BOUNDED COHOMOLOGY OF MEASURE-PRESERVING HOMEOMORPHISM GROUPS OF NON-ORIENTABLE SURFACES

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Abstract. Let N_g be a closed non-orientable surface of genus $g \geq 3$. Let $\mathrm{Homeo}_0(N_g, \mu)$ be the identity component of the group of measure-preserving homeomorphisms of N_g. In this work we prove that the third bounded cohomology of $\mathrm{Homeo}_0(N_g, \mu)$ is infinite dimensional.

1. Introduction

Bounded cohomology was defined in a seminal work by Gromov in 1982 [9]. Since then it has become an important tool in several fields of mathematical research including the geometry of manifolds, stable commutator length, amenable groups and symplectic geometry. Bounded cohomology of finitely generated groups is well understood in many cases, especially in low degrees. However, for higher degrees, even for the non-abelian free group F_2, nothing is known and seems to be a very difficult problem.

Since early 2000s, the study of bounded cohomology of transformation groups of smooth manifolds received a considerable attention. It is, in particular, due to the fact that second bounded classes are related to quasimorphisms, and many of those detect many interesting properties of groups and have deep connection to dynamics (entropy) and symplectic geometry (spectral invariants), see [2, 10]. However, until recent works by Brandenbursky-Marcinkowski [3], Kimura [11] and Nitsche [13], not much was known about higher bounded classes of transformation groups. They considered orientable manifolds, and the goal of this work is to complete their results in the non-orientable case.

Let N_g be a non-orientable closed surface of genus g. Let Σ_g be a closed orientable surface of genus g and let $\mathrm{Homeo}(N_g, \mu)$ be the group of all measure preserving homeomorphisms of N_g where μ is the Lebesgue measure induced by the two sheeted covering $\Sigma_{g-1} \to N_g$. We regard $\mathrm{Homeo}(N_g, \mu)$ as a topological group equipped with the compact open topology and denote by $\mathrm{Homeo}_0(N_g, \mu)$ its identity component. Our main result is the following:

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Theorem 1. *Let $g \geq 3$. Then the dimensions of $H_b^2(\mathrm{Homeo}_0(N_g, \mu))$ and $H_b^3(\mathrm{Homeo}_0(N_g, \mu))$ are infinite.*

Remark 1.1. Kumar in his thesis [12] proved that the space of homogeneous quasimorphisms on $\mathrm{Homeo}_0(N_g, \mu)$ is infinite dimensional, which implies that $\dim(H_b^2(\mathrm{Homeo}_0(N_g, \mu))) = \infty$. We would like to mention that our proof of this fact is different.

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2. Preliminaries

2.1. **Bounded cohomology.** Let G be a group and let $C_b^n(G)$ be the linear space of bounded functions $f : G^{n+1} \to \mathbb{R}$. An action of G on $C_b^n(G)$ is defined by the formula $g \cdot f(g_0, \ldots, g_n) := f(gg_0, \ldots gg_n)$. The space $C_b^n(G)^G :=\{f \in C_b^n(G) \mid g \cdot f = f \text{ for every } g \in G\}$ is the subspace of homogeneous bounded functions, and $\delta^n : C_b^n(G)^G \to C_b^{n+1}(G)^G$ is the usual coboundary map defined by:

$$\delta^n(f)(g_0, \ldots, g_{n+1}) = \Sigma_{i=0}^{n+1}(-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}).$$

The complex $\{C_b^n(G)^G, \delta^n\}$ is called *homogeneous* chain complex and its homology is called the bounded cohomology of G denoted by $H_b^n(G)$.

An equivalent way of defining $H_b^n(G)$ is as follows: set $\bar{C}_b^n(G) := C_b^{n-1}(G)$, where $G^0 = \{1\}$. The map $\phi : C_b^n(G)^G \to \bar{C}_b^n(G)$ defined by

$$\phi(f)(g_1, \ldots, g_n) := f(1, g_1, g_1 g_2 \ldots, g_1 \ldots g_n)$$

is an isomorphism. Let $\bar{\delta}^n : \bar{C}_b^n(G) \to \bar{C}_b^{n+1}(G)$ be the map $\phi\delta^n\phi^{-1}$. The complex $\{\bar{C}_b^n(G), \bar{\delta}^n\}$ is the *non-homogenous* chain complex whose homology is isomorphic to $H_b^n(G)$.

2.1.1. *The exact bounded cohomology.* The exact bounded cohomology of G is the kernel of the *comparison* map $c^n : H_b^n(G) \to H^n(G)$, and is denoted by $EH_b^n(G)$. The supremum norm on $C_b^n(G)$ induces a seminorm on $H_b^n(G)$. If $c \in H_b^n(G)$ then $\|c\| := \inf\{\|a\| \mid [a] = c\}$. Let $N^n(G)$ be the subspace of $H_b^n(G)$ consisting of classes of zero norm. The reduced bounded cohomology is the normed space $\bar{H}_b^n(G) := H_b^n(G)/N^n(G)$. The exact reduced bounded cohomology denoted by $\overline{EH}_b^n(G)$ is the quotient $EH_b^n(G)/(EH_b^n(G) \cap N^n(G))$.

2.2. **Quasimorphisms.** A map $f : G \to \mathbb{R}$ is called a *quasimorphism* if there exists a real constant D such that $|f(g_1)+f(g_2)-f(g_1g_2)| \leq D$ for every $g_1, g_2 \in G$. The infimum of all such constants is called the *defect* of f and denoted by $D(f)$. A quasimorphism is called *homogeneous* if $f(g^n) = nf(g)$

for every $g \in G, n \in \mathbb{Z}$. The space of homogeneous quasimorphisms is denoted by $Q^h(G, \mathbb{R})$. There exists an exact sequence:

$$0 \to \bar{C}_b^1(G) \oplus \mathrm{Hom}(G, \mathbb{R}) \hookrightarrow Q(G, \mathbb{R}) \xrightarrow{\phi} EH_b^2(G) \to 0$$

where $\phi(f) = \left[\bar{\delta}^1(f)\right]$. Therefore, $Q(G, \mathbb{R})/(\bar{C}_b^1(G) \oplus \mathrm{Hom}(G, \mathbb{R})) \cong EH_b^2(G)$, and $Q^h(G, \mathbb{R})/\operatorname{Hom}(G, \mathbb{R})$ is isomorphic to $EH_b^2(G)$.

