THE AUTONOMOUS NORM ON $\mathrm{Ham}(\mathbf{R}^{2n})$ IS BOUNDED

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Abstract. We prove that the autonomous norm on the group of compactly supported Hamiltonian diffeomorphisms of the standard $\mathbf{R}^{2n}$ is bounded.

Let (M,ω) be a symplectic manifold and let $\mathrm{Ham}(M,\omega)$ be the group of compactly supported Hamiltonian diffeomorphisms of (M,ω). Recall that a Hamiltonian diffeomorphism f is the time-one map of the flow generated by the vector field X_{F_t} defined by $\omega(X_{F_t},-) = dF_t$. Here $F\colon M \times S^1 \to \mathbf{R}$ is a smooth compactly supported function and $F(x,t) = F_t(x)$ (see [9, Section 5.1] for details). The function F is called a Hamiltonian of f. If F does not depend on time then f is called autonomous. It is known that every Hamiltonian diffeomorphism is a product of autonomous ones [3]. The autonomous norm on $\mathrm{Ham}(M,\omega)$ is defined by:

$$\|f\| = \min\{k \in \mathbf{N} \mid f = a_1 \cdots a_k, \text{ where } a_i \text{ is autonomous}\}.$$

It is a conjugation invariant norm and is known to be unbounded on the group of compactly supported Hamiltonian diffeomorphisms of an oriented surface of finite area [2, 4, 3, 6].

This paper is concerned with the group $\mathrm{Ham}(\mathbf{R}^{2n})$ of compactly supported Hamiltonian diffeomorphisms of the Euclidean space equipped with the standard symplectic form. We prove the following result.

Theorem. *The diameter of the autonomous norm on* $\mathrm{Ham}(\mathbf{R}^{2n})$ *is bounded above by* 3.

Proof. Let $f \in \mathrm{Ham}(\mathbf{R}^{2n})$. Let $f = a_m \cdots a_1$, where $a_i \in \mathrm{Ham}(\mathbf{R}^{2n})$ are autonomous diffeomorphisms with compactly supported Hamiltonian functions $F_i\colon \mathbf{R}^{2n} \to \mathbf{R}$. Let $B(r)$ be the Euclidean ball centered at the origin, of radius $r > 0$ large enough so that it contains the union of the supports of the functions F_i.

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Lemma. *There exists an autonomous diffeomorphism* $h \in \mathrm{Ham}\,(\mathbf{R}^{2n})$ *such that* $h^i(B(r)) \cap h^j(B(r)) = \emptyset$ *for* $0 \leq i \neq j \leq m$.

The statement of the lemma means that h displaces the ball $B(r)$ m times. It follows from [5, Lemma 2.6] that there exists $g \in \mathrm{Ham}\,(\mathbf{R}^{2n})$ such that the following equality holds:

$$f = a_m \cdots a_1 = [h, g] a_1^h a_2^{h^2} \cdots a_m^{h^m},$$

where $a_i^{h^i} = h^i a_i h^{-i}$ and h is the diffeomorphism from the Lemma. Observe that, since the supports of $F_i \circ h^i$ are pairwise disjoint for $i \in \{1, \ldots, m\}$, we obtain that the composition $a_1^h a_2^{h^2} \cdots a_m^{h^m}$ is autonomous with the Hamiltonian function equal to

$$F_1 \circ h + F_2 \circ h^2 + \cdots + F_m \circ h^m.$$

Since the commutator $[h, g] = h \cdot (h^{-1})^g$ is a product of two autonomous diffeomorphisms we obtain that f is a product of three autonomous diffeomorphisms. □

Proof of the Lemma. Let $H_1 \colon \mathbf{R} \to \mathbf{R}$ be a smooth function satisfying the following conditions:

(1) $H_1(y) = 0$ for $|y| > r + 1$
(2) $H_1'(y) = r$ for $|y| \leq r$.

Let $H(x_1, y_1, \ldots, x_n, y_n) = H_1(y_1)$. We have that $dH = r dx_1$ and that the induced Hamiltonian vector field X is equal to $r \frac{\partial}{\partial x_1}$. Thus the induced Hamiltonian diffeomorphism displaces the ball $B(r)$ as many times as we like. Taking an appropriate cut off function we obtain the required compactly supported diffeomorphism h. □

Remarks. If f in the Theorem is contained in the kernel of the Calabi homomorphism (see Section 8.B of [1] for a definition) then the same argument shows that it is a product of up to three autonomous diffeomorphisms with trivial Calabi invariant.

It is known that the Hofer norm on $\mathrm{Ham}\,(\mathbf{R}^{2n})$ is unbounded [7]. The kernel of the Calabi homomorphism does not admit nontrivial quasi-morphisms, however, it is stably unbounded [8].

It is not difficult to see that the diameter of the autonomous norm on $\mathrm{Ham}\,(\mathbf{R}^{2n})$ is at least 2. To the best of our knowledge it is an open question whether there exists a Hamiltonian diffeomorphism of $\mathbf{R}^{2n}$ of autonomous norm equal to 3.

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