ON THE AUTONOMOUS METRIC ON GROUPS OF HAMILTONIAN DIFFEOMORPHISMS OF CLOSED HYPERBOLIC SURFACES

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Abstract. Let $\mathbf{\Sigma}_g$ be a closed hyperbolic surface of genus g and let $\mathrm{Ham}(\mathbf{\Sigma}_g)$ be the group of Hamiltonian diffeomorphisms of $\mathbf{\Sigma}_g$. The most natural word metric on this group is the autonomous metric. It has many interesting properties, most important of which is the bi-invariance of this metric. In this work we show that $\mathrm{Ham}(\mathbf{\Sigma}_g)$ is unbounded with respect to this metric.

1. Introduction and main result

Let $\mathbf{\Sigma}_g$ be a closed hyperbolic surface of genus g and let ω be an area form on $\mathbf{\Sigma}_g$. For every smooth normalized function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$, i.e. H has a zero mean with respect to ω, there exists a unique vector field X_H which satisfies

$$dH(\cdot) = \omega(X_H, \cdot).$$

It is easy to see that X_H is tangent to the level sets of H. Let h be the time-one map of the flow h_t generated by X_H. The diffeomorphism h is area-preserving and every diffeomorphism arising in this way is called *autonomous*. Such a diffeomorphism is relatively easy to understand in terms of its generating function.

Denote by $\mathrm{Ham}(\mathbf{\Sigma}_g)$ the group of Hamiltonian diffeomorphisms of $\mathbf{\Sigma}_g$. It follows from results of Banyaga that every Hamiltonian diffeomorphism is a composition of finitely many autonomous diffeomorphisms [2]. We define the *autonomous norm* on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ by

$$\|f\|_{\mathrm{Aut}} := \min\{m \in \mathbf{N} \mid f = h_1 \cdots h_m \text{ where each } h_i \text{ is autonomous}\}.$$

The associated metric is defined by $\mathbf{d}_{\mathrm{Aut}}(f,g) := \|fg^{-1}\|_{\mathrm{Aut}}$. Since the set of autonomous diffeomorphisms is invariant under conjugation, the autonomous metric is bi-invariant. Our main result is the following

Theorem 1. *The metric group* $(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathbf{d}_{\mathrm{Aut}})$ *is unbounded.*

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Remark 1.1. We must notice that Theorem 1 may be seen as a corollary from more general results of the author, which will appear in a subsequent paper [8]. However, we think that the proof given in this paper is much easier than the one presented in [8].

2. Preliminaries

In this paper we consider only normalized functions $\mathbf{\Sigma}_g \to \mathbf{R}$.

2.A. **Quasi-morphisms.** A function $\psi\colon G \to \mathbf{R}$ from a group G to the reals is called a *quasi-morphism* if there exists a real number $A \geq 0$ such that

$$|\psi(gg') - \psi(g) - \psi(g')| \leq C$$

for all $g, g' \in G$. The infimum of such C's is called the *defect* of ψ and is denoted by D_ψ. If $\psi(g^n) = n\psi(g)$ for all $n \in \mathbf{Z}$ and $g \in G$ then ψ is called *homogeneous.* Any quasi-morphism ψ can be homogenized by setting

$$\overline{\psi}(g) := \lim_{p \to +\infty} \frac{\psi(g^p)}{p}.$$

The vector space of homogeneous quasi-morphisms on G is denoted by $Q(G)$. The space of homogeneous quasi-morphisms on G modulo the space of homomorphisms on G is denoted by $\widehat{Q}(G)$. For more information about quasi-morphisms and their connections to different brunches of mathematics, see [9].

2.B. **Polterovich construction.** Let $\mathbf{M}$ be an oriented closed Riemannian manifold equipped with a Riemannian volume form, and denote by $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$ the identity component of the group of volume preserving diffeomorphisms of $\mathbf{M}$. Let us describe a construction, due to L. Polterovich, of quasi-morphisms on $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$.

