2) = \{(x_1, x_2, x_3, x_4) \in (S^2)^4 : x_i \neq x_j \text{ for } i \neq j\}$$

with the restriction of the product measure. This measure has total mass one, and $\mathcal{T}_{S^2}$ acts diagonally. We use the standard configuration-space cocycle with values in $\text{PMod}(S^2, 4) \subset \text{Mod}(S^2, 4)$.

Lemma 17. *Integration over $X_4(S^2)$ defines a bounded linear map*

$$\overline{\Gamma}_{b, S^2}^d : \overline{H}_b^d(\text{Mod}(S^2, 4)) \longrightarrow \overline{H}_b^d(\mathcal{T}_{S^2}), \quad \|\overline{\Gamma}_{b, S^2}^d\| \leq 1.$$

For smooth transformations this map preserves reduced exact bounded cohomology. Integration over $X_4(S^2)^k$ defines the corresponding product transfer, denoted by $\overline{\Gamma}_{b, S^2, k}^d$, and the transfer of an external product equals the cup product of the individual transferred classes. The product maps satisfy (26) with $X = X_4(S^2)$.

Proof. We explain the two points that differ from the disc: isotopy independence and the ordinary transfer. Choose a base configuration and measurable closing paths. An isotopy from the identity to f gives a braid in $P_4(S^2)$. If two isotopies have the same endpoint, the difference is obtained by evaluating a loop of surface homeomorphisms on the configuration. Such evaluation loops are central in the configuration fundamental group: for a loop ℓ of configurations, the square $(s, t) \mapsto f_t(\ell(s))$ gives a homotopy between the two orders of concatenation. By (43), this central ambiguity is contained in $\langle \Delta_4^2 \rangle$. Projection to $\text{PMod}(S^2, 4)$ removes it. The resulting measurable cocycle, denoted by γ_{S^2} , satisfies

$$\gamma_{S^2}(fg, \mathbf{x}) = \gamma_{S^2}(f, g\mathbf{x})\gamma_{S^2}(g, \mathbf{x})$$

in our right-homogeneous convention. This is the sphere construction in [29, Section 5, before Section 5.1]; the centrality argument also applies to homeomorphism isotopies.

For a bounded homogeneous cochain c on $\text{Mod}(S^2, 4)$, set

$$(46) \quad (\Gamma'_{b,S^2} c)(f_0, \dots, f_d) = \int_{X_4(S^2)} c(\gamma_{S^2}(f_0, \mathbf{x}), \dots, \gamma_{S^2}(f_d, \mathbf{x})) d\mu^4(\mathbf{x}).$$

The same cocycle identity and change of variables as in Lemma 14 show that this is a cochain map with norm at most one. It induces the asserted reduced bounded-cohomology map for all four transformation groups.

For the smooth groups, the reference paths can be chosen so that, for each fixed f , the cocycle $\gamma_{S^2}(f, \cdot)$ has finite image on a full-measure subset; see [9, Lemma 2.1] and [29, Section 5]. The induced bounded transfer is unchanged by this choice. Thus, for each tuple of transformations, the transfer integrand has finite image almost everywhere even for an unbounded ordinary cochain. Hence, the same integral defines an ordinary cochain map, which agrees with the bounded transfer on bounded cochains and commutes with comparison. In particular, an ordinary primitive of a bounded cocycle transfers to an ordinary primitive of its image. Hence the transfer preserves exact bounded cohomology and, by its norm bound, reduced exact bounded cohomology. No ordinary transfer is asserted here for area-preserving homeomorphisms.

Finally, the diagonal action on $X_4(S^2)^k$ uses k separate four-point configurations. The external-product cochain is a product of functions, each depending on one configuration. Fubini's theorem therefore gives

$$(47) \quad \bar{\Gamma}_{b,S^2,k}^{d_1+\dots+d_k}(u_1 \times \dots \times u_k) = \bar{\Gamma}_{b,S^2}^{d_1}(u_1) \smile \dots \smile \bar{\Gamma}_{b,S^2}^{d_k}(u_k).$$

This is the same cochain computation as (35). Permuting the k four-point configurations also gives (26). $\square$

Fix $k \geq 1$. In an area-coordinate chart of S^2 , choose k disjoint open discs E_j and congruent pairs of overlapping round discs $N_{j,1}, N_{j,2} \Subset E_j$, as in the disc construction. Choose them small enough that their union does not cover the sphere. Use exactly the radial Hamiltonian isotopies defined before (36), now extended by the identity on S^2 . For their time-one maps put

$$(48) \quad \begin{aligned} \rho_{j,\varepsilon}(a) &= f_{j,1,\varepsilon}, & \rho_{j,\varepsilon}(b) &= f_{j,2,\varepsilon}, \\ \rho_\varepsilon : F_2^k &\longrightarrow \text{Ham}(S^2, \omega), & \rho_\varepsilon(w_1, \dots, w_k) &= \prod_{j=1}^k \rho_{j,\varepsilon}(w_j). \end{aligned}$$

The factors commute because their supports are disjoint. Let B_ε be the union of the transition annuli and put $b_\varepsilon = \mu(B_\varepsilon) \rightarrow 0$. Every time-one map fixes $S^2 \setminus B_\varepsilon$ pointwise, and B_ε is invariant under all the time-one maps.

For each j , use the four subsets

$$\begin{aligned} U_{j,1,\varepsilon} &= (N_{j,1} \setminus \overline{N_{j,2}}) \setminus B_\varepsilon, & U_{j,2,\varepsilon} &= (N_{j,1} \cap N_{j,2}) \setminus B_\varepsilon, \\ U_{j,3,\varepsilon} &= (N_{j,2} \setminus \overline{N_{j,1}}) \setminus B_\varepsilon, & U_{j,4,\varepsilon} &= (S^2 \setminus (N_{j,1} \cup N_{j,2})) \setminus B_\varepsilon. \end{aligned}$$

Up to boundary sets of measure zero, they partition $S^2 \setminus B_\varepsilon$. The two distributions used below are illustrated in Figure 5; the points shown avoid the transition annuli.

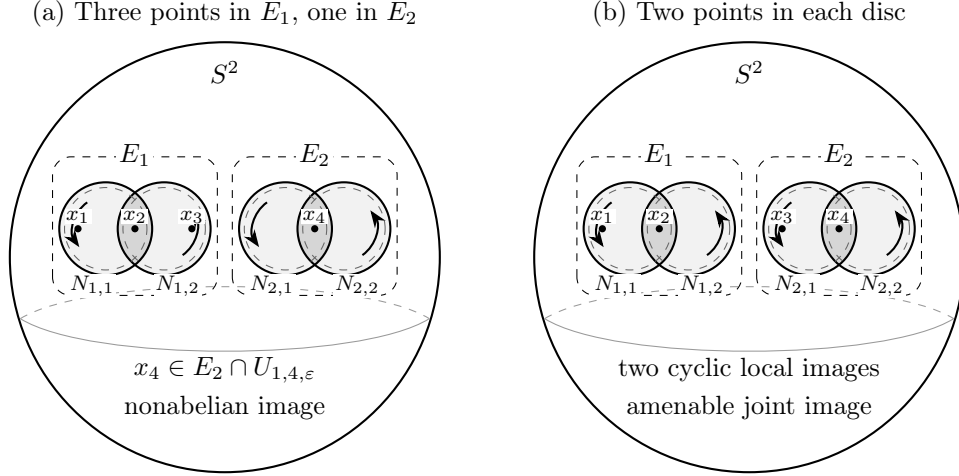


FIGURE 5. Two copies of the sphere rotation construction, viewed in a chart with the remaining sphere understood. Solid circles bound the rotating discs; the strips between solid and dashed circles indicate transition annuli. In (a), the first three points occupy the three distinct parts of the first pair, and the fourth lies outside that pair. Thus the quadruple has type $(1, 1, 1, 1)$ for the first pair, even when its fourth point lies in the second construction. In (b), each pair contains two points. Its local image is cyclic, and the two images commute; the joint image is amenable. Neither picture represents coincident points. The cyclic-image assertions are proved in Lemma 18.

The first three areas are independent of j . So is the fourth area, since $\mu(U_{j,4,\varepsilon}) = 1 - b_\varepsilon - \sum_{s=1}^3 \mu(U_{j,s,\varepsilon})$. Define

$$(49) \quad \lambda_{k,\varepsilon} = 4! \prod_{s=1}^4 \mu(U_{j,s,\varepsilon}), \quad \lambda_k = \lim_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} > 0.$$

Positivity follows because all three parts of each overlap have positive area and $S^2 \setminus (N_{j,1} \cup N_{j,2})$ has positive area. The notation allows these constants to depend on k , since the discs may be chosen smaller when k increases. Put $p_j : F_2^k \rightarrow F_2$ for the j th projection.

Lemma 18. *With the choices above, for every $d \geq 2$ and $u \in \overline{H}_b^d(\text{Mod}(S^2, 4))$,*

$$(50) \quad \left\| \rho_\varepsilon^* \overline{\Gamma}_{b,S^2}^d(u) - \lambda_k \sum_{j=1}^k p_j^* j_{S^2}^* u \right\| \leq (4b_\varepsilon + k|\lambda_{k,\varepsilon} - \lambda_k|) \|u\|.$$

In particular, the left-hand side tends to zero.

Proof. Kimura's computation for a single pair of rotating discs is [29, Lemma 5.2]. We give the simultaneous computation, including configurations meeting several of the disjoint discs E_j . Let $\mathcal{E}_\varepsilon \subset X_4(S^2)$ consist of quadruples with at least one point in B_ε . It is invariant under the diagonal action of the time-one maps, and $\mu^4(\mathcal{E}_\varepsilon) = 1 - (1 - b_\varepsilon)^4 \leq 4b_\varepsilon$. For a bounded cocycle c representing u , the integral over this set is a cochain map and has norm at most $4b_\varepsilon \|c\|_\infty$, exactly as in Lemma 15.

Fix a quadruple $\mathbf{x}$ outside $\mathcal{E}_\varepsilon$. Since every time-one map fixes $\mathbf{x}$, the map $w \mapsto \gamma_{S^2}(\rho_\varepsilon(w), \mathbf{x})$ is a homomorphism. We first analyze its restriction to one factor j . A complete rotation of a disc with r marked points represents the Dehn twist about its boundary in $\text{Mod}(S^2, 4)$. It is trivial if $r = 0, 1, 3, 4$, since one side of the boundary contains at most one marked point. Only $r = 2$ can give a nontrivial twist. These are statements in the central quotient: a full twist on three or four strands can be nontrivial in $P_4(S^2)$.

Let n_s be the number of points of $\mathbf{x}$ in $U_{j,s,\varepsilon}$. The two rotation discs contain $n_1 + n_2$ and $n_2 + n_3$ points. If either number is not two, at least one generator is trivial and the local image is cyclic. If both numbers equal two, then

$$(n_1, n_2, n_3, n_4) = (2, 0, 2, 0), \quad (1, 1, 1, 1), \quad \text{or} \quad (0, 2, 0, 2).$$

In the first case the overlap contains no marked point; in the last case each exclusive part contains no marked point. The two round boundary curves meet in two points, and in either case a puncture-free bigon can be removed. They become disjoint essential curves on the four-punctured sphere and hence are isotopic. The two positive Dehn twists are consequently equal. Their image is cyclic. This verifies directly the two exceptional cyclic types in [29, proof of Lemma 5.2].

For type $(1, 1, 1, 1)$, the first twist encloses the points in regions one and two and the second encloses those in regions two and three. The two boundary curves, their four complementary regions, and their marked points admit an orientation-preserving identification with the standard model of the sphere with four marked points. The two twists are thereby identified simultaneously with $j_{S^2}(a)$ and $j_{S^2}(b)$. An arbitrary labeling or closing-path choice changes this by a common conjugation in the full group $\text{Mod}(S^2, 4)$. Thus the local homomorphism is conjugate to j_{S^2} , as also shown in [29, Lemma 5.2].

Now restore all k factors. A quadruple of type $(1, 1, 1, 1)$ for the j th pair has three points in E_j and only one outside E_j . Every other E_ℓ therefore contains at most one point. Rotations supported there give the trivial mapping class. The full associated homomorphism is consequently

$$(51) \quad w \mapsto \eta_{\mathbf{x}} j_{S^2}(w_j) \eta_{\mathbf{x}}^{-1} \quad \text{for some } \eta_{\mathbf{x}} \in \text{Mod}(S^2, 4).$$

In particular, the sets of these quadruples for different j are disjoint. If no pair has type $(1, 1, 1, 1)$, every local image is cyclic. The local images

commute, since the surface homomorphisms commute and the cocycle defines a homomorphism at $\mathbf{x}$. Their joint image is therefore abelian. This includes the distribution of two points in each of two different E_j , see Figure 5(b).

