Non-commutative Barge-Ghys quasimorphisms: an erratum

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Abstract. This is an erratum to the paper "Non-commutative Barge-Ghys quasimorphisms". The effective version of Ambrose-Singer theorem used in this paper was mis-stated, making Theorem 3.11 false as stated. However, this statement remains true for a semidirect product $G = A \rtimes K$ of a non-compact Lie group with a compact Lie group K, as long as the curvature of the connection belongs to the Lie algebra of K. Theorems 3.12 and 3.22 remain true under this assumptions, the statements of Theorems 4.28 and 4.31 are modified accordingly. Theorem 5.2 is valid only for compact Lie groups.

1 Effective Ambrose-Singer theorem

The proof of Theorem 3.11 in [BV1] is based on the following observation, called "effective Ambrose-Singer theorem". Let (B, ∇) be a principal bundle with a connection. Then its holonomy along a contractible path is linearly expressed in terms of the integral of the curvature over a disk D filling this path ([RW, Theorem 1], [Y]).

This observation is true, but we used it to obtain a false conclusion: we claimed that if the curvature has a universal bound, then the norm of holonomy is bounded by a constant times the area of G.

This conclusion is false, because, before we integrate the curvature, we need to transport the curvature elements to the origin using the holonomy of ∇. Since the path which is used to transport the curvature elements is not bounded, the resulting curvature term is conjugated by the holonomy action. Unless the adjoint action of G on the curvature elements is unitary, this can result in an integral which is unbounded.

To preserve the validity of our paper, we are forced to consider a Lie group $G = A \rtimes K$, obtained as a a semidirect product of a non-compact Lie group A with a compact Lie group K, and assume that the curvature of the connection belongs to the Lie algebra of K. In that case, the adjoint action preserves the norm of the curvature elements, and the effective Ambrose-Singer theorem can be applied to obtain the results we announced in [BV1].

The last section of [BV1], generalizing Kazhdan's results about ε-representations ([Kaz]) to an arbitrary Lie group G, remains valid as long as G is assumed compact (or a semidirect product as above).

2 Corrections to the main results

In [BV2], we give corrected versions of the main results of [BV1].

Theorem 4.31, [BV1] (false): Let G be a simply connected, connected, non-abelian rational real nilpotent Lie group, and $\Gamma := \pi_1(M)$, where M is a closed manifold of strictly negative sectional curvature, $\dim_{\mathbb{R}} M > 1$. Then there exists a non-constructible HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$.

Theorem 4.14, [BV2] (corrected version): Let $G := A \rtimes K$ be a semidirect product of a compact group K and an abelian Lie group $A = \mathbb{R}^n, n \geqslant 2$, and K acts transitively on a sphere in A. Let $\Gamma := \pi_1(M)$, where M is a closed manifold of strictly negative sectional curvature, $\dim_{\mathbb{R}} M > 1$. Then there exists a non-constructible HBG-quasimorphism $q_\nabla : \Gamma \longrightarrow G$.

Theorem 3.11 and 3.12, [BV1] remain valid under an extra assumption: $G = A \rtimes K$ is a semidirect product of a Lie group A with a compact Lie group K, and the curvature 2-form of ∇ takes values in the Lie algebra $\operatorname{Lie}(G)$ of K.

Remark 4.33 of [BV1] is invalid.

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References

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