Let $z \in \mathbf{M}$. Suppose that the group $\pi_1(\mathbf{M}, z)$ has a trivial center and it admits a *non-trivial* homogeneous quasi-morphism

$$\psi\colon \pi_1(\mathbf{M}, z) \to \mathbf{R}.$$

For each $x \in \mathbf{M}$ let us choose a short geodesic path from x to z. In [12] Polterovich constructed the induced *non-trivial* homogeneous quasi-morphism $\overline{\Psi}$ on $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$ as follows:
For each $x \in \mathbf{M}$ and an isotopy $\{g_t\}_{t\in[0,1]}$ between Id and g, let g_x be a closed loop in $\mathbf{M}$ which is a concatenation of a geodesic path from z to x, the path $g_t(x)$ and a described above geodesic path from $g(x)$ to z. Denote by $[g_x]$ the corresponding element in $\pi_1(\mathbf{M}, z)$ and set

$$\Psi(g) := \int_{\mathbf{M}} \psi([g_x])\,\mathrm{vol} \qquad \overline{\Psi}(g) := \lim_{p\to\infty} \frac{1}{p} \int_{\mathbf{M}} \psi([(g^p)_x])\,\mathrm{vol}\,.$$

The maps Ψ and $\overline{\Psi}$ are well-defined quasi-morphisms because every diffeomorphism in $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$ is volume-preserving. They do not depend on a choice of the path $\{g_t\}$ because the evaluation map

$$\mathrm{ev}\colon \mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}) \to \mathbf{M},$$

where $\mathrm{ev}(g) = g(z)$, induces a map between $\pi_1(\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}), \mathrm{Id})$ and $\pi_1(\mathbf{M}, z)$ whose image lies in the center of $\pi_1(\mathbf{M}, z)$, which is trivial by our assumption. In addition, the quasi-morphism $\overline{\Psi}$ neither depends on the choice of a family of geodesic paths, nor on the choice of a base point z. For more details see [12]. We abuse the notation and sometimes write $\pi_1(\mathbf{M})$ instead of $\pi_1(\mathbf{M}, z)$.

Denote by $\mathfrak{Polt}_{\mathbf{M}}$ the linear map, induced by Polterovich construction, from $\widehat{Q}(\pi_1(\mathbf{M})) \to \widehat{Q}(\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}))$. Since every homogeneous quasi-morphism on $\pi_1(\mathbf{M})$, which is not a homomorphism, defines a homogeneous quasi-morphism on $\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol})$, which is also not a homomorphism, we obtain the following

Corollary 2.1. *The linear map* $\mathfrak{Polt}_{\mathbf{M}}\colon \widehat{Q}(\pi_1(\mathbf{M})) \to \widehat{Q}(\mathrm{Diff}_0(\mathbf{M}, \mathrm{vol}))$ *is injective.*

Let $g > 1$. Since the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$ is simple [2, 3], the linear space $\widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$ coincides with $Q(\mathrm{Ham}(\mathbf{\Sigma}_g))$. We denote by $\mathfrak{Polt}_g$ the linear map $\widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \to \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$ induced by Polterovich construction.

Proposition 2.2. *The linear map* $\mathfrak{Polt}_g\colon \widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \to \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$ *is injective.*

Proof. Suppose that $\mathfrak{Polt}_g$ is not injective. Then there exists a quasi-morphism ψ in $Q(\pi_1(\mathbf{\Sigma}_g))$, which is not a homomorphism, such that the induced homogeneous quasi-morphism $\overline{\Psi}$ on $\mathrm{Diff}_0(\mathbf{\Sigma}_g, \mathrm{area})$ vanishes on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$.

Lemma 2.3. *Let G be a group and G' its commutator subgroup. Then every homogeneous quasi-morphism on G that vanishes on G' is a homomorphism.*

Proof. Let $\varphi\colon G \to \mathbf{R}$ be a homogeneous quasi-morphism such that $\varphi(g') = 0$ for every $g' \in G'$. Since G' is a normal subgroup of G, the map $\widehat{\varphi}\colon G/G' \to \mathbf{R}$, given by $\widehat{\varphi}(gG') = \varphi(g)$, is a well-defined homogeneous quasi-morphism. The group G/G' is abelian and hence $\widehat{\varphi}$ is a homomorphism and so is φ. □