In his thesis [12] Rishi Kumar proved the following result: Let $g \geq 3$, then the space of nontrivial homogeneous quasimorphisms on $\mathrm{Homeo}_0(N_g, \mu)$ is infinite-dimensional. As an immediate consequence, we find that the dimension of $H_b^2(\mathrm{Homeo}_0(N_g, \mu))$ is infinite.

2.3. **Hyperbolically embedded subgroups.** Let G be a group, H a subgroup of G and X a subset of G. Given $h, k \in H$ we define $\hat{d}(h, k)$ to be the length of the shortest path in $\Gamma(G, X \sqcup H)$ between h and k with no edges in the complete graph $\Gamma(H, H)$. If there is no such path, we define $\hat{d}(h, k) = \infty$. It is easy to check that $\hat{d} : H \times H \to [0, \infty]$ satisfies axioms of a metric.

Definition 2.1. We say that H is hyperbolically embedded in G with respect to X if the following conditions hold.
(a) G is generated by $X \cup H$.
(b) $\Gamma(G, X \sqcup H)$ is hyperbolic.
(c) Every ball of finite radius in $(H, \hat{d})$ is finite.
We say that H is hyperbolically embedded subgroup of G and write $H \hookrightarrow_h G$ if H is hyperbolically embedded in G with respect to some $X \subseteq G$.

If G is hyperbolic with respect to a set of generators $S \subseteq X \sqcup H$, then its Cayley graph is a hyperbolic subgraph of $\Gamma(G, X \sqcup H)$ with the same set of vertices. The following proposition is useful for proving the hyperbolicity of $\Gamma(G, X \sqcup H)$ and will be used later.

Proposition 2.2. ([14, Lemma 5.5]). Let Σ be a graph obtained from a hyperbolic graph Γ by adding edges. Suppose that there exist $M > 0$ such that for every two vertices x, y connected by an edge in Σ and every geodesic p in Γ from x to y, the diameter of p in Σ is at most M. Then Σ is hyperbolic.

Definition 2.3. A subgroup H of a group G is called *malnormal* if $H \cap xHx^{-1} = e_G$ for every $x \in G \backslash H$. It is called *almost malnormal* if $xHx^{-1} \cap H$ is finite for every $x \in G \backslash H$.

Proposition 2.4. ([6, Proposition 2.10]) If $H \hookrightarrow_h G$, then H is an almost malnormal subgroup.

2.3.1. *Examples.*

- The subgroup of F_2 generated by a^2, ab, ab^{-1} is not hyperbolically embedded. It is not almost malnormal since $ba^{2n}b^{-1}$ belongs to this subgroup for every $n \in \mathbb{N}$.
- The group $\mathrm{F}_\infty \cong \langle a^n b a^{-n} \,|\, n \in \mathbb{N}\rangle$ is not an almost malnormal subgroup of F_2, therefore, it is not hyperbolically embedded. Indeed, the intersection of this subgroup with its conjugation by a is infinite.
- For every $n > 2$, the group $\mathrm{F}_2 = \langle a_1, a_2\rangle$ is hyperbolically embedded in $\mathrm{F}_n = \langle a_1, \dots, a_n\rangle$. Let $X = \{a_3, \dots, a_n\}$. Conditions (a) and (c) in Definition 2.1 hold immediately. For (b), we need to show that $Y = \Gamma(\mathrm{F}_n, X \sqcup \mathrm{F}_2)$ is hyperbolic. Let $h \in \mathrm{F}_2$, the diameter in Y of the unique geodesic from e to h in $\Gamma(\mathrm{F}_n, \{a_1, \dots, a_n\})$ is 1. Therefore, by Proposition 2.2, Y is hyperbolic.

In what follows, we show that for every $g \geq 3$ there exists a specific hyperbolic embedding of F_2 in $\pi_1(N_g)$, see Proposition 4.4. Moreover, this fact is heavily used in the proof of our main result.

3. GAMBAUDO-GHYS MAP

In this section we define a map $\Gamma_b^* : H_b^*(\pi_1(N_g)) \to H_b^*(\mathrm{Homeo}_0(N_g, \mu))$ which can be seen as a generalization of a construction given by Gambaudo-Ghys [8] and Polterovich [1]. This map for orientable manifolds was defined in [3]. Here we repeat the construction in the case of non-orientable surfaces.

Let $G := \mathrm{Homeo}_0(N_g, \mu)$. Let us define a map $\gamma : G \times N_g \to \pi_1(N_g, z)$ in the following way: for $(f, x) \in G \times N_g$ let $\{f_t\}_{t\in[0,1]}$ be an isotopy in G connecting the identity with f, and let $\alpha_{z,x}$ be a geodesic from z to x with respect to the Riemannian metric on N_g that is induced from the hyperbolic metric on Σ_{g-1}. Let $\gamma(f, x) := [\alpha_{z,x} * f_t(x) * \alpha_{f(x),z}]$ be the element in $\pi_1(N_g, z)$ represented by the concatenation of paths $\alpha_{z,x}$, $f_t(x)$ and $\alpha_{f(x),z}$.

Proposition 3.1. The map γ is independent of the choice of the isotopy.