For each j , the quadruples of type $(1, 1, 1, 1)$ form the union of $4!$ labelled product sets, with total measure $\lambda_{k,\varepsilon}$. Because u is a class on the full mapping class group, conjugation acts trivially. Integrating its bounded cochain homotopy, whose bound is independent of the conjugator [30, Lemma A.2], makes the contribution of this set equal to $\lambda_{k,\varepsilon} p_j^* j_{S^2}^* u$.

All remaining quadruples outside $\mathcal{E}_\varepsilon$ give homomorphisms with amenable image, so their positive-degree bounded-cohomology contribution is zero. To justify this also for the union, partition by the tuple of images of the generators of F_2^k . There are countably many such tuples and the parts are measurable. Each part has one homomorphism with amenable image. Finite partial sums give the zero class; the norm of the remaining integral is at most its measure times $\|c\|_\infty$ and tends to zero. Hence the whole contribution is zero in reduced bounded cohomology. We have proved

$$\left\| \rho_\varepsilon^* \bar{\Gamma}_{b,S^2}^d(u) - \lambda_{k,\varepsilon} \sum_j p_j^* j_{S^2}^* u \right\| \leq 4b_\varepsilon \|u\|.$$

Each restriction has norm at most one; replacing $\lambda_{k,\varepsilon}$ by λ_k gives (50). $\square$

Remark (The fixed Brooks class for the sphere). We choose the fixed class in the sphere proof so that its restriction to F_2 is the same Brooks class as in the disc case.

Let $Q_0 : F_2 \rightarrow \mathbb{R}$ be the invariant homogeneous quasimorphism constructed for the disc, so that

$$[\delta Q_0] = [\delta \phi_{ab}] + [\delta \phi_{ba}] \neq 0.$$

Put $\Gamma = B_3 / \langle \Delta_3^2 \rangle$, $\Delta_3^2 = (\sigma_1 \sigma_2)^3$, and write $\bar{\sigma}_1, \bar{\sigma}_2$ for the images of the standard braid generators in Γ . We identify F_2 with the subgroup generated by $a = \bar{\sigma}_1^2$ and $b = \bar{\sigma}_2^2$. The invariant quasimorphism Q_0 extends to a homogeneous quasimorphism

$$\bar{Q}_3 : \Gamma \rightarrow \mathbb{R}, \quad \bar{Q}_3|_{F_2} = Q_0,$$

by the same finite-index extension argument used for the disc.

Let $\delta_1, \delta_2, \delta_3$ be the standard generators of $B_4(S^2)$. Define $r : B_4(S^2) \rightarrow \Gamma$,

$$r(\delta_1) = \bar{\sigma}_1, \quad r(\delta_2) = \bar{\sigma}_2, \quad r(\delta_3) = \bar{\sigma}_1.$$

The braid relations are preserved. The additional spherical relation $\delta_1 \delta_2 \delta_3^2 \delta_2 \delta_1 = 1$ is preserved as well, since

$$r(\delta_1 \delta_2 \delta_3^2 \delta_2 \delta_1) = \bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_1^2 \bar{\sigma}_2 \bar{\sigma}_1 = (\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_1)^2 = 1.$$

Thus r is a well-defined homomorphism; see [29, Section 5.2] for the spherical presentation. Consequently, $\overline{Q}_4 = \overline{Q}_3 \circ r$ is a homogeneous quasimorphism on $B_4(S^2)$.

Since Δ_4^2 is central of order two, $\overline{Q}_4$ descends to a homogeneous quasimorphism $Q_{S^2} : \text{Mod}(S^2, 4) \rightarrow \mathbb{R}$. Set

$$\alpha_{S^2} = [\delta Q_{S^2}] \in \overline{EH}_b^2(\text{Mod}(S^2, 4)).$$

The inclusion j_{S^2} sends a, b to the mapping classes represented by δ_1^2, δ_2^2 . Their images under r are a, b , respectively. Hence

$$\theta := j_{S^2}^* \alpha_{S^2} = [\delta Q_0] = [\delta \phi_{ab}] + [\delta \phi_{ba}] \neq 0.$$

By [1, Theorem A(b)] and bilinearity,

$$\theta \smile u = 0 \quad (u \in \overline{H}_b^s(F_2), s \geq 1).$$

Choose $z \in Z_2^{\ell^1}(F_2)$ with $\langle \theta, z \rangle = 1$, using the duality recalled in Lemma 7.

With this choice of α_{S^2} , every repeated-coordinate term in the spherical product calculation vanishes, just as in the disc case. Thus the projection Π_k is unnecessary in that proof.

Proof of Theorem 2 for the sphere. Choose $\alpha = \alpha_{S^2}$ and z as in the preceding remark, and put $\theta = j_{S^2}^* \alpha$. Then $\langle \theta, z \rangle = 1$ and

$$\theta \smile u = 0 \quad (u \in \overline{H}_b^s(F_2), s \geq 1).$$

Use T_m^q from Proposition 9 with this θ, z . Let $m \geq 0$, $q = 2, 3$, $k = m + 1$, and $d = 2m + q$. For $\beta \in \overline{H}_b^q(\text{Mod}(S^2, 4))$, set $\xi_\beta = \text{Sym}_k(\alpha^{\times m} \times \beta)$. The class ξ_β is symmetric, and its restriction under $(j_{S^2}^k)^*$ is $T_m^q(j_{S^2}^* \beta)$. Define

$$(52) \quad \Psi_{m,q}^{S^2}(\beta) = \frac{1}{k!} \overline{\Gamma}_{b,S^2,k}^d(\xi_\beta).$$

Permutation invariance (26) and (47) identify this map with

$$\Psi_{m,q}^{S^2}(\beta) = (\overline{\Gamma}_{b,S^2}^2(\alpha))^{\smile m} \smile \overline{\Gamma}_{b,S^2}^q(\beta).$$

Thus we use exactly the normalized symmetric construction of the disc. Write $b = j_{S^2}^* \beta$ for the remainder of this proof.

Two factors: $m = 1$, $k = 2$. Choose the two rotation pairs defining $\rho_\varepsilon : F_2^2 \rightarrow \text{Ham}(S^2, \omega)$ and the constant $\lambda_2 > 0$ in (49). Apply Lemma 18 separately to α and β , with the same homomorphism. It gives

$$\rho_\varepsilon^* \overline{\Gamma}_{b,S^2}^2(\alpha) \longrightarrow \lambda_2(p_1^* \theta + p_2^* \theta), \quad \rho_\varepsilon^* \overline{\Gamma}_{b,S^2}^q(\beta) \longrightarrow \lambda_2(p_1^* b + p_2^* b).$$

The product estimate used in the disc proof justifies multiplication of these norm limits. Thus

$$(53) \quad \rho_\varepsilon^* \Psi_{1,q}^{S^2}(\beta) \longrightarrow \lambda_2^2[p_1^*(\theta \smile b) + \theta \times b + b \times \theta + p_2^*(\theta \smile b)].$$

The first and fourth terms in (53) vanish because $\theta \smile b = 0$. The two mixed terms have the same positive sign, since $(-1)^{2q} = 1$. Therefore

$$(54) \quad \rho_\varepsilon^* \Psi_{1,q}^{S^2}(\beta) \longrightarrow \lambda_2^2(\theta \times b + b \times \theta) = \lambda_2^2 T_1^q(b).$$

If the transferred class is zero, this identity gives $T_1^q(b) = 0$. Proposition 9 gives $b = 0$, and injectivity of $J_{S^2}^*$ gives $\beta = 0$.

An arbitrary number of factors. For $k = m+1$, Lemma 18 and the cup-product norm estimate give

$$\rho_\varepsilon^* \Psi_{m,q}^{S^2}(\beta) \longrightarrow \lambda_k^k \left(\sum_{j=1}^k p_j^* \theta \right)^{\smile^m} \smile \left(\sum_{j=1}^k p_j^* b \right).$$

The constants may depend on k . In the expansion there are $k = m+1$ labelled factors: m copies of θ and one copy of b . If an assignment repeats a coordinate, that coordinate contains a factor θ and another positive-degree factor. Their internal cup product vanishes. Thus only bijective assignments remain, and their sum is $\text{Sym}_k(\theta^{\times m} \times b) = T_m^q(b)$. Consequently,

$$(55) \quad \rho_\varepsilon^* \Psi_{m,q}^{S^2}(\beta) \longrightarrow \lambda_k^k T_m^q(b).$$

If $\Psi_{m,q}^{S^2}(\beta) = 0$, the norm limit in (55) gives $T_m^q(b) = 0$. Since $\lambda_k > 0$, Proposition 9 and injectivity of $J_{S^2}^*$ imply $\beta = 0$. Hence every $\Psi_{m,q}^{S^2}$ is injective. The domains in degrees two and three are infinite-dimensional by Lemma 16, so the degree patterns $2m+2$ and $2m+3$ give infinite-dimensional reduced bounded cohomology in every degree at least two.

For the smooth groups, α and β are reduced exact by Lemma 16. Permutation pullbacks preserve exactness, so every ξ_β is reduced exact. Its normalized transfer in (52) is reduced exact by Lemma 17. For homeomorphisms we use only bounded cochains and assert the reduced bounded conclusion. The testing maps (48) take values in $\text{Ham}(S^2, \omega)$, so the same calculation applies to all four groups in the theorem. $\square$

5. PROOF OF THEOREM 3

We first establish the local three-point restriction and transfer statements used for every positive genus. We then prove the theorem for $g \geq 2$, and finally treat the two-point construction needed for the torus.

Let $S = \Sigma_g$, where $g \geq 1$, with normalized area measure μ . Let $\mathcal{T}_S$ denote any of the four area-preserving transformation groups defined at the beginning of the paper. Recall the group $K(S, 3)$ defined before Theorem 3. The Birman exact sequence identifies it with

$$K(S, 3) \cong \begin{cases} B_3(S), & g \geq 2, \\ B_3(T^2)/Z(B_3(T^2)), & g = 1; \end{cases}$$

see [29, Theorem 2.4]. Choose an embedded disc containing the three base points. Extension by the identity gives an inclusion $\iota : B_3 \hookrightarrow K(S, 3)$; see [37, Corollary 4.2(iii)] Recall the inclusion $j : F_2 \hookrightarrow B_3$ defined before Lemma 13. Orient and mark every copy of the three-punctured disc in the same way.

Lemma 19. *There is an infinite-dimensional subspace $E_S \subset \overline{EH}_b^2(K(S, 3))$ on which $j^* \iota^*$ is injective. One can choose $\alpha \in E_S$ and $z \in Z_2^{\ell^1}(F_2)$ such that*

$$(56) \quad \eta = j^* \iota^* \alpha \neq 0, \quad \langle \eta, z \rangle = 1.$$

Proof. Bestvina–Bromberg–Fujiwara [3, Proposition 4.9] implies that restriction from the 3 punctured mapping class group $\text{Mod}(S, 3)$ to its disc subgroup has infinite-dimensional image on homogeneous quasimorphisms modulo homomorphisms and bounded functions, which is EH_b^2 . This application to disc braid subgroups also appears in [5, Proposition 2.6 and Corollary 2.7] for ambient genus greater than one. For the marked torus we apply [3, Proposition 4.9] directly to the same three-punctured disc.

Restriction factors through $K(S, 3)$. Hence the image of

$$EH_b^2(K(S, 3)) \xrightarrow{\iota^*} EH_b^2(B_3)$$

is infinite-dimensional. The degree-two seminorm is a norm, by [21, Theorem 6.5], so this conclusion is unchanged on passing to reduced exact bounded cohomology. Lemma 13 makes j^* isometric onto its image. Consequently the composite $j^* \iota^*$ still has infinite-dimensional image. Choose linearly independent classes in that image, choose one preimage for each, and let E_S be their algebraic span. The composite is injective on E_S .