By a theorem of Banyaga [3] the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$ coincides with the commutator subgroup of $\mathrm{Diff}_0(\mathbf{\Sigma}_g, \mathrm{area})$. It follows from Lemma 2.3

that $\overline{\Psi}$ is a homomorphism, which contradicts the fact that the map

$$\mathfrak{Polt}_{\mathbf{\Sigma}_g}\colon \widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \to \widehat{Q}(\mathrm{Diff}_0(\mathbf{\Sigma}_g, \mathrm{area}))$$

is injective by Corollary 2.1. □

3. Proofs

3.A. **Curves traced by Morse autonomous flows.** Let h_t be an autonomous flow generated by a Morse function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ and set $h := h_1$. Let $x \in \mathbf{\Sigma}_g$ which satisfies the following conditions:

- x is a regular point of H,
- x belongs to only one connected component, i.e. a simple closed curve in $\mathbf{\Sigma}_g$, of the set $H^{-1}(H(x))$.

Such a set of points in $\mathbf{\Sigma}_g$ is denoted by Reg_H. Note that the measure of $\mathbf{\Sigma}_g \setminus \mathrm{Reg}_H$ is zero. For each $x \in \mathrm{Reg}_H$ let $c_x\colon [0,1] \to \mathbf{\Sigma}_g$ be an injective path (on $(0,1)$), such that $c_x(0) = c_x(1) = x$ and its image is a simple closed curve which is a connected component of $H^{-1}(H(x))$. For every $y_1, y_2 \in \mathbf{\Sigma}_g$ choose an injective map $s_{y_1 y_2}\colon [0,1] \to \mathbf{\Sigma}_g$ whose image is a short geodesic path from y_1 to y_2. Define

$$\gamma_x(t) := \begin{cases} s_{zx}(3t) & \text{for } t \in \left[0, \frac{1}{3}\right] \\ c_x(3t-1) & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ s_{xz}(3t-2) & \text{for } t \in \left[\frac{2}{3}, 1\right] \end{cases}. \tag{1}$$

Denote by $[\gamma_x]$ the corresponding element in $\pi_1(\mathbf{\Sigma}_g, z)$. Let $x \in \mathrm{Reg}_H$ and let $[h_x]$ be an element in $\pi_1(\mathbf{\Sigma}_g, z)$ represented by a path which is a concatenation of paths s_{zx}, $h_t(x)$ and s_{xz}. Then for each $p \in \mathbf{N}$ the element $[h_x^p]$ can be written as a product

$$[h_x^p] = \alpha'_{p,x} \circ [\gamma_x]^{k_{h,p}} \circ \alpha''_{p,x}\,, \tag{2}$$

where $k_{h,p}$ is an integer which depends only h, p and x, and the word length of elements $\alpha'_{p,x}\,, \alpha''_{p,x}$ in $\pi_1(\mathbf{\Sigma}_g, z)$ is bounded by some constant K which is independent of h, x and p.

Denote by $\mathcal{MCG}_g^1$ the mapping class group of a surface $\mathbf{\Sigma}_g$ with one puncture z. Recall that there is a following short exact sequence due to Birman [5]

$$1 \to \pi_1(\mathbf{\Sigma}_g, z) \to \mathcal{MCG}_g^1 \to \mathcal{MCG}_g \to 1, \tag{3}$$

where $\mathcal{MCG}_g$ is the mapping class group of a surface $\mathbf{\Sigma}_g$. Hence we view $\pi_1(\mathbf{\Sigma}_g, z)$ as a normal subgroup of $\mathcal{MCG}_g^1$.

Proposition 3.1. *Let $g > 1$. There exists a finite set S_g of elements in $\mathcal{MCG}_g^1$, such that for every Morse function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ and every $x \in \mathrm{Reg}_H$ the loop $[\gamma_x] \in \pi_1(\mathbf{\Sigma}_g, z) < \mathcal{MCG}_g^1$ is conjugated to some element in S_g.*