Proof. Let $ev_z : G \to N_g$ be the evaluation map at a base point $z \in N_g$ and let $ev_{z*} : \pi_1(G, \mathrm{Id}) \to \pi_1(N_g, z)$ be its induced map at the level of fundamental groups. We show that $\mathrm{Im}\, ev_{z*}$ is a subgroup of the center $Z(\pi_1(N_g, z))$, and hence is trivial. Let $[c] \in \mathrm{Im}\, ev_{z*}$, hence

$$[c] = ev_{z*}([\alpha_t]) = [ev_z \circ \alpha_t] = [\alpha_t(z)],$$

where $[\alpha_t] \in \pi_1(G, \mathrm{Id})$. For every $[\delta] \in \pi_1(N_g, z)$, $\delta * \alpha_t(z) \simeq \alpha_t(z) * \delta$ by the homotopy $H(s, t) = \alpha_{st}(z) * \delta(t) * \alpha_{st+(1-s)t}(z)$, hence $[c] \in Z(\pi_1(N_g, z))$, which is trivial. Now, let $\{f_t\}, \{h_t\}$ be two isotopies in $\mathrm{Homeo}_0(N_g, \mu)$ connecting the identity and f. We have $ev_{z*}([f_t * h_t^{-1}]) = [z]$, therefore $\alpha_{z,x} * f_t(x) * \alpha_{f(x),z} \simeq \alpha_{z,x} * h_t(x) * \alpha_{f(x),z}$. □

Proposition 3.2. For every $f, h \in G$ and every $x \in N_g$ we have:

$$\gamma(fh, x) = \gamma(f, h(x))\gamma(h, x).$$

Proof. Let $\{f_t\}, \{h_t\}$ be isotopies connecting the identity with f and h, respectively. Then $h_t * f_t(h)$ is an isotopy connecting the identity with fh, and we obtain:

$$\begin{aligned}\gamma(f, h(x))\gamma(h, x) &= [\alpha_{z,x} * h_t(x) * \alpha_{h(x),z}] * [\alpha_{z,h(x)} * f_t(h(x)) * \alpha_{f(h(x)),z}] \\ &= [\alpha_{z,x} * h_t(x) * f_t(h(x)) * \alpha_{f(h(x)),z}] = \gamma(fh, x).\end{aligned}$$ □

Let $c \in C_b^n(\pi_1(N_g))$ and the map γ described above. We define the function $I_b^n(\gamma)(c) : G^{n+1} \to \mathbb{R}$ by

$$I_b^n(\gamma)(c)(g_0, ...g_n) := \int_{N_g} c(\gamma(g_0, x), \dots, \gamma(g_n, x))d\mu(x).$$

The map $x \mapsto \gamma(g, x)$ is μ-measurable for every $g \in G$, hence the function under the integral is measurable. For full explanation see, e.g., [4, Section 3.B] or [13, Section 3].

Proposition 3.3. The following statements hold:

1. The map $I_b^n(\gamma)$ commutes with the coboundary map.
2. The map $I_b^n(\gamma)(c)$ is homogeneous.

Proof. Let $g_0, \dots, g_{n+1} \in G$ and $c \in C_b^n(\pi_1(N_g))$.

Proof of 1. It follows from the definition of the coboundary map given in Subsection 2.1 that:

$$\begin{aligned}&I_b^n(\delta^n(c))(g_0, \dots, g_{n+1}) = \\ &\int_{N_g} \sum_{i=0}^{n+1} (-1)^i c\big(\gamma(g_0, x), \dots, \hat{\gamma}(g_i, x), \dots, \gamma(g_{n+1}, x)\big)d\mu(x) = \\ &\sum_{i=0}^{n+1} (-1)^i \int_{N_g} c\big(\gamma(g_0, x), \dots, \hat{\gamma}(g_i, x), \dots, \gamma(g_{n+1}, x)\big)d\mu(x) = \\ &\sum_{i=0}^{n} (-1)^i I_b^n(c)(g_0, \dots, \hat{g}_i, \dots, g_{n+1}) = \delta^n(I_b^n(c))(g_0, \dots, g_{n+1}).\end{aligned}$$

Proof of 2. Let $h \in G$. Since h^{-1} is a μ-preserving homemorphism, $h_*\mu = \mu$. We get:

$$\begin{aligned} I_b^n(\gamma)(c)(g_0h, ..., g_nh) = \int_{N_g} c\big(\gamma(g_0h, x), \dots, \gamma(g_nh, x)\big) d\mu(x) = \\ \int_{N_g} c\big(\gamma(g_0, h(x))\gamma(h, x), \dots, \gamma(g_n, h(x))\gamma(h, x)\big) d\mu(x) = \\ \int_{N_g} c\big(\gamma(g_0, h(x)), \dots, \gamma(g_n, h(x))\big) d\mu(x) = \\ \int_{N_g} c\big(\gamma(g_0, x), \dots, \gamma(g_n, x)\big) d(h_*\mu)(x) = I_b^n(\gamma)(c)(g_0, \dots, g_n). \qquad \square \end{aligned}$$

Proposition 3.3 implies that $I_b^n(\gamma)$ induces $\Gamma_b^n : H_b^n(\pi_1(N_g, z)) \to H_b^n(G)$, and Γ_b^n induces the map on the exact reduced bounded cohomology, which is denoted by $\overline{E\Gamma}_b^n$.

4. Proofs

Let us discuss the idea of the proof. Brooks [5] and Soma [15] showed that $\dim \overline{EH}_b^2(\mathrm{F}_2) = 2^{\aleph_0}$ and $\dim \overline{EH}_b^3(\mathrm{F}_2) = 2^{\aleph_0}$. We proceed as in [3], find a map $\overline{EH}_b^n(\mathrm{F}_2) \to \overline{EH}_b^n(\mathrm{Homeo}_0(N_g, \mu))$ whose properties help us to show:

$$\dim H_b^2(\mathrm{Homeo}_0(N_g, \mu)) \geq 2^{\aleph_0}, \dim H_b^3(\mathrm{Homeo}_0(N_g, \mu)) \geq 2^{\aleph_0}.$$

In what follows, we present an embedding $i : S^1 \vee S^1 \hookrightarrow N_g$ such that $i^n : \overline{EH}_b^n(\pi_1(N_g)) \to \overline{EH}_b^n(\mathrm{F}_2)$ is a surjection. Note that the proof of this fact is immediate for an orientable hyperbolic surface Σ_g, but non-trivial for N_g (for $g = 3, 4$) and requires the notion of hyperbolic embeddings of groups.