Choose $0 \neq \alpha \in E_S$. Its restriction η is nonzero, and the degree-two duality in Lemma 7 supplies an ℓ^1 -cycle with nonzero evaluation on η . Rescaling this cycle gives (56). $\square$

Let $X_3(S)$ denote the ordered configuration space, with measure $\nu = \mu^3|_{X_3(S)}$, and let $\nu_k = \nu^{\otimes k}$ on $X_3(S)^k$. Both measures have mass one. Choose measurable closing paths from a base configuration. Tracing an isotopy defines the endpoint cocycle

$$\gamma_{S,3} : \mathcal{T}_S \times X_3(S) \longrightarrow K(S, 3).$$

Its values lie in the image of the pure braid group. The central quotient makes it independent of the chosen isotopy: the evaluation of a loop of unmarked isotopies is central and is in the kernel of the Birman map. Equivalently, the cocycle is obtained directly from that map applied to the tracing braid; see [29, Theorem 2.4 and Section 3.1]. It satisfies $\gamma_{S,3}(fg, \mathbf{x}) = \gamma_{S,3}(f, g\mathbf{x})\gamma_{S,3}(g, \mathbf{x})$. For $\mathbf{X} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)})$, define

$$\gamma_{S,3,k}(f, \mathbf{X}) = (\gamma_{S,3}(f, \mathbf{x}^{(1)}), \dots, \gamma_{S,3}(f, \mathbf{x}^{(k)})).$$

As in Lemma 14, different triples are independent.

Lemma 20. *The diagonal action on $X_3(S)^k$ defines a norm non-increasing map*

$$\Gamma_{S,3,k}^d : \overline{H}_b^d(K(S, 3)^k) \longrightarrow \overline{H}_b^d(\mathcal{T}_S).$$

Write $\Gamma_{S,3}^d = \Gamma_{S,3,1}^d$. These maps satisfy the permutation identity (26) with $X = X_3(S)$. They also satisfy

$$(57) \quad \Gamma_{S,3,k}^{d_1+\dots+d_k}(u_1 \times \dots \times u_k) = \Gamma_{S,3}^{d_1}(u_1) \smile \dots \smile \Gamma_{S,3}^{d_k}(u_k).$$

For the smooth transformation groups they preserve reduced exact bounded cohomology. The bounded maps are also defined for $\text{Homeo}_0(S, \mu)$.

Proof. For a tuple $\mathbf{f} = (f_0, \dots, f_d)$, write $\gamma_k(\mathbf{f}, \mathbf{X}) = (\gamma_{S,3,k}(f_i, \mathbf{X}))_{i=0}^d$. For a bounded homogeneous cochain c on $K(S, 3)^k$, define

$$(58) \quad (\Gamma'_{S,3,k}c)(\mathbf{f}) = \int_{X_3(S)^k} c(\gamma_k(\mathbf{f}, \mathbf{X})) d\nu_k(\mathbf{X}).$$

The integrand is measurable and bounded. In the right-homogeneous convention of Lemma 10, the cocycle identity and the measure-preserving substitution $\mathbf{Y} = h\mathbf{X}$ prove right invariance, exactly as in Lemma 14, and $\delta\Gamma'_{S,3,k} = \Gamma'_{S,3,k}\delta$. The estimate $\|\Gamma'_{S,3,k}c\|_\infty \leq \|c\|_\infty$ gives the map on reduced bounded cohomology.

We recall the product calculation with its independent variables. Let c, e have degrees p, q and fix $\mathbf{f} = (f_0, \dots, f_{p+q})$. Set

$$c_{\mathbf{f}}(\mathbf{x}) = c((\gamma_{S,3}(f_i, \mathbf{x}))_{i=0}^p), e_{\mathbf{f}}(\mathbf{y}) = e((\gamma_{S,3}(f_i, \mathbf{y}))_{i=p}^{p+q}).$$

Substitution in (58) gives the cochain identity

$$\begin{aligned} (\Gamma'_{S,3,2}(c \times e))(\mathbf{f}) &= \int_{X_3(S)^2} c_{\mathbf{f}}(\mathbf{x}) e_{\mathbf{f}}(\mathbf{y}) d\nu(\mathbf{x}) d\nu(\mathbf{y}) = \\ &= \left(\int_{X_3(S)} c_{\mathbf{f}} d\nu \right) \left(\int_{X_3(S)} e_{\mathbf{f}} d\nu \right) = ((\Gamma'_{S,3}c) \smile (\Gamma'_{S,3}e))(\mathbf{f}). \end{aligned}$$

Fubini applies because both functions are bounded and the measure is finite. Permuting the independent configuration variables gives (26), so normalized symmetrization does not change any transferred class. Repeating the product calculation proves (57) on reduced bounded cohomology.

For smooth transformations, use the section of the mapping-class-valued configuration cocycle in [9, Lemma 2.1]; see also [29, Section 3.2]. For each fixed transformation that cocycle has finite image outside a null set. On the identity component its values lie in $K(S, 3)$, so the same is true as a $K(S, 3)$ -valued cocycle, and coordinatewise on the product. Hence, formula (58) is a finite sum even for an ordinary, unbounded cochain. It defines an ordinary cochain map with the same formula on bounded cochains. Thus, if $c = \delta B$ with c bounded,

$$\Gamma'_{S,3,k}c = \delta(\Gamma'_{S,3,k}B).$$

This proves preservation of exactness, and the norm estimate proves the reduced assertion. No ordinary configuration-space transfer is asserted here for arbitrary homeomorphisms. $\square$

We use the disc rotations of Lemma 15 in pairwise disjoint embedded discs $D_1, \dots, D_k \subset S$. Each D_j contains two overlapping rotation discs with closures in $\text{int}(D_j)$. The time-one maps of their Hamiltonian rotations define $\rho_{j,\varepsilon} : F_2 \rightarrow \text{Ham}(S, \omega)$, extended by the identity outside D_j . Their commuting images define

$$\rho_\varepsilon(w_1, \dots, w_k) = \rho_{1,\varepsilon}(w_1) \cdots \rho_{k,\varepsilon}(w_k).$$

Let B_ε be the union of all transition collars and put $b_\varepsilon = \mu(B_\varepsilon)$. Let $U_{j,1,\varepsilon}, U_{j,2,\varepsilon}, U_{j,3,\varepsilon}$ be, in order, the first exclusive region, the overlap, and the second exclusive region, with the collars removed. The construction can be chosen so that

$$\lambda_{j,\varepsilon} = 6 \prod_{s=1}^3 \mu(U_{j,s,\varepsilon}) \longrightarrow \lambda_j > 0, \quad b_\varepsilon \longrightarrow 0.$$

Let $p_j : F_2^k \rightarrow F_2$ denote the coordinate projections.

Lemma 21. *For $d \geq 2$ and $u \in \overline{H}_b^d(K(S, 3))$, the above homomorphisms satisfy*

$$(59) \quad \left\| \rho_\varepsilon^* \Gamma_{S,3}^d(u) - \sum_{j=1}^k \lambda_j p_j^* j^* \iota^* u \right\| \leq \left(3b_\varepsilon + \sum_{j=1}^k |\lambda_{j,\varepsilon} - \lambda_j| \right) \|u\|.$$

In particular, the expression on the left tends to zero.

Proof. We check that the configuration decomposition used for the disc remains valid after inclusion into S and, for the torus, after the central quotient. The set of triples having a point in B_ε is invariant under the time-one maps, with measure at most $3b_\varepsilon$. Its integral is a cochain map of norm at most $3b_\varepsilon$, by the same cocycle and change-of-variables argument as in Lemma 15.

Outside this set every marked point is fixed by every time-one map. Tracing therefore gives a homomorphism on F_2^k . A supporting disc containing no more than one of the selected points gives the trivial braid: a one-point trajectory contracts inside that disc, without meeting either of the other selected points. A triple has two or more points in at most one of the pairwise disjoint discs. Hence at most one factor $\rho_{j,\varepsilon}$ can contribute nontrivially.

Within that factor the complete rotations give exactly the local braids listed in Lemma 15. A disc enclosing zero or one point gives the identity. Enclosing all three gives the full twist, which is central in the *local* B_3 ; it need not be central in $K(S, 3)$. Enclosing two points gives the twist about that pair. If the two rotations enclose the same pair, the two curves are isotopic in the

local marked disc. If one generator is trivial or the local full twist, or both twist about the same pair, the image is abelian. Its image under inclusion into $K(S, 3)$ is still amenable. The only remaining case has one point in each of the three regions $U_{j,1,\varepsilon}, U_{j,2,\varepsilon}, U_{j,3,\varepsilon}$.

In their given order, this case reads σ_1^2, σ_2^2 on the two generators. A different closing path or label assignment conjugates both generators by the same element of $K(S, 3)$. To identify different supporting discs, move the marked model disc by an ambient isotopy from the identity and use its traced braid as the connecting path. Thus these local inclusions are conjugate to the fixed ι in $K(S, 3)$. Consequently the homomorphism on this region is conjugate to ιp_j . There are six label assignments, with total measure $\lambda_{j,\varepsilon}$. The uniformly bounded inner-automorphism homotopy [30, Lemma A.2] removes the measurable conjugators. The contribution is therefore $\lambda_{j,\varepsilon} p_j^* j^* \iota^* u$.

All other regions, except of possibly the region that contain triples having a point in B_ε , give zero in positive-degree bounded cohomology because their image is amenable. Combining the regions and integrating gives the stated estimate with $\lambda_{j,\varepsilon}$ in place of λ_j and error $3b_\varepsilon \|u\|$. Pullbacks have norm at most one, so replacing those constants by their limits proves (59). $\square$

5.1. Hyperbolic surfaces. We first prove Theorem 3 for $g \geq 2$. The essential neighborhood is fixed once and for all, while the number of additional supporting discs is unrestricted. The next degree-two splitting lemma will also be used in the torus proof.

Lemma 22. *Let $p_j : F_2^k \rightarrow F_2$ and $e_j : F_2 \rightarrow F_2^k$ denote the coordinate projections and inclusions. For every $\xi \in \overline{H}_b^2(F_2^k)$,*

$$(60) \quad \xi = \sum_{j=1}^k p_j^* e_j^* \xi.$$

Proof. Note that the comparison map

$$c_{F_2^k}^d : H_b^d(F_2^k) \longrightarrow H^d(F_2^k; \mathbb{R})$$

vanishes for every $k \geq 1$ and $d \geq 1$. Indeed, the ordinary Künneth formula shows that $H_d(F_2^k; \mathbb{R})$ is generated by images of fundamental classes of d -tori; see [27, Section 3.B]. Every bounded class restricts trivially to their fundamental groups, since these groups are amenable. Naturality of comparison and the universal coefficient theorem therefore imply the assertion. For the same argument in degree two, see [23, Remark 5.6 and proof of Theorem 5.15].

It follows that every degree-two bounded class is represented by δq , where q is a homogeneous quasimorphism on F_2^k , and hence $q(xy) = q(x) + q(y)$

whenever $xy = yx$. Consequently,

$$q(g_1, \dots, g_k) = \sum_{j=1}^k q(e_j(g_j)).$$

Taking coboundaries gives (60) already in bounded cohomology and hence in reduced bounded cohomology. $\square$

Let $S = \Sigma_g$, $g \geq 2$, and choose a compact one-holed torus $Q \subset S$ with nonempty complementary interior. Choose a basepoint in Q and simple meridian and longitude loops giving $i : F_2 \hookrightarrow \pi_1(S)$. In a standard surface presentation, write these as a_1, b_1 and complete them to

$$\pi_1(S) = \left\langle a_1, b_1, \dots, a_g, b_g \left| \prod_{j=1}^g [a_j, b_j] = 1 \right. \right\rangle.$$

Define $r : \pi_1(S) \rightarrow F_2 = \langle a, b \rangle$ by

$$\begin{aligned} r(a_1) &= a, & r(b_1) &= b, & r(a_2) &= b, & r(b_2) &= a, \\ r(a_j) &= r(b_j) = 1 & (3 \leq j \leq g). \end{aligned}$$

The relator maps to $[a, b][b, a] = 1$, so r is a homomorphism and $r \circ i = \text{id}$. The surface group is center free, since $g \geq 2$. Thus it is the group G in Lemma 10. For $q = 2, 3$ define the one-point transfer classes

$$(61) \quad \Theta_q(v) = \bar{\Gamma}_{b,1}^q(r^*v), \quad v \in \bar{H}_b^q(F_2).$$

Since $H^q(F_2; \mathbb{R}) = 0$, the map r^* has exact image. The comparison identity in Lemma 10 gives

$$\Theta_q(v) \in \bar{E}H_b^q(\mathcal{T}_S), \quad \|\Theta_q(v)\| \leq \|v\|.$$

Here the one-point transfer is defined on the compact surface. Its finite-image proof uses continuity of isotopies and compactness, not smoothness or the dimension assumption in Theorem 1. Thus this exactness statement includes $\text{Homeo}_0(S, \mu)$.