Proof. Let $x \in \mathrm{Reg}_H$. If the loop $\gamma_x(t)$ is homotopically trivial in $\mathbf{\Sigma}_g$, then $[\gamma_x] = 1_{\mathcal{MCG}_g^1}$. Suppose that $\gamma_x(t)$ is homotopically non-trivial in $\mathbf{\Sigma}_g$. We say that simple closed curves $\delta, \delta' \in \mathbf{\Sigma}_g$ are equivalent $\delta \cong \delta'$, if there exists a homeomorphism $f\colon \mathbf{\Sigma}_g \to \mathbf{\Sigma}_g$ such that $f(\delta) = \delta'$. It follows from classification of surfaces that the set of equivalence classes $\mathcal{E}_g$ is finite. Let c_x be a simple closed curve defined in (1). Since $\mathbf{\Sigma}_g$ and c_x are oriented, the curve c_x splits in $\mathbf{\Sigma}_g \setminus \{x\}$ into two simple closed curves $c_{x,+}$ and $c_{x,-}$ which are homotopic in $\mathbf{\Sigma}_g$, see Figure 1.

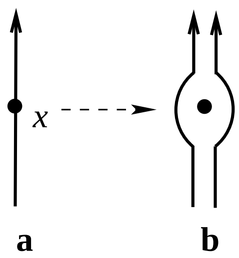

FIGURE 1. Part of the curve c_x is shown in Figure **a**. Its splitting into curves $c_{x,+}$ and $c_{x,-}$ is shown in Figure **b**. The left curve in Figure **b** is $c_{x,+}$ and the right curve is $c_{x,-}$.

The image of the element $[\gamma_x]$ in $\mathcal{MCG}_g^1$, under the Birman embedding (3) of $\pi_1(\mathbf{\Sigma}_g, z)$ into $\mathcal{MCG}_g^1$, is conjugated to $t_{c_{x,+}} \circ t_{c_{x,-}}^{-1}$, where $t_{c_{x,+}}$ and $t_{c_{x,-}}$ are Dehn twists in $\mathbf{\Sigma}_g \setminus \{x\}$ about curves $c_{x,+}$ and $c_{x,-}$ respectively.

Note that if $c_x \cong \delta$ then there exists a homeomorphism $f\colon \mathbf{\Sigma}_g \to \mathbf{\Sigma}_g$ such that $f(c_x) = \delta$, hence $f(c_{x,+}) = \delta_+$ and $f(c_{x,-}) = \delta_-$. We have

$$t_{\delta_+} = f \circ t_{c_{x,+}} \circ f^{-1} \qquad t_{\delta_-} = f \circ t_{c_{x,-}} \circ f^{-1}.$$

This yields

$$f \circ (t_{c_{x,+}} \circ t_{c_{x,-}}^{-1}) \circ f^{-1} = t_{\delta_+} \circ t_{\delta_-}^{-1}.$$

In other words, an element $[\gamma_x]$ is conjugated in $\mathcal{MCG}_g^1$ to some $t_{\delta_+} \circ t_{\delta_-}^{-1}$, where δ is a representative of an equivalence class in $\mathcal{E}_g$. Let $\{\delta_i\}_{i=1}^{\#\mathcal{E}_g}$ be a set of simple closed curves in $\mathbf{\Sigma}_g$, such that each equivalence class in $\mathcal{E}_g$ is represented by some δ_i. Let

$$S_g := \{t_{\delta_{1,+}} \circ t_{\delta_{1,-}}^{-1}, \ldots, t_{\delta_{\#\mathcal{E}_g,+}} \circ t_{\delta_{\#\mathcal{E}_g,-}}^{-1}\}. \tag{4}$$

It follows that $[\gamma_x]$ is conjugated to some element in S_g. Noting that the set S_g neither depend on H nor on x, we conclude the proof of the proposition. □

3.B. **Continuity of Polterovich quasi-morphisms.** The aim of this subsection is to prove the following technical result which will be used in the proof of Theorem 1.

Theorem 3.2. *Let* $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$ *and* $\{H_k\}_{k=1}^{\infty}$ *be a sequence of functions such that each* $H_k\colon \mathbf{\Sigma}_g \to \mathbf{R}$ *and* $H_k \xrightarrow[k\to\infty]{} H$ *in* C^1*-topology. Let* h *and* h_k *be the time-one maps of the Hamiltonian flows generated by* H *and* H_k *respectively. Then*

$$\lim_{k\to\infty} \overline{\Psi}(h_k) = \overline{\Psi}(h),$$

where $\overline{\Psi}$ *is a homogeneous quasi-morphism on* $\mathrm{Ham}(\mathbf{\Sigma}_g)$ *induced by Polterovich construction.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.4 in [7]. For the reader convenience, we present the proof below.