Then, we follow [3] and construct a family of maps $\rho_\epsilon : \mathrm{F}_2 \to \mathrm{Homeo}_0(N_g, \mu)$ such that for every $n \in \mathbb{N}$ the following diagram is commutative up to scalar multiplication and an arbitrary small error.

$$\begin{tikzcd} \overline{EH}_b^n(\pi_1(N_g)) \arrow[r, "\overline{E\Gamma}_b^n"] \arrow[d, "i^n"'] & \overline{EH}_b^n(\mathrm{Homeo}_0(N_g, \mu)) \arrow[dl, "\rho_\epsilon^n"] \\ \overline{EH}_b^n(\mathrm{F}_2) & \end{tikzcd}$$

Finally, by approaching ϵ to zero we show that

$$\dim \overline{EH}_b^n(\mathrm{Homeo}_0(N_g, \mu)) \geq \dim \overline{EH}_b^n(\mathrm{F}_2),$$

and obtain the proof of our main result.

4.1. **Easy case $g \geq 5$.**

Recall that $N_3 = \mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is homeomorphic to $\mathbb{T}^2\#\mathbb{R}\mathrm{P}^2$.

Proposition 4.1. Let $g \geq 5$. There exists an embedding $i : S^1 \vee S^1 \hookrightarrow N_g$ and a surjection $\pi : N_g \twoheadrightarrow S^1 \vee S^1$ such that $\pi \circ i = Id_{S^1 \vee S^1}$.

Proof. Since $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2 \cong \mathbb{T}^2\#\mathbb{R}\mathrm{P}^2$ we have $N_5 \cong \mathbb{T}^2\#\mathbb{T}^2\#\mathbb{R}\mathrm{P}^2$. For $g = 5$, the maps are presented in the following figure:

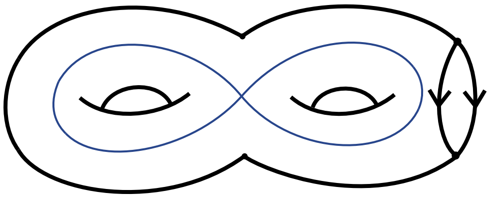

FIGURE 1. $i : S^1 \vee S^1 \hookrightarrow N_5$ and $\pi : N_5 \twoheadrightarrow S^1 \vee S^1$

The image of the map $i : S^1 \vee S^1 \hookrightarrow N_5$ is the blue figure-eight, and the map $\pi : N_5 \twoheadrightarrow S^1 \vee S^1$ is an obvious projection. The case $g > 5$ follows immediately since $N_g \cong \mathbb{T}^2\#\mathbb{T}^2\#\underbrace{\mathbb{R}\mathrm{P}^2\#\ldots\#\mathbb{R}\mathrm{P}^2}_{g-4 \text{ times}}$. □

Corollary 4.2. On the level of the exact reduced bounded cohomology $i^n \circ \pi^n = Id$, therefore i^n is onto and π^n is injective.

4.2. **General case.**

In order to prove our theorem for every $g \geq 3$ we need another approach. The idea is based on the following result due to Frigerio, Pozzeti and Sisto:

Theorem 4.3. ([7, Corollary 2]) Let G be a group and let H be a hyperbolically embedded subgroup of G. Then for every $n \geq 2$ the restriction map $i^n : \overline{EH}_b^n(G) \to \overline{EH}_b^n(H)$ is surjective.

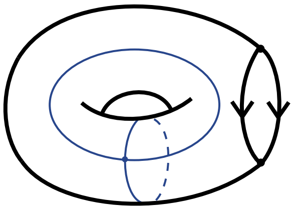

FIGURE 2. $i : S^1 \vee S^1 \hookrightarrow N_3$

Consider the following embedding $i : S^1 \vee S^1 \hookrightarrow N_3$, see Figure 2. Since $\pi_1(N_g) = \langle a_1, \dots, a_g \,|\, a_1 a_2 a_1^{-1} a_2^{-1} a_3^2 \dots a_g^2 = 1\rangle$, the image of the induced map between fundamental groups is $\langle a_1, a_2\rangle$. For $g > 3$, an embedding $i : S^1 \vee S^1 \hookrightarrow N_g$ is defined similarly.

Proposition 4.4. Let $g \geq 3$. Then $i_* : \mathrm{F}_2 \to \pi_1(N_g)$ is a hyperbolic embedding.

Proof. Case 1: $g = 3$. We prove that $H = \langle a, b\rangle$ is hyperbolically embedded in $G = \langle a, b, c \,|\, aba^{-1}b^{-1}c^2 = 1\rangle$. Let $X = \{c\}$. We show that conditions (a),(b) and (c) in Definition 2.1 are satisfied. Condition (a) is immediate.

Proof of condition (b): We need to prove that the graph $\Sigma = \Gamma(G, \{c\} \sqcup \langle a, b\rangle)$ is hyperbolic. Let $\Gamma = \Gamma(G, \{a, b, c\})$ be the Cayley graph of G. The graph Γ is hyperbolic and Σ is obtained from Γ by adding edges.

Let x, y be two vertices connected by an edge in Σ and let $[x, y]$ be a geodesic from x to y in Γ. Proposition 2.2 implies that it is enough to show that the diameter of $[x, y]$ in Σ is less or equal then 3. Now, if x, y are connected by an edge in Γ then $\operatorname{diam}_\Sigma[x, y] = 1$. If x, y are connected by an edge in Σ which does not belong to Γ then $y = xh$ for some $h \in H$. Since $d(x, xh) = d(e, h)$ it is enough to prove that for every $h \in H$, $\operatorname{diam}_\Sigma[e, h] \leq 3$.