Lemma 23. *There are homomorphisms $s_\delta : F_2 \rightarrow \text{Ham}(S, \omega)$ supported in $\text{int}(Q)$ and a number $\kappa > 0$, independent of v and $q \in \{2, 3\}$, such that*

$$(62) \quad s_\delta^* \Theta_q(v) \longrightarrow \kappa v \quad \text{in } \bar{H}_b^q(F_2) \quad (\delta \longrightarrow 0).$$

Proof. Choose thin annuli $N_a, N_b \subset \text{int}(Q)$ about the two simple generator loops, with intersection a disc. On each annulus use coordinates (ϑ, p) with $\omega = d\vartheta \wedge dp$, where $\vartheta \in S^1$, and use the compactly supported Hamiltonian $H_\delta = -p\chi_\delta(p)$. Here χ_δ equals one on intervals exhausting the annulus and is zero near its boundary. Its vector field is $(\chi_\delta + p\chi'_\delta)\partial_\vartheta$ under the convention $\iota_{X_H}\omega = -dH$. Its flow makes one complete circuit on the region where $\chi_\delta = 1$. It is the annular Hamiltonian construction of Lemma 11 in dimension two. Send the two free generators to the two time-one maps to obtain s_δ .

Outside the union C_δ of the transition collars every point is fixed by both endpoints. The set C_δ is invariant under the endpoints: a measure-preserving bijection fixing its complement pointwise preserves it. Its measure tends to zero. On the fixed overlap U_δ , the tracing homomorphism is conjugate to i . On each exclusive annular region its image is cyclic, and outside both annuli it is trivial. This is the one-point calculation of [10, Lemma 3.1]; the simple loops chosen above satisfy its surface hypothesis. Thus we obtain

$$\|s_\delta^* \Theta_q(v) - \mu(U_\delta)v\| \leq \mu(C_\delta)\|r^*v\|.$$

The cyclic and trivial images have zero positive-degree bounded cohomology. Also $i^*r^* = \text{id}$ and both pullbacks are norm non-increasing, so $\|r^*v\| = \|v\|$. Set $\kappa = \mu(N_a \cap N_b) > 0$. Since $\mu(U_\delta) \rightarrow \kappa$, the last estimate proves (62). $\square$

Choose m pairwise disjoint closed discs $D_1, \dots, D_m \subset S \setminus Q$ and use the local rotations of Lemma 21 inside them. The supports commute with each other and with s_δ . Hence

$$(63) \quad \begin{aligned} \rho_{\varepsilon, \delta} : F_2^{m+1} &\longrightarrow \text{Ham}(S, \omega), \\ \rho_{\varepsilon, \delta}(g_1, \dots, g_m, h) &= \rho_{1, \varepsilon}(g_1) \cdots \rho_{m, \varepsilon}(g_m) s_\delta(h). \end{aligned}$$

Let $p_j : F_2^{m+1} \rightarrow F_2$ again denote the projections.

Lemma 24. *For the homomorphism (63) and every $v \in \overline{H}_b^q(F_2)$, $q = 2, 3$, we have the following identity*

$$(64) \quad \rho_{\varepsilon, \delta}^* \Theta_q(v) = p_{m+1}^* s_\delta^* \Theta_q(v).$$

Proof. Choose the one-point closing path for $x \in D_j$ by first going from the global basepoint to a fixed point of D_j and then taking a path inside D_j . Outside the finitely many discs use the bounded-length closing paths from Lemma 10. The additional local paths may also be chosen with bounded length. Thus these choices retain the finite-image property.

If an isotopy is supported in D_j , its closed trajectory for a point of D_j is conjugate to a loop entirely inside D_j . This loop is contractible. Therefore its one-point cocycle has value 1 everywhere. This holds also for products of isotopies supported in the pairwise disjoint discs, including points in their transition collars. The isotopy defining $s_\delta(h)$ is stationary on every D_j .

Write $\gamma_{S,1}$ for the one-point cocycle and $e = \rho_{1, \varepsilon}(g_1) \cdots \rho_{m, \varepsilon}(g_m)$. Using the concatenated isotopies, the cocycle identity gives, for every point where the closing paths are defined,

$$\gamma_{S,1}(es_\delta(h), x) = \gamma_{S,1}(e, s_\delta(h)x) \gamma_{S,1}(s_\delta(h), x) = \gamma_{S,1}(s_\delta(h), x).$$

The one-point cocycle is independent of isotopy because the center of the surface group is trivial. Let $\mathbf{w}_i = (g_{1,i}, \dots, g_{m,i}, h_i) \in F_2^{m+1}$. For a bounded homogeneous cochain c on $\pi_1(S)$, put

$$c_{\mathbf{h}}(x) = c((\gamma_{S,1}(s_\delta(h_i), x))_{i=0}^q).$$

The preceding cocycle identity and the transfer formula give

$$(\rho_{\varepsilon,\delta}^* \Gamma'_{S,1} c)(\mathbf{w}_0, \dots, \mathbf{w}_q) = \int_S c_{\mathbf{h}}(x) d\mu(x) = (p_{m+1}^* s_{\delta}^* \Gamma'_{S,1} c)(\mathbf{w}_0, \dots, \mathbf{w}_q).$$

Here $\Gamma'_{S,1}$ is the one-point cochain map of Lemma 10 for this choice of closing paths. Take c to be the pullback by r of a cocycle representing v . This proves (64) on cochains for the chosen section, and therefore on reduced bounded cohomology. Independence of closing paths gives the same cohomology identity for any section. $\square$

Proof of Theorem 3 for $g \geq 2$. Let $S = \Sigma_g$ with $g \geq 2$, and fix the one-point maps (61). Choose α, η, z from Lemma 19 and put $A = \Gamma_{S,3}^2(\alpha) \in \overline{H}_b^2(\mathcal{T}_S)$. The variable class will come from $\overline{H}_b^q(F_2)$ through the one-point map, not from degree-three cohomology of a braid group. For $m \geq 0$ and $q \in \{2, 3\}$, define $\Phi_{m,q}(v)$ as the transfer over $X_3(S)^m \times S$ of the symmetric class

$$\frac{1}{m!} (\text{Sym}_m(\alpha^{\times m})) \times r^* v \in \overline{H}_b^{2m+q}(K(S, 3)^m \times \pi_1(S)).$$

Here Sym_m permutes only the first m group coordinates and fixes the last one. Since all m fixed factors have degree two and equal α , this class is $\alpha^{\times m} \times r^* v$. The factor $1/m!$ is the same normalized-symmetrization convention as (26), applied to the identical factors. The cochain calculation below shows

$$(65) \quad \begin{aligned} \Phi_{m,q} : \overline{H}_b^q(F_2) &\longrightarrow \overline{H}_b^{2m+q}(\mathcal{T}_S), \\ \Phi_{m,q}(v) &= A^{\smile m} \smile \Theta_q(v). \end{aligned}$$

For $m = 0$ this is Θ_q . We will prove injectivity, then verify that every image is reduced exact, including for homeomorphisms.

The mixed product transfer on cochains. The symmetric construction just defined has the same transfer as the ordered product $\alpha^{\times m} \times r^* v$. Unlike the other product transfers, its last integration factor is S , not $X_3(S)$. Thus we do not permute it with a configuration factor. The diagonal action has the product cocycle whose first m entries are $\gamma_{S,3}$ and whose last entry is $\gamma_{S,1}$, with values in $K(S, 3)^m \times \pi_1(S)$. Its integration is a bounded cochain map by the same cocycle and identities as (58). For an explicit verification of the product identity, choose a bounded 2-cocycle c representing α and a bounded q -cocycle e representing $r^* v$. For fixed $\mathbf{f} = (f_0, \dots, f_{2m+q})$ set

$$C_{j,\mathbf{f}}(\mathbf{x}) = c((\gamma_{S,3}(f_i, \mathbf{x}))_{i=2j-2}^{2j}), \quad E_{\mathbf{f}}(y) = e((\gamma_{S,1}(f_i, y))_{i=2m}^{2m+q}).$$

Writing $\mathbf{X} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)})$, Fubini gives

$$\begin{aligned} & \int_{X_3(S)^m \times S} \left(\prod_{j=1}^m C_{j,\mathbf{f}}(\mathbf{x}^{(j)}) \right) E_{\mathbf{f}}(y) d\nu_m(\mathbf{X}) d\mu(y) \\ &= \left(\prod_{j=1}^m \int_{X_3(S)} C_{j,\mathbf{f}} d\nu \right) \left(\int_S E_{\mathbf{f}} d\mu \right). \end{aligned}$$

The last line is the evaluation of the cup product of the transferred cochains, with the successive degree intervals displayed in the definitions of $C_{j,\mathbf{f}}$ and $E_{\mathbf{f}}$. All integrands are bounded. Permuting the first m configuration variables in this integral leaves it unchanged, exactly as in (26). Hence its value also equals the transfer of the normalized symmetric class used in the definition of $\Phi_{m,q}$. No permutation involving the last variable is used. In particular, the mixed bounded transfer is defined for homeomorphisms without integrating an unbounded primitive.

Two coordinates: one disc and one essential neighborhood. For $m = 1$ choose a supporting disc outside Q and use $\rho_{\varepsilon,\delta}$ from (63). Write

$$\eta_{\varepsilon} = \rho_{1,\varepsilon}^* A, \quad E_{\delta} = s_{\delta}^* A, \quad v_{\delta} = s_{\delta}^* \Theta_q(v).$$

Lemmas 22 and 24 give

$$\rho_{\varepsilon,\delta}^* A = p_1^* \eta_{\varepsilon} + p_2^* E_{\delta}, \quad \rho_{\varepsilon,\delta}^* \Theta_q(v) = p_2^* v_{\delta}.$$

These are exact identities in reduced bounded cohomology. Their product is

$$(66) \quad \rho_{\varepsilon,\delta}^* \Phi_{1,q}(v) = \eta_{\varepsilon} \times v_{\delta} + 1 \times (E_{\delta} \smile v_{\delta}).$$

To remove the second summand, let $\tau_1 : F_2^2 \rightarrow F_2^2$ be given by $\tau_1(g, h) = (1, h)$, and put $\Pi_1 = \text{id} - \tau_1^*$. The pullback τ_1^* annihilates $\eta_{\varepsilon} \times v_{\delta}$ and fixes $1 \times (E_{\delta} \smile v_{\delta})$. Therefore

$$(67) \quad \Pi_1 \rho_{\varepsilon,\delta}^* \Phi_{1,q}(v) = \eta_{\varepsilon} \times v_{\delta} \longrightarrow \lambda_1 \kappa \eta \times v.$$

The limit follows from $\eta_{\varepsilon} \rightarrow \lambda_1 \eta$ and $v_{\delta} \rightarrow \kappa v$ in Lemmas 21 and 23, since the external product is bounded and bilinear. If $\Phi_{1,q}(v) = 0$, (67) gives $\eta \times v = 0$. By Lemma 8 and $\langle \eta, z \rangle = 1$, external product with η is injective, so $v = 0$. This proves injectivity for $q = 2, 3$, without any condition on the class E_{δ} .

An arbitrary number of disc factors. For $m \geq 2$ choose m pairwise disjoint supporting discs in $S \setminus Q$. Such a finite collection exists for every m ; its areas and the positive constants below may depend on m . Put $\eta_{j,\varepsilon} = \rho_{j,\varepsilon}^* A$ and retain E_{δ}, v_{δ} as above. The same two lemmas give

$$(68) \quad \rho_{\varepsilon,\delta}^* \Phi_{m,q}(v) = \left(\sum_{j=1}^m p_j^* \eta_{j,\varepsilon} + p_{m+1}^* E_{\delta} \right) \smile^m \smile p_{m+1}^* v_{\delta}.$$

Removing terms with an unused coordinate. Fix $k \geq 1$. For $1 \leq j \leq k$, let $\tau_j : F_2^k \rightarrow F_2^k$ replace coordinate j by the identity and leave every other coordinate unchanged. For $0 \leq r \leq k$, set

$$\Pi_r = (\text{id} - \tau_1^*) \cdots (\text{id} - \tau_r^*), \quad \Pi_0 = \text{id}.$$

The product F_2^k is always specified; Π_r acts only on its first r coordinates. The pullbacks commute and are idempotent, so Π_r is a bounded projection with $\|\Pi_r\| \leq 2^r$. A positive-degree class pulled back by the trivial homomorphism is zero. Therefore τ_j^* annihilates an external product whose j th factor has positive degree and fixes one whose j th factor has degree zero. It follows that Π_r fixes a product having positive degree in all of its first r coordinates, and annihilates a product with degree zero in any of those coordinates.

