ON THE ENTROPY NORM ON THE GROUP OF DIFFEOMORPHISMS OF CLOSED ORIENTED SURFACE

MICHAEL BRANDENBURSKY AND ARPAN KABIRAJ

Abstract. We prove that the entropy norm on the group of diffeomorphisms of a closed orientable surface of positive genus is unbounded.

1. Introduction

Let $M$ be a smooth compact manifold with some fixed Riemannian metric. Let $f: M \to M$ be a continuous function. Recall that the topological entropy of $f$ may be defined as follows. Let $d$ be the metric on $M$ induced by some Riemannian metric. For $p \in \mathbb{N}$ define a new metric $d_{f,p}$ on $M$ by

$$d_{f,p}(x,y) = \max_{0 \leq i \leq p} d(f^i(x), f^i(y)).$$

Let $M_f(p, \epsilon)$ be the minimal number of $\epsilon$-balls in the $d_{f,p}$-metric that cover $M$. The topological entropy $h(f)$ is defined by

$$h(f) = \lim_{\epsilon \to 0} \limsup_{p \to \infty} \frac{\log M_f(p, \epsilon)}{p},$$

where the base of log is two. It turns out that $h(f)$ does not depend on the choice of Riemannian metric, see [3, 10].

In this note we consider the case when $M$ is a closed oriented surface $\Sigma_g$ of genus $g$. Denote by $\text{Diff}(\Sigma_g)$ the group of orientation preserving diffeomorphisms of $\Sigma_g$. Let

$$\text{Ent}(\Sigma_g) \subset \text{Diff}(\Sigma_g)$$

be the set of entropy-zero diffeomorphisms. This set is conjugation invariant and it generates $\text{Diff}(\Sigma_g)$, see Lemma 2.1. In other words, a diffeomorphism of $\Sigma_g$ is a finite product of entropy-zero diffeomorphisms. One may ask for a minimal decomposition and this question leads to the concept of the entropy norm defined by

$$\|f\|_{\text{Ent}} := \min\{k \in \mathbb{N} \mid f = h_1 \cdots h_k, h_i \in \text{Ent}(\Sigma_g)\}.$$ 

It is the word norm associated with the generating set $\text{Ent}(\Sigma_g)$. This set is conjugation invariant, so is the entropy norm. The associated bi-invariant 2000 Mathematics Subject Classification. Primary 53; Secondary 57.

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metric is denoted by $d_{\text{Ent}}$. It follows from the work of Burago-Ivanov-Polterovich [9] and Tsuboi [17, 18] that for many manifolds all conjugation invariant norms on $\text{Diff}(\mathcal{M})$ are bounded. Hence the entropy norm is bounded in those cases. In particular, it is bounded in case $g = 0$.

Entropy metric may be defined in the same way on the group $\text{Ham}(\Sigma_g)$ of Hamiltonian diffeomorphisms of $\Sigma_g$, and on groups $\text{Diff}(\Sigma_g, \text{area})$ and $\text{Diff}_0(\Sigma_g, \text{area})$. It is related to the autonomous metric [4, 5, 6, 8, 13]. Recently, the first author in collaboration with Marcinkowski showed that the entropy metric is unbounded on groups: $\text{Ham}(\Sigma_g)$, $\text{Diff}_0(\Sigma_g, \text{area})$ and on $\text{Diff}(\Sigma_g, \text{area})$, see [7]. On the other hand, it is not known, and seems to be a difficult problem, whether $\text{Diff}_0(\Sigma_g)$ is unbounded in case $g > 0$. In this work we discuss the case of $\text{Diff}(\Sigma_g)$ where $g > 0$. Our main result is the following

**Theorem 1.** Let $\Sigma_g$ be a closed oriented Riemannian surface of positive genus. Then the diameter of $(\text{Diff}(\Sigma_g), d_{\text{Ent}})$ is infinite.

**Remarks.**

- The above theorem holds for non-sporadic surfaces with punctures. The proof is exactly the same.

- In [7] the first author in collaboration with Marcinkowski showed that the diameter of $(\text{Diff}(\Sigma_g, \text{area}), d_{\text{Ent}})$ is infinite. Our proof of Theorem 1, which is simpler than the one given in [7], is applicable to the case of $\text{Diff}(\Sigma_g, \text{area})$.

- It would be interesting to know whether the entropy metric, or the autonomous metric are unbounded on $\text{Diff}_0(\Sigma_g)$ in case $g > 0$.