Let $h \in H$. We can write h as a reduced word of the form:

$$h = h_1(aba^{-1}b^{-1})^{m_1} h_2(aba^{-1}b^{-1})^{m_2} \dots h_k(aba^{-1}b^{-1})^{m_k} h_{k+1}$$

where $k \in \mathbb{N}$, $h_1, \dots h_{k+1} \in H$ and $m_1, \dots, m_k \in \mathbb{Z}$. Note that the geodesic $[e, h]$ in Γ is the geodesic from e to h in $\Gamma(H, \{a, b\})$, shortened by the relation $aba^{-1}b^{-1} = c^{-2}$. Let u, v be two vertices in $[e, h]$. Since every two vertices in H are connected by an edge in Σ we get:

$$d_\Sigma(u, v) \leq \begin{cases} 1, & u, v \in H \\ 2, & u \in H, v \notin H \\ 3, & u, v \notin H. \end{cases}$$

The calculation for $h = a^2b^{-1}(aba^{-1}b^{-1})^2 ab(aba^{-1}b^{-1})^{-1}a^{-1}ba$ is shown in Figure 3.

Proof of condition (c): We need to prove that every ball of finite radius in $(H, \hat{d})$ is finite. It is enough to show that for every $r > 0$, the ball $B_{\hat{d}}(e, r)$ is finite. Denote by Γ_H the complete graph $\Gamma(H, H)$. We say that a path in $\Gamma(G, X \sqcup H)$ is *admissible* if it does not contain edges in Γ_H. We construct inductively an admissible geodesic from e to an element in H and show that its path is of the form $e, c, c^2, \dots, c^{2n}$ or $e, c^{-1}, c^{-2}, \dots, c^{-2n}$. The only two edges starting at e and are not contained in Γ_H are labeled by c and c^{-1}. Note that a geodesic ending in H can not start in an edge which belongs to one of the orbits $c\Gamma_H$ or $c^{-1}\Gamma_H$. Hence, the only two edges which begin at c or c^{-1}, such that the resulting geodesic ends in H are labeled by c and c^{-1}.

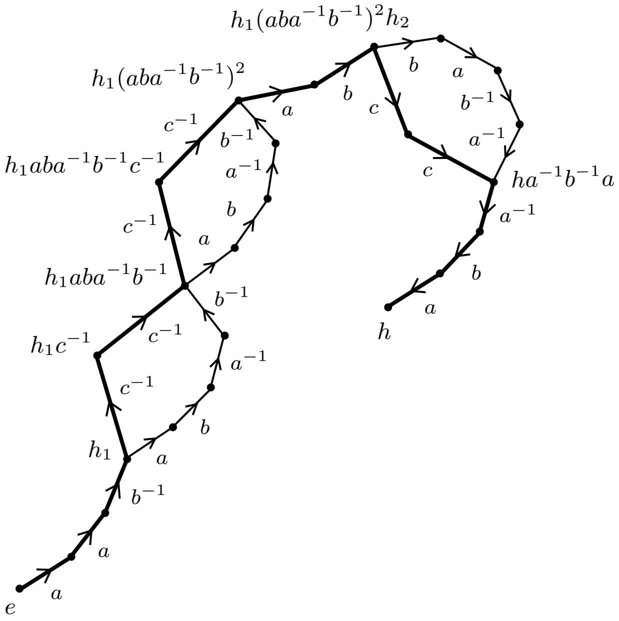

FIGURE 3. The geodesic in Γ from e to h

We proceed with the path e, c, c^2 inductively: an element c^{2n} belongs to H, hence the only two edges staring at c^{2n} and are not contained in Γ_H are labeled by c^{2n+1} and c^{2n-1}. A geodesic that ends in H can not start in an edge which belongs to the orbit $c^{2n+1}\Gamma_H$. Therefore, the only two edges which start at c^{2n+1}, such that the resulting geodesic ends in H are again labeled by c and c^{-1}. Consequently, $\hat{d}(e,h) = n$ if and only if n is even and $h = c^{\frac{n}{2}}$ or $h = c^{-\frac{n}{2}}$. Hence, $B_{\hat{d}}(e, 2n) = \{(aba^{-1}b^{-1})^l | l \in \mathbb{Z}, |l| < n\}$. For $r > 0$, let $n \in \mathbb{N}$ such that $2n-2 < r \leq 2n$. We get a finite ball

$$B_{\hat{d}}(e,r) = \left\{(aba^{-1}b^{-1})^l \,|\, l \in \mathbb{Z}, |l| < \frac{\lceil r \rceil}{2}\right\}.$$

Case 2: $g = 4$. We prove that $H = \langle a, b\rangle$ is hyperbolically embedded in $G = \langle a, b, c, d \,|\, aba^{-1}b^{-1}c^2d^2 = 1\rangle$. Let $X = \{c, d\}$. We need to prove that the graph $\Sigma = \Gamma(G, \{c, d\} \sqcup \langle a, b\rangle)$ is hyperbolic and every ball of finite radius in $(H, \hat{d})$ is finite. For the first argument, it is enough to show that for every $h \in H$, $\operatorname{diam}_\Sigma[e, h] \leq 5$. Note that for every $h \in H$ there are two geodesics in $\Gamma = \Gamma(G, \{a, b, c, d\})$ from h to $haba^{-1}b^{-1}$ which are $h, ha, hab, haba^{-1}, haba^{-1}b^{-1}$ and $h, hd^{-1}, hd^{-2}, hd^{-2}c^{-1}, hd^{-2}c^{-2}$. Let $h \in H$, p a geodesic in Γ from e to h and let $u, v \in p$. Similarly to the previous case we have:

$$d_\Sigma(u,v) \leq \begin{cases} 1, & u,v \in H \\ 3, & u \in H, v \notin H \\ 5, & u,v \notin H. \end{cases}$$