In particular, let $k = m + 1$, let θ, z satisfy the hypotheses of Proposition 9, let $v \in \overline{H}_b^q(F_2)$ with $q = 2, 3$, and let $c_1, \dots, c_k$ be real numbers. Then

$$\Pi_k \left[\left(\sum_{j=1}^k c_j p_j^* \theta \right)^{\smile^m} \smile \left(\sum_{j=1}^k c_j p_j^* v \right) \right] = \left(\prod_{j=1}^k c_j \right) T_m^q(v).$$

Here $p_j : F_2^k \rightarrow F_2$ is projection. To verify the identity, expand the product and regard its m fixed factors as separately labelled. There are k factors of positive degree to assign to k coordinates. The projection removes any term with an unused coordinate, so only the bijective assignments remain. Each has coefficient $\prod_j c_j$, and their sum is precisely $\text{Sym}_k(\theta^{\times m} \times v)$. No cup-product vanishing is assumed in this calculation. We use Π_m acting on the first m coordinates of F_2^{m+1} . The last coordinate, carrying v_δ , is kept fixed. The bound is $\|\Pi_m\| \leq 2^m$.

Expand the power in (68). There are exactly m degree-two factors to distribute. A term using E_δ or assigning two factors to the same disc coordinate leaves another disc coordinate unused, and is therefore sent to zero by Π_m . The remaining terms assign one factor to each of the first m coordinates. There are $m!$ such assignments. They agree, with no sign change because all assigned factors have even degree. Hence

$$(69) \quad \Pi_m \rho_{\varepsilon, \delta}^* \Phi_{m, q}(v) = m! \eta_{1, \varepsilon} \times \cdots \times \eta_{m, \varepsilon} \times v_\delta.$$

For $m = 2$, (69) is $2 \eta_{1, \varepsilon} \times \eta_{2, \varepsilon} \times v_\delta$; Π_2 here acts on the first two coordinates of F_2^3 . The approximation lemmas and boundedness of the external product show that the right side converges in norm to

$$(70) \quad m! \kappa \left(\prod_{j=1}^m \lambda_j \right) \eta^{\times m} \times v.$$

If $\Phi_{m, q}(v) = 0$, (69) and the positive coefficient in (70) give $\eta^{\times m} \times v = 0$ in reduced bounded cohomology. Lemma 8 again gives $v = 0$.

For the lower norm estimate, there is a ℓ^1 -cycle $z^{\times m}$, and $\langle \eta^{\times m}, z^{\times m} \rangle = 1$ by Lemma 8. Its product identity and norm bound give

$$\|\eta^{\times m} \times v\| \geq \frac{\|v\|}{\binom{2m+q}{2m} \|z^{\times m}\|_1}.$$

Since $\|\Pi_m\| \leq 2^m$ and pullback is norm non-increasing, (69)–(70) yield

$$\|\Phi_{m,q}(v)\| \geq \frac{m! \kappa \prod_{j=1}^m \lambda_j}{2^m \binom{2m+q}{2m} \|z^{\times m}\|_1} \|v\| \quad (m \geq 1).$$

The denominator is positive by the pairing identity. For $m = 0$, the lower bound is $\kappa\|v\|$. The upper bound is $\|\Phi_{m,q}(v)\| \leq \|A\|^m \|v\|$.

Exactness, including for homeomorphisms. By (61) and Lemma 10, the reduced class $\Theta_q(v)$ is the image of an exact bounded-cohomology class. Choose a bounded cocycle d_v representing that exact class and an ordinary cochain B_v on $\mathcal{T}_S$ with $d_v = \delta B_v$. Choose a bounded 2-cocycle c_A representing A . The cochain B_v may be unbounded, but the Leibniz identity for ordinary cochains gives

$$(71) \quad c_A^{\smile m} \smile d_v = \delta(c_A^{\smile m} \smile B_v).$$

The cocycle on the left is bounded and represents $\Phi_{m,q}(v)$, while the expression on the right gives its ordinary primitive directly on the transformation group. Formula (71) proves that the class is reduced exact without assuming that A is exact. In particular, for homeomorphisms we never integrate an unbounded cochain over $X_3(S)$.

Both $\overline{H}_b^2(F_2)$ and $\overline{H}_b^3(F_2)$ are infinite-dimensional, and $2m+2, 2m+3$, $m \geq 0$, exhaust all degrees at least two. The injective maps in (65) give both conclusions of the theorem. Every testing homomorphism takes values in $\text{Ham}(S, \omega)$, and all cocycle calculations apply to each of the four stated transformation groups. This proves Theorem 3 for $g \geq 2$. $\square$

5.2. Torus case. We finish the proof of Theorem 3 with the torus. The quotient of its fundamental group by its center is trivial, so the one-point factor used above cannot be used here. We replace it by Kimura's two-point transfer. The first task is to keep the annular construction disjoint from a fixed disc; the second is to check that adding local disc isotopies does not change the two-point factor in positive-degree bounded cohomology.

Put $T = T^2$ and use $K(T, 2)$ as defined before Theorem 3. The Birman map and the two-strand braid identifications give

$$K(T, 2) = B_2(T)/Z(B_2(T)), \quad P_2(T)/Z(P_2(T)) \cong F_2.$$

Let $j_T : F_2 \hookrightarrow K(T, 2)$ denote the inclusion of this pure quotient. Its index is two. With a base configuration (z_1, z_2) , choose the generators a, b by moving z_1 once along a meridian or longitude in $T \setminus \{z_2\}$, while keeping z_2 fixed. Moving the second point instead gives braids a_2, b_2 ; the simultaneous motions

aa_2, bb_2 generate the center of $P_2(T)$. In particular, the free group here is the fundamental group of a punctured torus, see [29, Section 5.3].

Lemma 25. *For $q = 2, 3$, the space $\overline{H}_b^q(K(T, 2))$ is infinite-dimensional and equals $\overline{EH}_b^q(K(T, 2))$. Restriction*

$$j_T^* : \overline{H}_b^q(K(T, 2)) \longrightarrow \overline{H}_b^q(F_2)$$

is an isometric embedding.

Proof. As recalled in [29, proof of Corollary 1.4, Section 5.4],

$$K(T, 2) \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2.$$

This group is non-elementary hyperbolic. Its second bounded cohomology is infinite-dimensional by [16], and its third reduced exact bounded cohomology is infinite-dimensional by [22, Corollary 3]. The degree-two seminorm is a norm, as recalled in Lemma 7. This proves infinite-dimensionality before using restriction to the free subgroup. The subgroup $j_T(F_2)$ is normal of index two in $K(T, 2)$. Hence restriction is injective and isometric in bounded cohomology by [29, Theorem 2.1]. Thus $\|j_T^* u\| = \|u\|$, and passing to the quotients by zero-seminorm classes gives the asserted isometric embedding on reduced bounded cohomology.

Ordinary restriction to the subgroup of index two is injective over $\mathbb{R}$: composition with ordinary transfer is multiplication by two. Since $H^q(F_2; \mathbb{R}) = 0$ for $q \geq 2$, it follows that $H^q(K(T, 2); \mathbb{R}) = 0$ in these degrees. Thus the comparison map is zero, and all the bounded classes in the statement are exact, also after reduction. $\square$

Let $X_2(T)$ denote the ordered configuration space of two points, with measure $\nu_2 = \mu^2|_{X_2(T)}$. The cocycle, followed by the Birman map, is

$$\gamma_{T,2} : \mathcal{T}_T \times X_2(T) \longrightarrow K(T, 2).$$

It is independent of isotopy after taking this quotient. Its values lie in the pure subgroup $j_T(F_2)$; allowing the full group $K(T, 2)$ in the domain of cohomology accounts for the exchange of the two labels. For a bounded homogeneous cochain c and $\mathbf{f} = (f_0, \dots, f_q)$, define

$$(\Gamma'_{T,2}c)(\mathbf{f}) = \int_{X_2(T)} c((\gamma_{T,2}(f_i, \mathbf{x}))_{i=0}^q) d\nu_2(\mathbf{x}).$$

The cocycle identity, invariance of ν_2 , and the differential show that this is a bounded cochain map, exactly as in Lemma 20. Its norm is at most one. Let

$$(72) \quad \Theta_{T,q} : \overline{H}_b^q(K(T, 2)) \longrightarrow \overline{H}_b^q(\mathcal{T}_T)$$

denote the induced map. For smooth transformations it preserves reduced exact bounded cohomology. Indeed, the finite-image result [9, Lemma 2.1] applies also to two marked points. For that section the same formula is a finite sum for each transformation tuple, even for an ordinary cochain. Hence

it transfers ordinary primitives, as in Lemma 20. No ordinary two-point transfer on all cochains is asserted for arbitrary homeomorphisms. Lemma 28 below gives ordinary primitives for a specified family of degree-two classes. The bounded transfer is independent of the closing paths by the uniformly bounded change-of-section homotopy.

Lemma 26 (Localized two-point approximation). *There are a closed disc $D_0 \subset T$, homomorphisms $s_\delta : F_2 \rightarrow \text{Ham}(T, \omega)$ whose defining isotopies are supported in $T \setminus D_0$, and a constant $\kappa_T > 0$ such that*

$$(73) \quad \|s_\delta^* \Theta_{T,q}(u) - \kappa_T J_T^* u\| \leq e_\delta \|u\|, \quad e_\delta \longrightarrow 0,$$

for $q = 2, 3$ and $u \in \overline{H}_b^q(K(T, 2))$. The choices of D_0 , the homomorphisms, and the constants are independent of u and q .

Proof. We use the two-point calculation of [29, Lemma 5.3]. Choose a small closed disc U about z_1 , and choose meridian and longitude paths passing through U . Outside U their two bands are disjoint. Choose a closed disc D_0 about z_2 disjoint from a fixed small neighborhood of U and these paths.

Adjoining one thin band to U gives a meridional annulus, and adjoining the other gives a longitudinal annulus. Shrink the widths of the bands and the boundary collars to zero. This gives nested annuli $W_{a,\delta} \Subset V_{a,\delta}$ and $W_{b,\delta} \Subset V_{b,\delta}$, all avoiding D_0 , and disjoint open discs $U_{1,\delta}, U_{2,\delta}$ with the following properties: $U_{1,\delta}$ contains z_1 and lies in both inner annuli; $U_{2,\delta}$ contains D_0 and is disjoint from the two outer annuli; and

$$\begin{aligned} \mu(U_{1,\delta}) &\longrightarrow a_0 := \mu(U) \in (0, 1), & \mu(U_{2,\delta}) &\longrightarrow 1 - a_0, \\ b_\delta &:= \mu(T \setminus (U_{1,\delta} \cup U_{2,\delta})) &\longrightarrow 0. \end{aligned}$$

For clarity, $U_{1,\delta}$ is a slightly smaller disc in U . The complement of the two outer annuli is a disc because their union is a regular neighborhood of a meridian and longitude meeting in one disc. Take $U_{2,\delta}$ slightly inside this complementary disc, retaining D_0 . The area left outside the two good discs consists of the shrinking bands and collars, see Figure 6. This also explains why the unused disc D_0 can be fixed while δ tends to zero.

On each outer annulus choose area coordinates (ϑ, p) , $\vartheta \in S^1$, in which its inner annulus is a subannulus and $\omega = d\vartheta \wedge dp$. The area-preserving annulus parametrization permits these coordinates; equivalently, straighten the two inner boundary curves first and use area as the transverse coordinate. Take

$$H_\delta(\vartheta, p) = -p\chi_\delta(p),$$

where $\chi_\delta = 1$ on the inner annulus and vanishes near the outer boundary. With $\iota_{X_H} \omega = -dH$, the vector field is $(\chi_\delta + p\chi'_\delta)\partial_\vartheta$. Its flow makes one complete positive circuit on the inner annulus and is stationary off the outer annulus. The transition collars, including the compensating motion required for a Hamiltonian, are retained in the exceptional set. Extending H_δ by zero

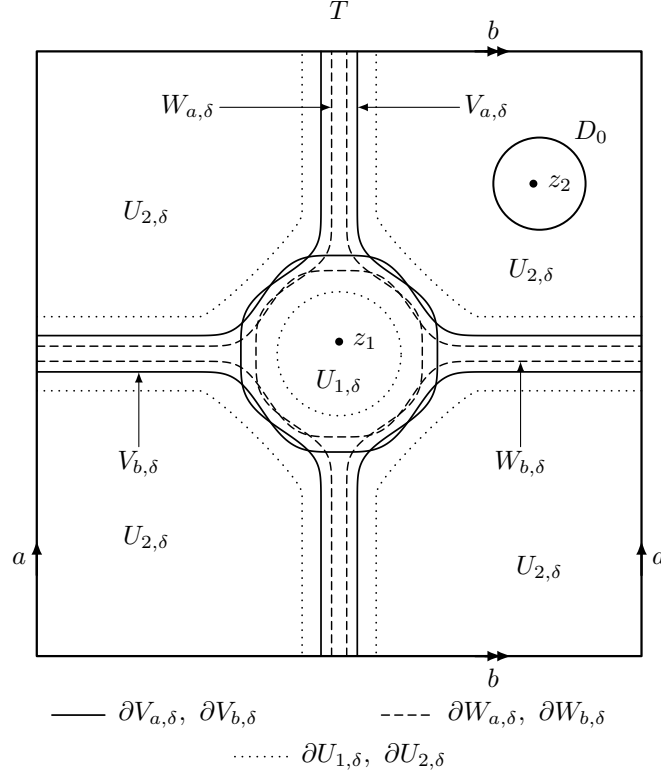


FIGURE 6. Annuli for the localized two-point construction on T . Solid and dashed curves bound the outer and inner annuli, respectively; dotted curves bound the two discs $U_{1,\delta}$ and $U_{2,\delta}$. The four portions labelled $U_{2,\delta}$ form one disc containing D_0 and z_2 . The disc $U_{1,\delta}$ contains z_1 and lies in both inner annuli, while $U_{2,\delta}$ is disjoint from both outer annuli.

gives a Hamiltonian on T . Send a, b to the two time-one maps to define s_δ . Their chosen isotopies are stationary on D_0 .