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2. Preliminaries

Let us start with the following

**Lemma 2.1.** Let $\Sigma_g$ be a closed oriented surface of genus $g$. Then $\text{Diff}(\Sigma_g)$ is generated by the set $\text{Ent}(\Sigma_g)$ of entropy zero diffeomorphisms.

**Proof.** The group $\text{Diff}_0(\Sigma_g)$ is simple and hence is generated by entropy zero diffeomorphisms. It is enough to prove the lemma in case $g > 0$ since $\text{Diff}(\Sigma_0) = \text{Diff}_0(\Sigma_0)$. In addition, Dehn twists have entropy zero and they
generate $\text{Diff}(\Sigma_g)/\text{Diff}_0(\Sigma_g)$ in case $g > 1$. Hence in this case $\text{Diff}(\Sigma_g)$ is generated by entropy zero diffeomorphisms. In case $g = 1$ we have that
\[
\text{Diff}(\Sigma_1)/\text{Diff}_0(\Sigma_1) \cong \text{SL}_2(\mathbb{Z}),
\]
which in turn is generated by two matrices of finite order. Hence in this case $\text{Diff}(\Sigma_g)$ is also generated by entropy zero diffeomorphisms. \hfill \Box

Let $\Sigma_g$ be a closed oriented surface of genus $g > 1$.

2.A. Translation length in Teichmüller space. We denote the Teichmüller space associated to $\Sigma_g$ by $\mathcal{T}(\Sigma_g)$. We equip $\mathcal{T}(\Sigma_g)$ with the Teichmüller metric $d_T$. Let $\text{MCG}(\Sigma_g)$ be the mapping class group of $\Sigma_g$, i.e., $\text{MCG}(\Sigma_g) := \text{Diff}(\Sigma_g)/\text{Diff}_0(\Sigma_g)$. Note that it acts naturally on $\mathcal{T}(\Sigma_g)$. Let $[f] \in \text{MCG}(\Sigma_g)$. The translation length of $[f]$ in $\mathcal{T}(\Sigma_g)$ is defined by
\[
\tau_T([f]) = \lim_{n \to \infty} \frac{d_T([f]^n(X), X)}{n}
\]
where $X \in \mathcal{T}(\Sigma_g)$. It is independent of the choice of $X$.

Let $[f] \in \text{MCG}(\Sigma_g)$ be a pseudo-Anosov element with dilatation $\lambda([f])$. According to Bers [1] proof of Thurston’s classification theorem of elements of mapping class group we have:

- there exists $X \in \mathcal{T}(\Sigma_g)$ such that $\tau_T([f]) = d_T([f](X), X),$
- $\tau_T([f]) = \log(\lambda([f])).$

2.B. Translation length in curve complex. Given a surface $\Sigma_g$, we associate to it a simplicial complex as follows: its vertices are free homotopy classes of essential simple closed curves; a collection of $n + 1$ vertices form an $n$-simplex whenever it can be realized by pairwise disjoint closed curves in $\Sigma_g$. This complex is called the curve complex of $\Sigma_g$ and is denoted by $\mathcal{C}(\Sigma_g)$. It is known that $\mathcal{C}(\Sigma_g)$ is connected. We consider the path metric on the 1-skeleton of $\mathcal{C}(\Sigma_g)$ and denote it by $d_C$.

Mapping class group $\text{MCG}(\Sigma_g)$ acts by isometry on $\mathcal{C}(\Sigma_g)$. Given a mapping class $[f] \in \text{MCG}(\Sigma_g)$, the translation length of $[f]$ in $\mathcal{C}(\Sigma_g)$ is defined by
\[
\tau_C([f]) = \lim_{n \to \infty} \frac{d_C([f]^n(\alpha), \alpha)}{n}
\]
where $\alpha$ is a vertex in $\mathcal{C}(\Sigma_g)$. The translation length is independent of $\alpha$ and is non-zero if and only if $[f]$ is a pseudo-Anosov mapping class [15].
2.C. 

**Bestvina-Fujiwara quasimorphisms.** Let $G$ be a group. Recall that a function $\psi : G \to \mathbb{R}$ is called a quasimorphism if there exists $D > 0$ such that

$$|\psi(ab) - \psi(a) - \psi(b)| < D$$

for all $a, b \in G$. A quasimorphism $\psi$ is called homogeneous if $\psi(a^n) = n\psi(a)$ for all $n \in \mathbb{Z}$ and all $a \in G$. Given a quasimorphism $\psi$ we can always construct a homogeneous quasimorphism $\overline{\psi}$ by setting

$$\overline{\psi}(a) := \lim_{p \to \infty} \psi(a^p)$$

In [2], Bestvina and Fujiwara constructed infinitely many homogeneous quasimorphisms on $\text{MCG}(\Sigma_g)$. Let us recall their construction.