At this point we recall a definition of the L^1-metric on the group $\mathrm{Ham}(\mathbf{\Sigma}_g)$. It is defined as follows. Let

$$\mathcal{L}_1\{h_t\} := \int_0^1 \int_{\mathbf{\Sigma}_g} |\dot{h}_t(x)| \omega dt$$

be the L^1-length of a path $\{h_t\}_{t\in[0,1]} \in \mathrm{Ham}(\mathbf{\Sigma}_g)$, where $|\dot{h}_t(x)|$ denotes the length of the tangent vector $\dot{h}_t(x) \in T_x\mathbf{\Sigma}_g$ induced by the Riemannian metric. Observe that this length is right-invariant, that is, $\mathcal{L}_1\{h_t \circ f\} = \mathcal{L}_p\{h_t\}$ for any $f \in \mathrm{Ham}(\mathbf{\Sigma}_g)$. It defines a non-degenerate right-invariant metric on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ by

$$\mathbf{d}_1(h_0, h_1) := \inf_{h_t} \mathcal{L}_1\{h_t\},$$

where the infimum is taken over all paths from h_0 to h_1. See Arnol'd-Khesin [1] for a detailed discussion. We set $\|h\|_1 := \mathbf{d}_1(\mathrm{Id}, h)$. In [6] the author proved the following

Theorem 3.3 ([6])**.** *Let* $\mathbf{\Sigma}_g$ *be a closed hyperbolic surface, and* $\overline{\Psi}$ *a homogeneous quasi-morphism on* $\mathrm{Ham}(\mathbf{\Sigma}_g)$ *induced by Polterovich construction. Then* $\overline{\Psi}$ *is Lipschitz with respect to the* L^1*-metric on the group* $\mathrm{Ham}(\mathbf{\Sigma}_g)$*, i.e. there exists* $C > 0$ *such that* $\forall h \in \mathrm{Ham}(\mathbf{\Sigma}_g)$

$$\overline{\Psi}(h) \leq C\|h\|_1.$$

Lemma 3.4. *Let* $G\colon \mathbf{\Sigma}_g \to \mathbf{R}$ *be smooth function. Then for any* $\epsilon > 0$ *and* $p \in \mathbf{N}$ *there exists* $\delta_p > 0$*, such that if* G *is* δ_p*-close to a smooth function* $F\colon \mathbf{\Sigma}_g \to \mathbf{R}$ *in* C^1*-topology, then*

$$\mathbf{d}_1(g^p, f^p) < \epsilon,$$

where g_t *and* f_t *are the Hamiltonian flows generated by* G *and* F*, and* g *and* h *are time-one maps of these flows.*

Proof. We replace $\mathbf{D}^2$ by $\mathbf{\Sigma}_g$ in the proof of Lemma 3.7 in [7]. Now the proof is identical to the proof of Lemma 3.7 in [7]. □

Proposition 3.5. *Let $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$. Then for any $\epsilon > 0$ there exists $\delta > 0$, such that if $F\colon \mathbf{\Sigma}_g \to \mathbf{R}$ is δ-close to H in C^1-topology then:*

$$\left|\overline{\Psi}(h) - \overline{\Psi}(f)\right| \leq \epsilon,$$

where h and f are time-one maps of flows generated by H and F.