For a good two-point configuration, tracing the endpoints defines a homomorphism from F_2 . If both points lie in $U_{2,\delta}$, this homomorphism is trivial. If both lie in $U_{1,\delta}$, one complete meridional rotation moves both points together and represents a conjugate of aa_2 ; the longitudinal rotation similarly represents a conjugate of bb_2 . These braids are central and hence trivial in $K(T, 2)$. With one point in each disc, the two rotations give, up to one common conjugation, the generators of $\mathcal{J}_T(F_2)$. This is exactly the three-type computation in [29, proof of Lemma 5.3]. The simultaneous conjugator is obtained by identifying the marked meridian and longitude with the chosen model; it works on both generators and therefore on every word. Interchanging the two labels conjugates this homomorphism in the full group $K(T, 2)$.

The two mixed good regions have total measure

$$\kappa_{T,\delta} = 2\mu(U_{1,\delta})\mu(U_{2,\delta}) \longrightarrow \kappa_T := 2a_0(1 - a_0) > 0.$$

Their cohomology contribution is $\kappa_{T,\delta}j_T^*u$. The other good regions contribute zero. The exceptional configurations have at least one point outside the two good discs, and measure at most $2b_\delta$. That set is invariant under the endpoints, since its complement is fixed pointwise. Its integral is a cochain map of norm at most $2b_\delta$. Taking the infimum over bounded cocycles representing u gives

$$\|s_\delta^*\Theta_{T,q}(u) - \kappa_{T,\delta}j_T^*u\| \leq 2b_\delta\|u\|.$$

Thus (73) holds with $e_\delta = 2b_\delta + |\kappa_{T,\delta} - \kappa_T|$. $\square$

Choose m pairwise disjoint closed discs $D_1, \dots, D_m \subset \text{int}(D_0)$. Use the three-point local rotations of Lemma 21 in these discs. Their supports are disjoint from those of s_δ for every δ . They give the homomorphism

$$(74) \quad \begin{aligned} \rho_{\varepsilon,\delta} : F_2^{m+1} &\longrightarrow \text{Ham}(T, \omega), \\ \rho_{\varepsilon,\delta}(g_1, \dots, g_m, h) &= \rho_{1,\varepsilon}(g_1) \cdots \rho_{m,\varepsilon}(g_m) s_\delta(h). \end{aligned}$$

Let p_j denote the coordinate projections of F_2^{m+1} .

Lemma 27 (Two-point factorization). *Let $q \in \{2, 3\}$ and $u \in \overline{H}_b^q(K(T, 2))$. For the homomorphism in (74), we have*

$$(75) \quad \rho_{\varepsilon,\delta}^*\Theta_{T,q}(u) = p_{m+1}^*s_\delta^*\Theta_{T,q}(u).$$

Proof. Unlike one-point tracing, two-point tracing of an isotopy supported in a disc need not be trivial. We prove the identity by showing that this additional contribution is a bounded coboundary.

Discard the null sets with a point on a disc boundary. Let $Y_j = X_2(\text{int}(D_j))$ for $1 \leq j \leq m$, and let $Y_0 = X_2(T) \setminus \bigcup_{j=1}^m Y_j$. These sets are invariant under the combined action and under s_δ . Choose closing paths adapted to this partition. On Y_j , first follow a fixed path to a base configuration in D_j , then a measurable path inside $X_2(D_j)$. On Y_0 , at most one selected point lies in each disc. Choose its local part of the closing path inside that disc; any other selected point is kept outside it. If neither point is in a supporting disc, no adaptation is needed. Otherwise, first place each disc point at a fixed reference point in its disc and choose the other point, if any, outside those discs. Then move the disc points along paths in their discs. The last paths lie in contractible fibres.

Write $\gamma_2 = \gamma_{T,2}$ and $e = \rho_{1,\varepsilon}(g_1) \cdots \rho_{m,\varepsilon}(g_m)$. For $\mathbf{x} \in Y_0$, each local trajectory closes to a contractible loop in a disc containing no other selected point. Hence $\gamma_2(e, \mathbf{x}) = 1$. Since s_δ is stationary on all the discs, the cocycle identity gives

$$\gamma_2(es_\delta(h), \mathbf{x}) = \gamma_2(s_\delta(h), \mathbf{x}) \quad (\mathbf{x} \in Y_0).$$

This is an exact cocycle identity on Y_0 , including trajectories in the transition collars.

On Y_j , only the isotopy supported in D_j moves the two points. For the chosen paths its cocycle values lie in a fixed conjugate C_j of the image of $P_2(D_j) \cong \mathbb{Z}$ in $K(T, 2)$. Thus C_j is amenable. Choose a normalized bounded homogeneous q -cocycle c representing u . Normalization does not affect its class, by the standard bounded normalization homotopy. Amenable vanishing supplies a bounded homogeneous cochain b_j on C_j with $c|_{C_j} = \delta b_j$; see [21, Corollary 3.7].

For $\mathbf{w}_i = (g_{1,i}, \dots, g_{m,i}, h_i)$, abbreviate $\gamma_\rho(\mathbf{w}_i, \mathbf{x}) = \gamma_2(\rho_{\varepsilon, \delta}(\mathbf{w}_i), \mathbf{x})$. Define the bounded cochain

$$\mathcal{B}_j(\mathbf{w}_0, \dots, \mathbf{w}_{q-1}) = \int_{Y_j} b_j((\gamma_\rho(\mathbf{w}_i, \mathbf{x}))_{i=0}^{q-1}) d\nu_2(\mathbf{x}).$$

Its norm is at most $\nu_2(Y_j) \|b_j\|_\infty$. Invariance of Y_j and the cocycle identity prove homogeneity. Write $I_j c$ for the same integral with c . We obtain

$$(\delta \mathcal{B}_j)(\mathbf{w}_0, \dots, \mathbf{w}_q) = \int_{Y_j} (\delta b_j)((\gamma_\rho(\mathbf{w}_i, \mathbf{x}))_{i=0}^q) d\nu_2(\mathbf{x}) = (I_j c)(\mathbf{w}_0, \dots, \mathbf{w}_q).$$

Combining this with the identity on Y_0 proves

$$\rho_{\varepsilon, \delta}^* \Gamma'_{T,2} c - p_{m+1}^* s_\delta^* \Gamma'_{T,2} c = \delta \left(\sum_{j=1}^m \mathcal{B}_j \right).$$

This is a bounded coboundary. It proves (75) first in bounded cohomology and then after reduction. Independence of closing paths gives the same class identity for every section. Ordinary exactness is proved using a finite-image section for smooth transformations. $\square$

Lemma 28 (Exact two-point classes for torus homeomorphisms). *There is an infinite-dimensional linear subspace $\mathcal{E}_T \subset \overline{H}_b^2(K(T, 2))$ of codimension at most one such that*

$$\Theta_{T,2}(\mathcal{E}_T) \subset \overline{EH}_b^2(\text{Homeo}_0(T, \mu)).$$

Proof. Put $K = K(T, 2)$ and $\mathcal{H} = \text{Homeo}_0(T, \mu)$ for this proof. By Lemma 25, every class in $\overline{H}_b^2(K)$ is exact. It is represented by the coboundary of a homogeneous quasimorphism $\varphi : K \rightarrow \mathbb{R}$. This homogeneous quasimorphism is unique since $K \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$, and hence $\text{Hom}(K, \mathbb{R}) = 0$. There is no further identification on passing to reduced cohomology, because the degree-two seminorm is a norm.

Let $t \in K$ be the image of a generator of the local subgroup $B_2(D)$ for a disc D containing the two base points. Define

$$\mathcal{E}_T = \{[\delta \varphi] \in \overline{H}_b^2(K) : \varphi \text{ is homogeneous and } \varphi(t) = 0\}.$$

Since $\overline{EH}_b^2(K(T, 2)) = \overline{H}_b^2(K(T, 2))$ is infinite-dimensional and the condition $\varphi(t) = 0$ imposes at most one linear equation, $\mathcal{E}_T$ is infinite-dimensional.

Lift φ to $B_2(T)$ and restrict it to $P_2(T)$. It vanishes on the center and on the local pure subgroup generated by t^2 . Choose a full-measure two-cell whose interior contains D . Its two-strand pure braid group has the same cyclic generator. Consequently the lift satisfies the hypotheses of [7, Theorem 1.1]. In the identification $P_2(T) \cong F_2 \times \mathbb{Z}^2$, this local pure generator is the commutator $[a, b]$ in the free factor; see [8, Lemma 2.6].

Use the closing paths of [7, Section 2] and write $\gamma = \gamma_{T,2}$. Theorem 1.1 and its proof in that paper show that

$$Q_\varphi(f) = \int_{X_2(T)} \varphi(\gamma(f, \mathbf{x})) d\nu_2(\mathbf{x}) \quad (f \in \mathcal{H})$$

is well defined and is a quasimorphism.

We check that its coboundary represents the bounded transfer already defined in (72), not merely a class agreeing with it on the diffeomorphism subgroup. In the right-homogeneous cochain convention, put

$$\begin{aligned} F_\varphi(k_0, k_1) &= \varphi(k_0 k_1^{-1}) \quad (k_0, k_1 \in K), \\ C_\varphi(f_0, f_1) &= Q_\varphi(f_0 f_1^{-1}) \quad (f_0, f_1 \in \mathcal{H}). \end{aligned}$$

These are ordinary homogeneous 1-cochains and need not be bounded. The cocycle identity gives

$$\gamma(f_0, \mathbf{x}) \gamma(f_1, \mathbf{x})^{-1} = \gamma(f_0 f_1^{-1}, f_1 \mathbf{x}).$$

Since f_1 preserves ν_2 , we obtain

$$\begin{aligned} \int_{X_2(T)} F_\varphi(\gamma(f_0, \mathbf{x}), \gamma(f_1, \mathbf{x})) d\nu_2(\mathbf{x}) &= \int_{X_2(T)} \varphi(\gamma(f_0 f_1^{-1}, f_1 \mathbf{x})) d\nu_2(\mathbf{x}) \\ &= Q_\varphi(f_0 f_1^{-1}) = C_\varphi(f_0, f_1). \end{aligned}$$

All three expressions are finite by the preceding integrability statement. The cochain δF_φ is bounded and represents $[\delta\varphi]$. We have

$$\begin{aligned} (76) \quad (\delta C_\varphi)(f_0, f_1, f_2) &= \int_{X_2(T)} (\delta F_\varphi)(\gamma(f_0, \mathbf{x}), \gamma(f_1, \mathbf{x}), \gamma(f_2, \mathbf{x})) d\nu_2(\mathbf{x}) \\ &= (\Gamma'_{T,2} \delta F_\varphi)(f_0, f_1, f_2). \end{aligned}$$

Thus (76) supplies an ordinary primitive of the transferred bounded cocycle. It proves the asserted exactness before reduction and hence after reduction. $\square$

Proof of Theorem 3 for the torus. Choose α, η, z from Lemma 19 for $S = T$, and put $A = \Gamma_{T,3}^2(\alpha)$. Use the two-point maps (72). For $m \geq 0$, $q = 2, 3$, and

$u \in \overline{H}_b^q(K(T, 2))$, define

$$(77) \quad \begin{aligned} \Phi_{m,q}^T : \overline{H}_b^q(K(T, 2)) &\longrightarrow \overline{H}_b^{2m+q}(\mathcal{T}_T), \\ \Phi_{m,q}^T(u) &= A^{\smile m} \smile \Theta_{T,q}(u). \end{aligned}$$

The variable degree-three classes now come from the full two-strand quotient.