Let $\omega$ be a finite oriented path in $\mathcal{C}(\Sigma_g)$. Denote the length of a path $\omega$ by $|\omega|$. For any finite path $\sigma$ in $\mathcal{C}(\Sigma_g)$, we define

$$|\sigma|_\omega := \{\text{the number of non-overlapping copies of } \omega \text{ in } \sigma\}.$$

Fix a positive integer $W < |\omega|$. Given any two vertices $\alpha, \beta \in \mathcal{C}(\Sigma_g)$, define

$$c_{\omega,W}(\alpha, \beta) = d_c(\alpha, \beta) - \inf(|\sigma| - W|\sigma|_\omega),$$

where the infimum is taken over all paths $\sigma$ between $\alpha$ and $\beta$.

It turns out that the function $\psi_\omega : \text{MCG}(\Sigma_g) \to \mathbb{R}$ defined by

$$\psi_\omega([f]) = c_{\omega,W}(\alpha, [f](\alpha)) - c_{\omega-1,W}(\alpha, [f](\alpha)),$$

where $\alpha$ is a vertex of $\mathcal{C}(\Sigma_g)$, is a quasimorphism [2]. The induced homogeneous quasimorphism is denoted by $\overline{\psi}_\omega$. We denote by $Q_{BF}(\text{MCG}(\Sigma_g))$ the space of homogeneous quasimorphisms on $\text{MCG}(\Sigma_g)$ which is spanned by Bestvina-Fujiwara quasimorphisms. In [3] it is proved that $Q_{BF}(\text{MCG}(\Sigma_g))$ is infinite dimensional whenever $\Sigma_g$ is a non-sporadic surface.

### 3. Proof of the main result

Let us start with the following well-known

**Lemma 3.1.** Let $G$ be a group generated by set $S$ and let $\psi : G \to \mathbb{R}$ be a non-trivial homogeneous quasimorphism which vanishes on $S$. Then the induced word norm $\| \cdot \|_S$ is unbounded.

For the reader convenience we present its proof.

**Proof.** Let $g \in G$ such that $\psi(g) \neq 0$. Then $g = s_1 \cdots s_{\|g\|_S}$. It follows that $|\psi(g)| \leq \|g\|_SD_\psi$. Hence for each $n$ we get $\|g^n\|_S \geq n|\psi(g)|/D_\psi$ and the proof follows. $\square$
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Now we prove Theorem 1.

Case 1. Let $g = 1$ and denote $T := Σ_1$. Let us consider homomorphism $F : \text{Diff}(T) \to \text{SL}_2(\mathbb{Z})$ induced by the action of a diffeomorphism on the first homology $H_1(T, \mathbb{Z})$. It is known that $F$ is surjective (see [11, Theorem 2.5]). By [14, Theorem 1], $\log(\text{spec}(f)) \leq h(f)$ where $\text{spec}(f)$ is the modulus of the largest eigenvalue of $F(f)$. Therefore if $f$ has entropy zero then the modulus of the eigenvalues of $F(f)$ is at most one.

There are three types of elements in $\text{SL}_2(\mathbb{Z})$: periodic (trace<2), parabolic (trace=2) and hyperbolic (trace>2). Therefore if $F(f)$ is hyperbolic then $\text{spec}(f) > 1$ and hence $h(f) > 0$. Hence if $f$ is an entropy zero diffeomorphism, then $F(f)$ is either parabolic or periodic.

The value of any homogeneous quasimorphism on a periodic element is zero. It follows from the work of Polterovich and Rudnick [16, Proposition 3] that there exists a non-trivial homogeneous quasimorphism on $\text{SL}_2(\mathbb{Z})$ which vanishes on parabolic elements. Therefore there exists a non-trivial homogeneous quasimorphism on $\text{Diff}(T)$ whose restriction on entropy-zero diffeomorphisms is zero. Hence by Lemma 3.1 the entropy norm on $\text{Diff}(T)$ is unbounded.

Case 2. Let $g > 1$. Given a homeomorphism $f$ of a surface $Σ_g$ define

$$H(f) = \inf\{h(f') : f' \text{ is isotopic to } f\}$$

The topological entropy of $[f] \in \text{MCG}(Σ_g)$ is defined to be $H(f)$.