Proof. Fix some $\epsilon > 0$. Let C be the constant which was defined in Theorem 3.3. Take $p \in \mathbf{N}$ such that $\frac{D_{\overline{\Psi}}+C}{p} < \epsilon$. It follows from Lemma 3.4 that there exists $\delta_p > 0$, such that if F is δ_p-close to H in C^1-topology, then $\mathbf{d}_1(f^p, h^p) < 1$. Thus we obtain

$$\left|\overline{\Psi}(f) - \overline{\Psi}(h)\right| = \frac{1}{p}\left|\overline{\Psi}(f^p) - \overline{\Psi}(h^p)\right| \leq \frac{D_{\overline{\Psi}} + \left|\overline{\Psi}(f^p h^{-p})\right|}{p}.$$

It follows from Theorem 3.3 that

$$\left|\overline{\Psi}(f^p h^{-p})\right| \leq C\mathbf{d}_1(Id, f^p h^{-p}) = C\mathbf{d}_1(f^p, h^p) < C.$$

Thus

$$\left|\overline{\Psi}(f) - \overline{\Psi}(h)\right| < \frac{D_{\overline{\Psi}} + C}{p} < \epsilon.$$

□

Proposition 3.5 concludes the proof of Theorem 3.2. □

3.C. **Proof of Theorem 1.** Recall that the group $\pi_1(\mathbf{\Sigma}_g)$ is a subgroup of $\mathcal{MCG}_g^1$. Denote by $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ the space of homogeneous quasi-morphisms on $\pi_1(\mathbf{\Sigma}_g)$ so that:

- For each $\varphi \in Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ there exists $\widehat{\varphi} \in Q(\mathcal{MCG}_g^1)$ such that $\widehat{\varphi}|_{\pi_1(\mathbf{\Sigma}_g)} = \varphi$,
- each φ vanishes on the finite set S_g,

where S_g is the set defined in (4). The group $\pi_1(\mathbf{\Sigma}_g)$ contains a non-abelian free group, and thus is not virtually abelian. It is an infinite normal subgroup of $\mathcal{MCG}_g^1$ and hence is a non-reducible subgroup of $\mathcal{MCG}_g^1$, see [10, Corollary 7.13]. Now, by a result of Bestvina-Fujiwara [4, Theorem 12] we have the following

Corollary 3.6. *The space $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ is infinite dimensional.*

Note that since every non-trivial quasi-morphism in $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ vanishes on S_g, it can not be a homomorphism. Hence the space

$Q_{\mathcal{MCG}^1_g}(\pi_1(\mathbf{\Sigma}_g), S_g)$ may be viewed as a linear subspace of $\widehat{Q}(\pi_1(\mathbf{\Sigma}_g))$. Recall that by Corollary 2.1 the linear map

$$\mathfrak{Polt}_g\colon \widehat{Q}(\pi_1(\mathbf{\Sigma}_g)) \hookrightarrow \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$$

is injective. Hence the map

$$\mathfrak{Polt}_g\colon Q_{\mathcal{MCG}^1_g}(\pi_1(\mathbf{\Sigma}_g), S_g) \hookrightarrow \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$$

is also injective.

Denote by $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$ the space of homogeneous quasi-morphisms on $\mathrm{Ham}(\mathbf{\Sigma}_g)$ that vanish on the set $\mathrm{Aut} \subset \mathrm{Ham}(\mathbf{\Sigma}_g)$ of all autonomous diffeomorphisms. Since $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$ contains no non-trivial homomorphisms, it is viewed as a linear subspace of $\widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$. Now we a ready to state and prove our key proposition.

Proposition 3.7. *The image of the map*

$$\mathfrak{Polt}_g\colon Q_{\mathcal{MCG}^1_g}(\pi_1(\mathbf{\Sigma}_g), S_g) \hookrightarrow \widehat{Q}(\mathrm{Ham}(\mathbf{\Sigma}_g))$$

lies in the linear space $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$.

Proof. Let $\psi \in Q_{\mathcal{MCG}^1_g}(\pi_1(\mathbf{\Sigma}_g), S_g)$ and $h \in \mathrm{Ham}(\mathbf{\Sigma}_g)$ an autonomous diffeomorphism. We need to show that $\overline{\Psi}(h) = 0$, where $\overline{\Psi} = \mathfrak{Polt}_g(\psi)$. Since Morse functions on $\mathbf{\Sigma}_g$ form a dense subset in the set of all smooth functions in C^1-topology [11], by Theorem 3.2 it is enough to show that $\overline{\Psi}(h) = 0$, where h is a time-one map of the flow generated by some Morse function $H\colon \mathbf{\Sigma}_g \to \mathbf{R}$. Recall that by definition we have

$$\overline{\Psi}(h) = \int_{\mathbf{\Sigma}_g} \lim_{p\to\infty} \frac{\psi([h^p_x])}{p}\omega\,.$$

Since the set Reg_H is of full measure in $\mathbf{\Sigma}_g$, it is enough to show that for each $x \in \mathrm{Reg}_H$ the following equality holds $\lim_{p\to\infty} \frac{|\psi([h^p_x])|}{p} = 0$.