We continue as in the previous case: an admissible geodesic in Σ from e to an element in H is a path of the form $e, c, c^2, c^2d, c^2d^2, \ldots, (c^2d^2)^n$ or $e, d^{-1}, d^{-2}, d^{-2}c^{-1}, d^{-2}c^{-2} \ldots, (c^2d^2)^{-n}$. Hence we obtain a finite ball:

$$B_{\hat{d}}(e,r) = \left\{ (aba^{-1}b^{-1})^l \,|\, l \in \mathbb{Z}, |l| < \frac{\lceil r \rceil}{4} \right\}.$$

Case 3: $g \geq 5$. Let $H = \langle a_1, a_2 \rangle$, $G = \langle a_1, \ldots, a_g \,|\, a_1a_2a_1^{-1}a_2^{-1}a_3^2 \ldots a_g^2 = 1 \rangle$ and $X = \langle a_3, \ldots, a_g \rangle$. For every $h \in H$ the geodesic in $\Gamma(G, \{a_1, \ldots, a_g\})$ from e to h is the geodesic in $\Gamma(H, \{a_1, a_2\})$ from e to h. Therefore, its diameter in $\Gamma(G, X \sqcup H)$ is equal to 1. Similarly, for every $r > 0$ we have:

$$B_{\hat{d}}(e,r) = \left\{ (a_1a_2a_1^{-1}a_2^{-1})^l \,|\, l \in \mathbb{Z}, |l| < \frac{\lceil r \rceil}{2g-4} \right\}$$

and hence H is a hyperbolically embedded subgroup of G. □

The corollary below follows immediately from Theorem 4.3:

Corollary 4.5. The map i^n is onto for every $n \geq 2$.

Remark 4.6. By a similar proof, the group $\mathrm{F}_2 = \langle a, b \rangle$ is hyperbolically embedded in $G = \langle a, b, c \,|\, a^2b^2c^2 = 1 \rangle$. Note, however, that most of the embeddings of F_2 into $G = \langle a, b, c \,|\, a^2b^2c^2 = 1 \rangle$ are not hyperbolic.

Consider $H = \langle a^2, b^2 \rangle < G$. Since $aHa^{-1} = H$, H is not an almost malnormal subgroup of G, thus it is not hyperbolically embedded. The proof of the previous proposition fails in the following step: Let $X = \{a, b, c\}$. Then G is generated by $X \cup H$ and the graph $\Sigma = \Gamma(G, X \sqcup H)$ is hyperbolic. But not every ball of finite radius in $(H, \hat{d})$ is finite. Let $n \in \mathbb{N}$. Consider the following path in $\Gamma(G, X \sqcup H)$: e, a, a^{2n-1}, a^{2n}, see Figure 4. Note that this path is admissible since it does not contain edges in the complete subgraph Γ_H. Hence $\hat{d}(e, a^{2n}) = 3$ for every $n \in \mathbb{N}$ and $B_{\hat{d}}(e, 4)$ is infinite. Every subset $X \subseteq G$ such that G is generated by $X \cup H$ has to contain a. Therefore, H is not hyperbolically embedded in G with respect to any subset of G.

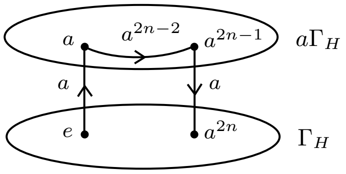

FIGURE 4. The admissible path e, a, a^{2n-1}, a^{2n} when $H = \langle a^2, b^2 \rangle$

4.3. **Construction of $\rho_\epsilon : \mathrm{F}_2 \to \mathrm{Homeo}_0(N_g, \mu)$.** Let i be an embedding of figure-eight discussed in the previous section, i.e., i^n is surjective. Now we proceed as in [3].

Lemma 4.7. *Let* $n \geq 2$ *and* $\epsilon \in (0,1)$*. Then there exists a real number* $\Lambda \neq 0$ *and a map* $\rho_\epsilon : \mathrm{F}_2 \to \mathrm{Homeo}_0(N_g, \mu)$ *such that*

$$\left\|\rho_\epsilon^n \overline{E\Gamma}_b^n([c]) - \Lambda i^n([c])\right\| \xrightarrow[\epsilon\to 0]{} 0$$

for every class $[c] \in \overline{EH}_b^n(\pi_1(N_g))$*.*

Proof. Let $\epsilon \in (0,1)$ and $N = \big\{(x,y) \in \mathbb{R}^2 \,|\, 1 \leq x^2 + y^2 \leq 4\big\}$. Define an isotopy $P_\epsilon^t : [0,1] \to \mathrm{Diff}(N, \kappa)$ by

$$P_\epsilon^t(r,\theta) = \big(r, \theta + 2\pi t f(r)\big),$$

where $f : [1,2] \to \mathbb{R}$ is a smooth function with $f(x) = 1$ for $x \in (1+\epsilon, 2-\epsilon)$ and $f(1) = f(2) = 0$, and κ is the Euclidean area form on $\mathbb{R}^2$. Note that the Lebesgue measure is preserved by the maps P_ϵ^t since $\det(D_{(r,\theta)} P_\epsilon^t) = 1$.

Denote $\mathrm{F}_2 = \langle a, b\rangle$ and $i(a), i(b)$ by embedded loops α, β in N_g based at z which intersect only at z. Let $N(\alpha)$ be a closed tubular neighborhood of α that is diffeomorphic to N. Note that for the embeddings described in Sections 4.1 and 4.2, we can choose $N(\alpha)$ to be an oriented neighborhood equipped with an area form induced by the covering map $\Sigma_{g-1} \to N_g$. We show that there exists a diffeomorphism $n_\alpha : N(\alpha) \to N$ such that the composition $n_\alpha^{-1} \circ P_\epsilon^t \circ n_\alpha$ is a measure preserving diffeomorphism.