The mixed transfer. Write $X = X_3(T)$, $Y = X_2(T)$, and use their probability measures ν_3, ν_2 . The product space $X^m \times Y$ has the diagonal action and the cocycle with values in $K(T, 3)^m \times K(T, 2)$ obtained by using $\gamma_{T,3}$ in its first m entries and $\gamma_{T,2}$ in its last entry. Integration defines a cochain map, by the cocycle and differential identities in Lemma 20. We transfer the class

$$\frac{1}{m!}(\text{Sym}_m(\alpha^{\times m})) \times u.$$

Only the m identical three-point factors are permuted. The two-point factor is distinguished. The normalized class equals $\alpha^{\times m} \times u$, since all the fixed factors have degree two. For $m = 0$, omit these factors and their symmetrization.

For the cochain calculation, choose bounded cocycles c, e representing α, u , and fix $\mathbf{f} = (f_0, \dots, f_{2m+q})$. Put

$$\begin{aligned} C_{j,\mathbf{f}}(\mathbf{x}) &= c((\gamma_{T,3}(f_i, \mathbf{x}))_{i=2j-2}^{2j}), \quad E_{\mathbf{f}}(\mathbf{y}) = e((\gamma_{T,2}(f_i, \mathbf{y}))_{i=2m}^{2m+q}), \\ C_{\mathbf{f}}(\mathbf{X}) &= \prod_{j=1}^m C_{j,\mathbf{f}}(\mathbf{x}^{(j)}), \quad \mathbf{X} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}). \end{aligned}$$

Fubini gives

$$\int_{X^m \times Y} C_{\mathbf{f}}(\mathbf{X}) E_{\mathbf{f}}(\mathbf{y}) d\nu_3^{\otimes m}(\mathbf{X}) d\nu_2(\mathbf{y}) = \left(\prod_{j=1}^m \int_X C_{j,\mathbf{f}} d\nu_3 \right) \left(\int_Y E_{\mathbf{f}} d\nu_2 \right).$$

The last line is the evaluation of the cup product in (77). All functions are bounded, so this identity also applies to homeomorphisms.

Two factors: one disc and the annular construction. For $m = 1$ use (74), and put

$$\eta_\varepsilon = \rho_{1,\varepsilon}^* A, \quad E_\delta = s_\delta^* A, \quad u_\delta = s_\delta^* \Theta_{T,q}(u).$$

Lemma 22 and (75) give the exact class identities

$$\rho_{\varepsilon,\delta}^* A = p_1^* \eta_\varepsilon + p_2^* E_\delta, \quad \rho_{\varepsilon,\delta}^* \Theta_{T,q}(u) = p_2^* u_\delta.$$

Their product is

$$\rho_{\varepsilon,\delta}^* \Phi_{1,q}^T(u) = \eta_\varepsilon \times u_\delta + 1 \times (E_\delta \smile u_\delta).$$

Use $\tau_1(g, h) = (1, h)$ and $\Pi_1 = \text{id} - \tau_1^*$. This fixes the first external product and annihilates the second. Therefore

$$(78) \quad \Pi_1 \rho_{\varepsilon,\delta}^* \Phi_{1,q}^T(u) = \eta_\varepsilon \times u_\delta \longrightarrow \lambda_1 \kappa_T \eta \times j_T^* u.$$

Here (59) gives $\eta_\varepsilon \rightarrow \lambda_1 \eta$, and (73) gives $u_\delta \rightarrow \kappa_T j_T^* u$. External product with η is injective by Lemma 8, since $\langle \eta, z \rangle = 1$. Restriction j_T^* is injective by

Lemma 25. Thus a zero class on the left of (78) forces $u = 0$, for both $q = 2$ and $q = 3$. No vanishing of E_δ or of its internal cup products is assumed.

An arbitrary number of disc factors. For any $m \geq 2$ place m disjoint discs in $\text{int}(D_0)$ and put $\eta_{j,\varepsilon} = \rho_{j,\varepsilon}^* A$. The same two lemmas give

$$\rho_{\varepsilon,\delta}^* \Phi_{m,q}^T(u) = \left(\sum_{j=1}^m p_j^* \eta_{j,\varepsilon} + p_{m+1}^* E_\delta \right) \smile^m \smile p_{m+1}^* u_\delta.$$

Apply $\Pi_m = \prod_{j=1}^m (\text{id} - \tau_j^*)$ from the coordinate-projection construction, only on the first m coordinates. Every term with an unused disc coordinate is annihilated. There are exactly m degree-two factors to distribute, so a surviving term uses each disc coordinate once and uses no factor E_δ . The $m!$ surviving assignments have the same positive sign. Hence

$$(79) \quad \Pi_m \rho_{\varepsilon,\delta}^* \Phi_{m,q}^T(u) = m! \eta_{1,\varepsilon} \times \cdots \times \eta_{m,\varepsilon} \times u_\delta.$$

The limit of (79) in reduced norm is

$$(80) \quad m! \kappa_T \left(\prod_{j=1}^m \lambda_j \right) \eta^{\times m} \times j_T^* u.$$

Its coefficient is strictly positive. Lemma 8 makes external product with $\eta^{\times m}$ injective, and j_T^* is injective. Thus (79)–(80) prove injectivity of every $\Phi_{m,q}^T$. For $m = 0$, use (73) directly.

The norm argument is the same as for $g \geq 2$, and we obtain nonvanishing after reduction.

Exactness and completion of the proof. For each smooth transformation group, the two-point transfer preserves exactness, and Lemma 25 makes its domain exact. Choose a bounded cocycle d_u representing an exact lift of $\Theta_{T,q}(u)$ and an ordinary cochain B_u with $d_u = \delta B_u$. If c_A is a bounded cocycle representing A , the ordinary Leibniz identity gives

$$c_A^{\smile m} \smile d_u = \delta(c_A^{\smile m} \smile B_u).$$

This proves that the image of (77) is reduced exact for the smooth groups.

For $\mathcal{T}_T = \text{Homeo}_0(T, \mu)$, first take $q = 2$ and $u = [\delta\varphi] \in \mathcal{E}_T$ from Lemma 28. Equation (76) gives the ordinary homogeneous cochain C_φ with $d_u = \delta C_\varphi$ bounded and representing $\Theta_{T,2}(u)$. The same computation is now valid on the homeomorphism group:

$$(81) \quad c_A^{\smile m} \smile d_u = \delta(c_A^{\smile m} \smile C_\varphi) \quad (m \geq 0).$$

For $m = 0$ the factor $c_A^{\smile 0}$ is the degree-zero unit. The primitive in (81) is an ordinary cochain on the transformation group. It need not be bounded, and no unbounded three-point cochain is integrated. Thus every restriction $\Phi_{m,2}^T|_{\mathcal{E}_T}$ has reduced exact image. Its injectivity and lower norm estimate have already been proved on the entire domain. Since $\mathcal{E}_T$ is infinite-dimensional,

this gives infinite-dimensional reduced exact bounded cohomology in every even degree for torus homeomorphisms. We have not proved exactness of the degree-three two-point factor class on this group.

Both spaces $\overline{H}_b^2(K(T, 2))$ and $\overline{H}_b^3(K(T, 2))$ are infinite-dimensional by Lemma 25. The degrees $2m + 2$ and $2m + 3$, $m \geq 0$, exhaust all integers at least two. Thus the maps (77) give reduced bounded infinite-dimensionality in every degree, reduced exact infinite-dimensionality in every degree for the smooth groups, and the additional even-degree reduced exact conclusion for homeomorphisms. All testing homomorphisms are Hamiltonian, and the same cochain computations apply directly to each of the four groups. Together with the preceding proof for $g \geq 2$, this proves Theorem 3. $\square$

6. ORDINARY COHOMOLOGY, FINAL COMMENTS AND REMARKS

6.1. Ordinary cohomology. For completeness, we record the following consequence of the classical mass-flow construction [19, Proposition 5.1 and Section 6]. Combined with the ordinary cohomology of $\mathbb{R}$ regarded as a discrete group, it gives infinite-dimensionality in every positive degree under the hypothesis below. In the orientable setting, this yields a strengthening of the ordinary-cohomology statement in [35, Theorem 4.5].

Theorem 29 (Ordinary cohomology of volume-preserving groups). *Let M be a compact connected oriented smooth manifold of dimension $n \geq 2$, possibly with boundary, and let μ be the probability measure induced by a smooth positive volume form. Set $G = \pi_1(M)/Z(\pi_1(M))$, and suppose that*

$$(82) \quad H^1(G; \mathbb{R}) \neq 0.$$

Let $\mathcal{T}_M$ denote either $\text{Homeo}_0(M, \mu)$ or $\text{Diff}_0(M, \mu)$.

Let $\mathbb{R}^\delta$ denote the additive group of real numbers with the discrete topology. There are group homomorphisms

$$F : \mathcal{T}_M \longrightarrow \mathbb{R}^\delta, \quad s : \mathbb{R}^\delta \longrightarrow \mathcal{T}_M$$

such that

$$(83) \quad F \circ s = \text{id}_{\mathbb{R}^\delta}.$$

The section s can be chosen to be a smooth volume-preserving flow, supported in $\text{int } M$ when the boundary is nonempty.

Consequently, pullback gives an embedding

$$F^* : H^*(\mathbb{R}^\delta; \mathbb{R}) \longrightarrow H^*(\mathcal{T}_M; \mathbb{R}),$$

and

$$\dim H^d(\mathcal{T}_M; \mathbb{R}) = \infty \quad \text{for every } d \geq 1.$$

All group cohomology in this statement is ordinary cohomology of discrete groups with trivial real coefficients.

Proof. We first construct a real-valued homomorphism on $\text{Homeo}_0(M, \mu)$ and a section whose image consists of smooth volume-preserving diffeomorphisms.

Since G is finitely generated, condition (82) gives a nonzero homomorphism $G \rightarrow \mathbb{Z}$. Composing with the quotient map, obtain

$$\chi : \pi_1(M) \longrightarrow \mathbb{Z}, \quad \chi|_{Z(\pi_1(M))} = 0.$$

Choose a smooth map $u : M \rightarrow S^1$ representing χ . For an isotopy $\{f_t\}_{t \in [0,1]}$ from the identity to $f \in \text{Homeo}_0(M, \mu)$, let $\Delta_f(x)$ be the endpoint of the real lift, starting at zero, of the path

$$t \longmapsto u(f_t(x)) - u(x) \in S^1.$$

The function $\Delta_f : M \rightarrow \mathbb{R}$ is continuous.

Changing the isotopy changes this endpoint by the value of χ on the evaluation of a loop of isotopies. Such an evaluation lies in the center of $\pi_1(M)$; see [10, Proposition 2.1]. Hence Δ_f is independent of the chosen isotopy. Concatenation gives

$$\Delta_{fg}(x) = \Delta_f(gx) + \Delta_g(x).$$

The scalar mass-flow construction [19, Section 5] therefore defines

$$F(f) = \int_M \Delta_f(x) d\mu(x).$$

The integral is finite by compactness. Since g preserves μ , the preceding identity gives

$$F(fg) = F(f) + F(g).$$

Thus $F : \text{Homeo}_0(M, \mu) \rightarrow \mathbb{R}$ is a group homomorphism.

We next construct a section of F . Write $d\mu = \Omega$, where Ω is a smooth positive volume form with $\int_M \Omega = 1$, and put $\alpha = u^*(d\vartheta)$, where $d\vartheta$ is the standard closed one-form on S^1 with integral one. The class $[\alpha]$ is nonzero. By Poincaré–Lefschetz duality, choose a smooth closed $(n-1)$ -form β , compactly supported in $\text{int } M$ when $\partial M \neq \emptyset$, such that

$$\int_M \alpha \wedge \beta = 1.$$

Define the smooth vector field X by $\iota_X \Omega = \beta$. Since $d\beta = 0$, Cartan’s formula gives $\mathcal{L}_X \Omega = 0$. Thus the flow $\{P_t\}_{t \in \mathbb{R}}$ of X preserves Ω . It exists for all real times: M is compact, and X vanishes near the boundary when the boundary is nonempty.

For this flow, we have

$$\Delta_{P_t}(x) = \int_0^t \alpha(X)(P_\tau(x)) d\tau.$$

Using preservation of μ and $\alpha(X)\Omega = \alpha \wedge \iota_X \Omega$, we obtain

$$F(P_t) = \int_M \int_0^t \alpha(X)(P_\tau(x)) d\tau d\mu(x) = t \int_M \alpha(X)\Omega = t \int_M \alpha \wedge \beta = t.$$

Consequently, $s(t) = P_t$ is a group homomorphism with $F \circ s = \text{id}$. Its image lies in $\text{Diff}_0(M, \mu)$. The same section therefore works for the restriction of F to $\text{Diff}_0(M, \mu)$.