The group $\pi_1(\mathbf{\Sigma}_g)$ admits the following presentation

$$\pi_1(\mathbf{\Sigma}_g) = \langle \alpha_i, \beta_i |\ 1 \le i \le g,\ \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1\rangle. \tag{5}$$

For every $\alpha \in \pi_1(\mathbf{\Sigma}_g)$ denote by $l(\alpha)$ the word length of α with respect to the set of generators given in (5). Since $\psi \in Q_{\mathcal{MCG}^1_g}(\pi_1(\mathbf{\Sigma}_g), S_g)$, it is constant on conjugacy classes and thus $\psi(\alpha_i) = \psi(\beta_i) = 0$ for each $1 \le i \le g$. In addition, for every $\alpha \in \pi_1(\mathbf{\Sigma}_g)$ we have $|\psi(\alpha)| \le D_\psi l(\alpha)$. It follows from (2) that for every $p \in \mathbf{N}$ and $x \in \mathrm{Reg}_H$ we have

$$[h^p_x] = \alpha'_{p,x} \circ [\gamma_x]^{k_{h,p}} \circ \alpha''_{p,x}\,,$$

where $k_{h,p}$ is an integer which depends only h, p and x, and $l(\alpha'_{p,x})$, $l(\alpha''_{p,x})$ are bounded by some constant $K > 0$ independent of h, x and p. Hence for every $p \in \mathbf{N}$ and $x \in \mathrm{Reg}_H$ we have

$$0 \leq \frac{|\psi([h_x^p])|}{p} \leq \frac{|\psi(\alpha'_{p,x})| + |k_{h,p}||\psi([\gamma_x])| + |\psi(\alpha''_{p,x})| + 2D_\psi}{p} .$$

Since $\psi \in Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$, by definition it extends to a homogeneous quasi-morphism on $\mathcal{MCG}_g^1$ and vanishes on the set S_g. Hence by Proposition 3.1 we have $\psi([\gamma_x]) = 0$, and hence

$$0 \leq \frac{|\psi([h_x^p])|}{p} \leq \frac{2K \cdot D_\psi + 2D_\psi}{p} = \frac{2D_\psi(K+1)}{p} .$$

By taking $p \to \infty$ we conclude the proof of the proposition. □

Since the map $\mathfrak{Polt}_g$ is injective and by Corollary 3.6 the linear space $Q_{\mathcal{MCG}_g^1}(\pi_1(\mathbf{\Sigma}_g), S_g)$ is infinite dimensional, an immediate consequence of Proposition 3.7 is the following

Corollary 3.8. *The space* $Q(\mathrm{Ham}(\mathbf{\Sigma}_g), \mathrm{Aut})$ *is infinite dimensional.*

Remark 3.9. The existence of a non-trivial homogeneous quasi-morphism $\overline{\Psi}\colon \mathrm{Ham}(\mathbf{\Sigma}_g) \to \mathbf{R}$, which is trivial on the set $\mathrm{Aut} \subset \mathrm{Ham}(\mathbf{\Sigma}_g)$ of all autonomous diffeomorphisms, implies that the autonomous norm is unbounded. Indeed, for every $f \in \mathrm{Ham}(\mathbf{\Sigma}_g)$ we have that

$$|\overline{\Psi}(f)| = |\overline{\Psi}(h_1 \circ \ldots \circ h_m)| \leq m D_{\overline{\Psi}}$$

and hence for every natural number n we get $\|f^n\|_{\mathrm{Aut}} \geq \frac{|\overline{\Psi}(f)|}{D_{\overline{\Psi}}} n > 0$, provided $\overline{\Psi}(f) \neq 0$.

Remark 3.9 and Corollary 3.8 conclude the proof of the theorem. □

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