Let ν be the area form on $N(\alpha)$ induced by the Lebesgue measure μ. Let φ be a diffeomorphism $N(\alpha) \to N$. We assume that the total areas with respect to κ and $(\varphi^{-1})^*(\nu)$ are equal. Now, by Moser's theorem there exists a diffeomorphism $\psi : N \to N$ such that $\psi^*(\kappa) = (\varphi^{-1})^*(\nu)$. Define $n_\alpha = \psi \circ \varphi$. Denote by A the Lebesgue measure on $\mathbb{R}^2$ and let $U \subseteq N$ be a measurable set. We get:

$$\psi^*(A)(U) = \int_U \psi^*(\kappa) = \int_U (\varphi^{-1})^*(\nu) = \int_{\varphi^{-1}(U)} \nu = \mu(\varphi^{-1}(U)) = \varphi_*(\mu)(U).$$

Let $B \subseteq N(\alpha)$ be a measurable set. By the following computation we get the required result:

$$\mu(\varphi^{-1}\psi^{-1}P_\epsilon^t\psi\varphi(B)) = \varphi_*(\mu)(\psi^{-1}P_\epsilon^t\psi\varphi(B)) = \psi^*(A)(\psi^{-1}P_\epsilon^t\psi\varphi(B)) =$$
$$A(P_\epsilon^t\psi\varphi(B)) = A(\psi\varphi(B)) = \psi^*(A)(\varphi(B)) = \varphi_*(\mu)(\varphi(B)) = \mu(B).$$

Now, define an isotopy $P_\epsilon^t(\alpha) : [0,1] \to \mathrm{Homeo}_0(N_g, \mu)$ in the following way:

$$P_\epsilon^t(\alpha)(x) = \begin{cases} n_\alpha^{-1} \circ P_\epsilon^t \circ n_\alpha(x), & x \in N(\alpha) \\ \qquad x, & x \notin N(\alpha) \end{cases}.$$

Let $A_\epsilon(\alpha) = n_\alpha^{-1}\{(r,\theta) \,|\, 1+\epsilon \leq r \leq 2-\epsilon\}$ and $B_\epsilon(\alpha) = N(\alpha) \backslash A_\epsilon(\alpha)$. Similarly, we define $P_\epsilon^t(\beta), A_\epsilon(\beta)$, $B_\epsilon(\beta)$, and $\rho_\epsilon : \mathrm{F}_2 \to \mathrm{Homeo}_0(N_g, \mu)$ by:

$$\rho_\epsilon(a) = P_\epsilon^1(\alpha), \;\; \rho_\epsilon(b) = P_\epsilon^1(\beta)$$

We compute $\gamma(\rho_\epsilon(\omega), x)$ for every $\omega \in \mathrm{F}_2$, $x \in N_g$. To simplify the notation, we identify F_2 with $i(\mathrm{F}_2)$. Let $h_a : \mathrm{F}_2 \to \langle a\rangle$ be the homomorphism which sends a to a and b to e. We define h_b in a similar way. Choose $g_t := P^t_\epsilon(\alpha)$ as an isotopy connecting the identity with $\rho_\epsilon(\alpha)$. If $x \in A_\epsilon(\alpha)$, then $\gamma(\rho_\epsilon(a), x)$ is conjugated to $[\alpha]$, and if $x \notin N(\alpha)$ then $\gamma(\rho_\epsilon(a), x) = [z]$, see Figure 5. Similarly, If $x \in A_\epsilon(\beta)$ then $\gamma(\rho_\epsilon(b), x)$ is conjugated to $[\beta]$, and if $x \notin N(\beta)$ then $\gamma(\rho_\epsilon(b), x) = [z]$.

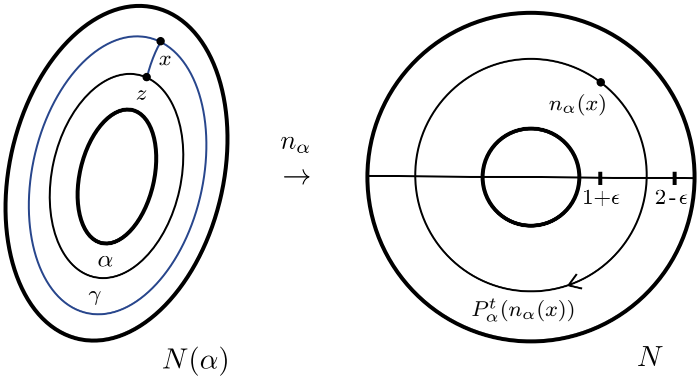

FIGURE 5. The loop α is free homotopic (or conjugated in $\pi_1(N_g, z)$) to the loop $\gamma(\rho_\epsilon(a), x)$ whenever $x \in N(\alpha)$.

Let $\omega \in \mathrm{F}_2$. By using Proposition 3.2 we get:

$$\gamma(\rho_\epsilon(\omega), x) = \begin{cases} [e], & x \in N_g \backslash (N(\alpha) \cup N(\beta)) \\ [u_x][\omega][u_x]^{-1}, & x \in A_\epsilon = A_\epsilon(\alpha) \cap A_\epsilon(\beta) \\ [u_{x,a}][h_a(\omega)][u_{x,a}]^{-1}, & x \in A^a_\epsilon = A_\epsilon(\alpha) \backslash N(\beta) \\ [u_{x,b}][h_b(\omega)][u_{x,b}]^{-1}, & x \in A^b_\epsilon = A_\epsilon(\beta) \backslash N(\alpha) \end{cases}$$

for some $[u_x], [u_{x,a}], [u_{x,b}] \in \pi_1(N_g, z)$. Let $x \in B_\epsilon = B_\epsilon(\alpha) \cup B_\epsilon(\beta)$. In this case we do not have a precise expression for $\gamma(\rho_\epsilon(\omega), x)$, but we show that this case is negligible as ϵ approaches to zero.