For either choice of $\mathcal{T}_M$, functoriality gives

$$s^* \circ F^* = (F \circ s)^* = \text{id} \quad \text{on } H^*(\mathbb{R}^\delta; \mathbb{R}).$$

Thus F^* is an embedding. Now $H^d(\mathbb{R}^\delta; \mathbb{R})$ is infinite-dimensional for every $d \geq 1$ and the proof follows. $\square$

For a closed negatively curved manifold of dimension at least two, the fundamental group has trivial center. Hence $H^1(G; \mathbb{R}) \cong H^1(M; \mathbb{R})$. If $b_1(M) > 0$, Theorem 29 therefore implies.

Corollary 30 (Negatively curved manifolds). *Let M be a closed connected oriented smooth manifold of dimension $n \geq 2$ admitting a Riemannian metric of strictly negative sectional curvature. If $n > 2$ suppose that*

$$b_1(M) = \dim H^1(M; \mathbb{R}) > 0.$$

Then, we have

$$\dim \mathcal{T}_M^d(\text{Homeo}_0(M, \mu); \mathbb{R}) = \infty, \quad \dim \mathcal{T}_M^d(\text{Diff}_0(M, \mu); \mathbb{R}) = \infty \quad (d \geq 1).$$

6.2. Final comments and remarks. The following proposition is a consequence of [32, Corollary 4.25] and [15, Theorem 6.14(c)], together with the standard fixed-point argument for finite groups acting on trees.

Proposition 31. *Every group satisfying (i) is acylindrically hyperbolic and has no nontrivial finite normal subgroup. In particular, it satisfies (iii). The hyperbolically embedded free subgroup need not coincide with the specified F_2 -vertex in (i).*

Proof. Write $G = F_2 *_A B$. Since A is an amenable subgroup of F_2 , it is trivial or infinite cyclic. In particular, it is finitely generated and has infinite index in F_2 . By [32, Corollary 4.25], G is either infinite cyclic or acylindrically hyperbolic. The first possibility is excluded because G contains F_2 .

We show that G has no nontrivial finite normal subgroup. If the inclusion $A \rightarrow B$ is onto, then $G \cong F_2$ and this is immediate. Otherwise both edge inclusions are proper, and the Bass–Serre action on the associated tree T is minimal; see [32, Lemma 4.14]. Let $N \triangleleft G$ be finite. Its fixed-point set T^N is a nonempty subtree; see [39, Chapter I, Section 6]. Normality makes T^N invariant under G , so minimality gives $T^N = T$. Hence N lies in every edge stabilizer, and in particular in A . Since A is torsion-free, $N = 1$.

By [36, Theorem 1.2] and [15, Theorem 6.14(c)], an acylindrically hyperbolic group with no nontrivial finite normal subgroup contains a hyperbolically embedded subgroup isomorphic to F_2 . Thus (iii) holds. This conclusion does not assert that the specified F_2 -vertex is hyperbolically embedded. $\square$

Remark 32. For $n \geq 3$, Proposition 31 shows that every group covered by Theorem 1 satisfies (ii) or (iii). Thus, allowing a different free embedding and algebraic rather than isometric lifts, the full space $\overline{H}_b^2(F_2)$ can be injected into $\overline{EH}_b^2(\mathcal{T}_M)$ in every case. The possible codimension-one loss in the direct (i) construction concerns that specific BBFIPP section and vertex; it is not a loss in the all-degree conclusion or an obstruction to another exact section.

Corollary 33. *The all-degree conclusion of Theorem 1 for $n \geq 3$ also holds whenever $G = \pi/Z(\pi)$ is acylindrically hyperbolic, even if G has a nontrivial finite normal subgroup and does not satisfy the condition (iii). In particular, the conclusion holds for any compact orientable smooth M of dimension at least three for which G is a non-elementary torsion-free word-hyperbolic group.*

Proof. By [36, Theorem 1.2] and [15, Theorem 6.14(c)], there is a hyperbolically embedded subgroup $H = F_2 \times K$ with K finite. Let $p : H \rightarrow F_2$ be the projection, $s : F_2 \rightarrow H$ the direct-factor inclusion, and $j : H \hookrightarrow G$. Put $i = j \circ s$; this is an injective homomorphism.

The reduction through $F_2 \times K$ is the one used in [22, Theorem 5.4 and Corollary 5.5] and [10, proof of Theorem B]. Fix $q \in \{2, 3\}$ and $u \in H_b^q(F_2)$. Since $H^q(F_2; \mathbb{R}) = 0$, the class u is exact. Naturality of comparison gives

$$c_H^q(p^*u) = p^*c_{F_2}^q(u) = 0.$$

The exact extension theorem [22, Proposition 5.1], applied to $j : H \hookrightarrow_h G$, therefore gives $v \in EH_b^q(G)$ with $j^*v = p^*u$. Restricting further gives

$$i^*v = s^*j^*v = s^*p^*u = (p \circ s)^*u = u.$$

Given a class in $\overline{H}_b^q(F_2)$, choose a representative $u \in H_b^q(F_2)$ and take the image of v in $\overline{EH}_b^q(G)$. This proves that $i^* : \overline{EH}_b^q(G) \rightarrow \overline{H}_b^q(F_2)$ is onto. Choose algebraic linear right inverses as in Lemma 6.

Apply Lemma 7 and Propositions 9 and 12 to this inclusion and the two right inverses. The proof of Theorem 1 for $n \geq 3$ then applies without change. Thus all four transformation-group conclusions follow. $\square$

Remark 34 (Comparison with Nitsche). For compact Riemannian manifolds of dimension at least three, [35, Theorem 4.2] proves

$$\dim \overline{EH}_b^d(\text{Homeo}_{\text{vol},0}(M)) = \infty \quad \text{for } d = 3 \text{ and even } d \geq 2,$$

assuming $\pi_1(M) \twoheadrightarrow F_2$. This is exactly condition (ii). On the compact oriented manifolds considered here, Theorem 1 also gives the degrees 5, 7, 9, $\dots$

and allows condition (iii), with the four groups and support conventions in its statement.

Strictness of the enlargement of the manifold hypothesis follows from the third construction in the proof of Proposition 35. It gives closed and compact manifolds with boundary satisfying (iii) but not (ii). That example gives closed and boundary manifolds satisfying (iii) but not (ii). They are therefore covered by Theorem 1 and not by the hypothesis of Nitsche's Theorem 4.2, even in its even-degree range. This comparison uses the stated theorem and the example; it requires no additional nonvanishing argument. The earlier degree-wise transfer results in [10, Theorems A and B] remain the one-point antecedents of our construction, so the comparison is not a claim of novelty for every individual consequence.

Proposition 35. *Among the three displayed conditions, the only universal implication between distinct conditions is (i) $\Rightarrow$ (iii). In particular, (ii) and (iii) are independent, and (i) and (ii) are independent, but the three conditions are not pairwise independent.*

Proof. Proposition 31 gives (i) $\Rightarrow$ (iii). We establish all the other nonimplications by the following three constructions.

Condition (i) does not imply (ii). Let Q be a closed oriented hyperbolic integral homology three-sphere, and put $P = \pi_1(Q)$. Hence P is infinite, torsion-free, and perfect. Fix $1 \neq t \in P$ and set $G_+ = P * \langle b \rangle$. Eliminating the generator a gives

$$(84) \quad G_+ \cong F(a, b) *_{\langle a \rangle = \langle t \rangle} P.$$

Both edge maps are injective because t has infinite order. Thus G_+ satisfies (i), and hence (iii) by Proposition 31. The free-product normal form gives $Z(G_+) = 1$; see [39, Chapter I, Section 4]. On the other hand,

$$(G_+)_{\text{ab}} \cong \mathbb{Z}.$$

An epimorphism $G_+ \twoheadrightarrow F_2$ would induce an epimorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}^2$, which is impossible. Hence G_+ does not satisfy (ii).

The closed oriented three-manifold $M_+ = Q \# (S^1 \times S^2)$ has fundamental group G_+ by van Kampen's theorem. Thus its center quotient satisfies (i) and (iii), but not (ii). The manifold $M_+ \times S^1$ has fundamental group $G_+ \times \mathbb{Z}$ and center quotient G_+ , while $M_+ \times S^{n-3}$ has fundamental group G_+ for every $n \geq 5$. These give closed manifold realizations in every dimension at least three.

Condition (ii) implies neither (i) nor (iii). Consider the closed symplectic four-manifold

$$M_- = \Sigma_2 \times \Sigma_2, \quad G_- = \pi_1(\Sigma_2) \times \pi_1(\Sigma_2),$$

with the product of area forms. The surface groups have trivial center, so G_- is the center quotient of $\pi_1(M_-)$; see [36, Corollary 7.2(a)]. The presentation

$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle$$

gives an epimorphism onto $F_2 = \langle a, b \rangle$ by sending a_1 to a , a_2 to b , and b_1, b_2 to the identity; see [27, Section 1.2]. Composing with projection onto the first surface-group factor proves (ii) for G_- .

A product of two infinite groups is not acylindrically hyperbolic; see [36, Corollary 7.2(b)]. Any subgroup of G_- isomorphic to F_2 must be proper, since G_- contains $\mathbb{Z}^2$, whereas a free group does not. A hyperbolically embedded copy of F_2 would therefore imply acylindrical hyperbolicity by [36, Theorem 1.2]. Thus (iii) fails, and Proposition 31 implies that (i) fails as well.

Condition (iii) implies neither (i) nor (ii). Let $P = \pi_1(Q)$ from the first construction. Since Q is closed and hyperbolic, P is center-free, torsion-free, and non-elementary word-hyperbolic. It has no nontrivial finite normal subgroup and satisfies (iii) by [15, Theorem 6.14(c)]. Since P is perfect, (ii) fails.

The group P is one-ended and has boundary S^2 . It has no nontrivial splitting over a finite group, and Bowditch's splitting criterion excludes a splitting over an infinite cyclic group: S^2 has no local cut point and P is not cocompact Fuchsian; see [4, Section 0 and Theorem 6.2]. In a splitting as in (i), the edge group would be trivial or infinite cyclic by (8). Neither nontrivial amalgam is therefore possible. The degenerate case would give $P \cong F_2$, which is excluded by $P_{\text{ab}} = 0$. Hence (i) also fails. The same splitting obstruction shows that the fundamental group of any closed real hyperbolic n -manifold, $n \geq 3$, fails (i), since its boundary S^{n-1} is connected and has no local cut point. The homology-sphere choice additionally ensures failure of (ii).

Thus Q realizes a center quotient satisfying only (iii). The center quotient of $Q \times S^1$ is again P , and $\pi_1(Q \times S^{n-3}) = P$ for $n \geq 5$. This gives such closed manifold realizations in every dimension at least three.

Compact manifolds with boundary and symplectic realizations. Deleting the interiors of finitely many pairwise disjoint embedded n -balls does not change the fundamental group when $n \geq 3$. Consequently, all the closed examples above also give compact manifolds with nonempty boundary and the same center quotients. The boundary conventions of Theorem 1 apply.

By Gompf's realization theorem [25], there are closed connected symplectic four-manifolds X_+ and X_3 with fundamental groups G_+ and P , respectively. Together with $M_- = \Sigma_2 \times \Sigma_2$, these realize all three patterns by symplectic four-manifolds. Their products with $\mathbb{CP}^s$, $s \geq 0$, retain the same fundamental groups and give symplectic examples in every even dimension at least four. Removing Darboux balls gives the corresponding symplectic examples with

boundary. Theorem 1 applies to all four transformation groups for these symplectic manifolds.

Finally, all three conditions hold for F_2 itself: use the amalgam $F_2 *_1 1$, the identity epimorphism, and $F_2 \hookrightarrow_h F_2$. For instance, $\#_2(S^1 \times S^2)$ and its ball complement realize this group. The three preceding counterexamples establish all the claimed nonimplications, completing the proof. $\square$

Theorem 3 gives infinite-dimensional reduced bounded cohomology of $\text{Homeo}_0(T, \mu)$ in every degree at least two, and infinite-dimensional reduced exact bounded cohomology in every even degree. We conclude with the following question.

Question. For every odd integer $d > 3$, is the reduced exact bounded cohomology $\overline{EH}_b^d(\text{Homeo}_0(T, \mu))$ infinite-dimensional?

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