Lemma 3.2. Each quasimorphism in $\mathcal{Q}_{BF}(\text{MCG}(Σ_g))$ is Lipschitz with respect to the topological entropy.

Proof. Let $ψ \in \mathcal{Q}_{BF}(\text{MCG}(Σ_g))$. If $[f]$ is reducible then $ψ([f]) = 0$ for all $ψ \in \mathcal{Q}_{BF}(\text{MCG}(Σ_g))$. Therefore it is enough to consider only pseudo-Anosov elements of $\text{MCG}(Σ_g)$. Since $ψ \in \mathcal{Q}_{BF}(\text{MCG}(Σ_g))$, then $ψ = \sum_{i=1}^{k} a_i ψ_{w_i}$, where $a_1, \ldots, a_k \in \mathbb{R}$ and $w_1, \ldots, w_k$ are some paths in $C(S)$. It follows from the definition of $ψ_{w_i}$ that $ψ_{w_i}(\lfloor f \rfloor) \leq τ_C(\lfloor f \rfloor)$ for each $[f] \in \text{MCG}(Σ_g)$ and each $i \in \{1, \ldots, k\}$. Therefore we have

$$|ψ([f])| \leq \sum_{i=1}^{k} |a_i| τ_C(\lfloor f \rfloor).$$

By setting $C_ψ := \sum_{i=1}^{k} |a_i|$ we get $|ψ([f])| \leq C_ψ τ_C(\lfloor f \rfloor)$.

Let $\text{sys} : \mathcal{T}(Σ_g) \to C(Σ_g)$ be the systole function, i.e., $X \in \mathcal{T}(Σ_g)$ goes to a vertex in $C(Σ_g)$ which corresponds to a simple closed curve of minimal length in $X$. By [15] there exist $K, C > 0$ such that for all $X, Y \in \mathcal{T}(Σ_g)$

$$d_C(\text{sys}(X), \text{sys}(Y)) \leq K d_{\mathcal{T}}(X, Y) + C.$$

It is immediate that $[f]^n(\text{sys}(X)) = \text{sys}([f]^n(X))$ for every $[f] \in \text{MCG}(Σ_g)$. 

Let \([f] \in \text{MCG}(\Sigma_g)\) be a pseudo-Anosov element with dilatation \(\lambda_{[f]}\). It follows from Bers [1] proof of Thurston’s theorem that \(\tau_T([f]) = \log \lambda_{[f]}\).

Therefore
\[
\frac{\tau_C([f])}{\tau_T([f])} = \lim_{n \to \infty} \frac{d_C(X, [f]^n(X))}{\frac{n}{d_T(X, [f]^n(X))}} = \lim_{n \to \infty} \frac{d_C(X, [f]^n(X))}{d_T(X, [f]^n(X))} \leq \lim_{n \to \infty} \frac{Kd_T(X, [f]^n(X)) + C}{d_T(X, [f]^n(X))} = K
\]

Thus
\[
\tau_C([f]) \leq K\tau_T([f]).
\]

It follows that for each \(\psi \in Q_{BF}(\text{MCG}(\Sigma_g))\) we have
\[
|\psi([f])| \leq C\psi \tau_C([f]) \leq C\psi K\tau_T([f]) = C\psi K \log \lambda_{[f]}.
\]

By Thuston’s result [12, Proposition 10.13], \(\log \lambda_{[f]} = H(f)\). Hence
\[
|\psi([f])| \leq C\psi K H(f)
\]

and the proof of the lemma follows.

Let \(\Pi : \text{Diff}(\Sigma_g) \to \text{MCG}(\Sigma_g)\) be the quotient map and let \(\psi \in Q_{BF}(\text{MCG}(\Sigma_g))\). It follows from the proof of Lemma 3.2 that for each \(f \in \text{Diff}(\Sigma_g)\) we have
\[
|\psi\Pi(f)| \leq C\psi K H(f) \leq C\psi K h(f).
\]

Hence for each non-trivial \(\psi \in Q_{BF}(\text{MCG}(\Sigma_g))\) the homogeneous quasimorphism
\[
\psi\Pi : \text{Diff}(\Sigma_g) \to \mathbb{R}
\]
is non-trivial and Lipschitz with respect to the topological entropy. It follows that it vanishes on the set of entropy-zero diffeomorphisms. Hence by Lemma 3.1 the entropy norm on \(\text{Diff}(\Sigma_g)\) is unbounded.

References


Ben-Gurion University, Israel

E-mail address: brandens@math.bgu.ac.il

Ben-Gurion University, Israel

E-mail address: kabiraj@post.bgu.ac.il