Let $n \in \mathbb{N}$, $[c] \in \overline{EH}^n_b(\pi_1(N_g))$ and $\bar{\omega} = (\omega_0, \ldots, \omega_n) \in F_2^{n+1}$. Denote $\bar{g} = (g_0, \ldots, g_n) \in \mathrm{Homeo}_0(N_g, \mu)^{n+1}$, $\gamma(\bar{g}, x) = (\gamma(g_0, x), \ldots, \gamma(g_n, x))$ an element in $\pi_1(N_g)^{n+1}$, and $\rho_\epsilon(\bar{\omega}) = (\rho_\epsilon(\omega_0), \ldots, \rho_\epsilon(\omega_n))$. We have:

$$\rho^n_\epsilon \overline{E\Gamma}^n_b([c])(\bar{\omega}) = \overline{E\Gamma}^n_b([c])(\rho_\epsilon(\bar{\omega})) = \int_{N_g} c(\gamma(\rho_\epsilon(\bar{\omega}), x)) d\mu(x).$$

We think about the integral $\int_{N_g} c(\gamma(\rho_\epsilon(\bar{\omega}), x)) d\mu(x)$ as a function of $\bar{\omega}$, i.e., a map $\mathrm{F}_2^{n+1} \to \mathbb{R}$ belongs to $C^n_b(\mathrm{F}_2)$. By linearity we compute the equivalent class of the integral on each subset. Conjugation acts trivially on cohomology, hence on A_ϵ we have:

$$\left[\bar{\omega} \mapsto \int_{A_\epsilon} c(\bar{\omega}) d\mu(x)\right] = \left[\bar{\omega} \mapsto \mu(A_\epsilon) c(\bar{\omega})\right] = \mu(A_\epsilon) i^n([c]).$$

For the set A^a_ϵ we consider the following sequence of homomorphisms:

$$\mathrm{F}_2 \xrightarrow{h_a} \mathbb{Z} \xrightarrow{i_{|\mathbb{Z}}} i(\mathbb{Z})$$

It induces the following sequence on the level of the reduced exact bounded cohomology: $\overline{EH}^n_b(i(\mathbb{Z})) \xrightarrow{i^n_{|\mathbb{Z}}} \overline{EH}^n_b(\mathbb{Z}) \xrightarrow{h^n_a} \overline{EH}^n_b(\mathrm{F}_2)$. Since $\overline{EH}^n_b(\mathbb{Z}) = 0$, and conjugation acts trivially on cohomology we get:

$$\left[\int_{A^a_\epsilon} c(h_a(\bar{\omega}))d\mu(x)\right] = \left[\mu(A^a_\epsilon)c(h_a(\bar{\omega}))\right] = \mu(A^a_\epsilon)(i_{|\mathbb{Z}} \circ h_a)^n([c]) = [0]$$

Similarly, we obtain $\left[\int_{A^b_\epsilon} c(h_b(\bar{\omega}))d\mu(x)\right] = [0]$. Note that:

$$\left\| \left[\int_{B_\epsilon} c(\gamma(\rho_\epsilon(\bar{\omega}), x))d\mu(x)\right]\right\| \leq \mu(B_\epsilon)\cdot\|c\| \xrightarrow[\epsilon\to 0]{} 0$$

and $\mu(A_\epsilon) \xrightarrow[\epsilon\to 0]{} \Lambda$, where $\Lambda := \mu(N(\alpha)\cap N(\beta))$. Finally, we have:

$$\left\|\rho^n_\epsilon \overline{E\Gamma}^n_b([c]) - \Lambda i^n([c])\right\| = \left\|(\mu(A_\epsilon) - \Lambda)i^n([c]) + \int_{B_\epsilon} c(\gamma(\rho_\epsilon(\bar{\omega}), x))d\mu(x)\right\|$$

$$\leq (\Lambda - \mu(A_\epsilon))\|i^n([c])\| + \mu(B_\epsilon)\|c\| \xrightarrow[\epsilon\to 0]{} 0. \qquad \square$$

We complete the proof of our main result. We showed that whenever $g \geq 5$ the map $i^n : \overline{EH}^n_b(\pi_1(N_g)) \to \overline{EH}^n_b(\mathrm{F}_2)$ is an injection, and the homomorphism $\pi^n : \overline{EH}^n_b(\mathrm{F}_2) \to \overline{EH}^n_b(\pi_1(N_g))$ is a surjection. Let $d \in \overline{EH}^n_b(\mathrm{F}_2)$ such that $\|d\| > 0$ and let $c = \pi^n(d)$. Then we have $\|i^n(c)\| = \|d\| > 0$. By Lemma 4.7 there exists a small ϵ such that $\|\rho^n_\epsilon \overline{E\Gamma}^n_b(c)\| > 0$. Therefore, $\overline{E\Gamma}^n_b(\pi^n(d)) = \overline{E\Gamma}^n_b(c) \neq 0$. Hence the map $\overline{E\Gamma}^n_b \circ \pi^n$ is injective which yields the proof of the case $g \geq 5$.

Let $g \geq 3$. Note that $\operatorname{Ker}(\overline{E\Gamma}^n_b) \subset \operatorname{Ker}(i^n)$. Indeed, let $c \in \overline{EH}^n_b(\pi_1(N_g))$ such that $\overline{E\Gamma}^n_b(c) = 0$. Then $\|\Lambda i^n(c)\| = \left\|\rho^n_\epsilon \overline{E\Gamma}^n_b(c) - \Lambda i^n(c)\right\| \xrightarrow[\epsilon\to 0]{} 0$, and so $i^n(c) = 0$. It follows that

$$\dim(\overline{EH}^n_b(\mathrm{Homeo}_0(N_g, \mu))) \geq \dim(\overline{EH}^n_b(\pi_1(N_g))/\operatorname{Ker}(i^n)).$$

Since i^n is surjective, $\dim(\overline{EH}^n_b(\pi_1(N_g))/\operatorname{Ker}(i^n)) = \dim(\overline{EH}^n_b(\mathrm{F}_2))$ and the proof follows.

Remark 4.8. An interesting and quite difficult problem is to compute the bounded cohomology group $H^n_b(\mathrm{Homeo}_0(N_g, \mu))$ when $n \geq 2$ and $g = 1, 2$. Here new ideas are required, since in this case $\pi_1(N_g)$ has a trivial bounded cohomology.

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