

ON QUASIMORPHISMS AND DISTORTION IN HOMEOMORPHISM GROUPS

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ABSTRACT. Let M be a smooth compact oriented connected manifold, and $\text{Homeo}_0(M, \mu)$ the group of homeomorphisms of M supported away from ∂M , which preserve a Borel probability measure μ induced by a volume form on M , and are isotopic to the identity. In this paper, we identify those Gambaudo-Ghys and Polterovich quasimorphisms $\Psi: \text{Diff}_0(M, \mu) \rightarrow \mathbf{R}$ which extend C^0 -continuously to $\text{Homeo}_0(M, \mu)$ as quasimorphisms, and to $\text{Homeo}_0(M)$ as group cochains whose differentials are semi-bounded cocycles.

We present several applications of this result which include unboundedness of certain bi-invariant metric on the commutator subgroup of $\text{Homeo}_0(M, \mu)$, and conditions under which a homeomorphism in $\text{Homeo}_0(M)$ is undistorted.

1. INTRODUCTION

Let M be a smooth compact oriented connected manifold equipped with a Borel probability measure μ defined by a volume form on M . Let $\text{Homeo}_0(M, \mu)$ denote the group of measure preserving homeomorphisms of M which act by the identity homeomorphism on a neighborhood of ∂M and are isotopic to the identity. Let $P_n(M, z) = \pi_1(C_n(M), z)$ be the pure braid group on M . Here $C_n(M) \subset M^n$ denotes the space of ordered configurations of n points in M , equipped with the product $\text{Homeo}_0(M)$ action. If $\dim(M) = 2$ then we let $n \in \mathbf{N}$ be an arbitrary number, and when $\dim(M) \geq 3$, we assume $n = 1$ throughout the paper. That is, for higher dimensional manifolds we consider only their fundamental groups $P_1(M, z) = \pi_1(M, z)$.

Let $\gamma: \text{Homeo}_0(M) \times C_n(M) \rightarrow P_n(M)$ be a measurable cocycle defined by

$$\gamma(f, x) = [\ell_{z, f(x)} * \{f_t(x)\} * \ell_{x, z}],$$

where $\{f_t\}$ is an isotopy from the identity to f and $\ell_{x, y}$ is a certain path in $C_n(M)$ from x to y which is defined in Section 2. By convention, the concatenation of paths is read from right to left.

The main object of study in the present paper is a family of maps $\Psi: \text{Homeo}_0(M) \rightarrow \mathbf{R}$ defined by

$$(1.1) \quad \Psi(f) = \int_{C_n(M)} \varphi(\gamma(f, x)) dx,$$

where $\varphi: P_n(M) \rightarrow \mathbf{R}$ is a homogeneous quasimorphism vanishing on the centre of $P_n(M)$. On $C_n(M)$ we consider the product measure μ^n . The formula (1.1) was used by Gambaudo-Ghys to define non-trivial quasimorphisms on groups of area preserving diffeomorphisms of compact surfaces [22] and by Polterovich for volume preserving diffeomorphisms of compact manifolds of higher dimension [30]. It was further exploited by several authors in various configurations [7, 9, 11, 12, 31].

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In general, it is not well defined for homeomorphisms and we identify quasimorphisms $\varphi: P_n(M) \rightarrow \mathbf{R}$ for which it is. Here is our main result.

Theorem 1.1. *Let $\varphi: P_n(M) \rightarrow \mathbf{R}$ be a non-trivial homogeneous quasimorphism vanishing on the centre of $P_n(M)$. If M is a surface, we assume that M is closed of positive genus and φ vanishes on the subgroup $P_n(\Delta) \leq P_n(M)$ of braids supported in an interior of a full measure two-cell Δ , see Section 2 for definition. Then the map $\Psi: \text{Homeo}_0(M) \rightarrow \mathbf{R}$ given by (1.1) is well defined if and only if $\varphi: P_n(M) \rightarrow \mathbf{R}$ is as above, and moreover:*

(1) *The differential $\delta\Psi$ is a semi-bounded 2-cocycle, that is*

$$D(f) := \sup_g |\delta\Psi(f, g)| = \sup_g |\Psi(g) - \Psi(fg) + \Psi(f)| < \infty.$$

(2) *The restriction of Ψ to the subgroup of homeomorphisms preserving the measure μ is a quasimorphism whose homogenization is C^0 -continuous.*

(3) *If φ extends to an unbounded quasimorphism on the full braid group $B_n(M)$ then the homogeneous quasimorphism from the previous item is non-trivial.*

Remark 1.2. Let Σ be a closed surface of a positive genus and $n > 1$. Recall that $B_n(\Sigma)/Z(B_n(\Sigma))$, where $Z(B_n(\Sigma))$ is the center of $B_n(\Sigma)$, is a non-reducible subgroup of the n punctured mapping class group MCG_n [26, Corollary 7.13] and obviously $P_n(\Delta) \leq \text{MCG}_n$ is a reducible subgroup. It follows from Bestvina-Fujiwara construction [2, Theorem 12] that the space of quasimorphisms on $B_n(\Sigma)$ which vanish on $P_n(\Delta)$ is infinite-dimensional. Hence there are infinitely many linearly independent quasimorphisms $\varphi: B_n(\Sigma) \rightarrow \mathbf{R}$ which satisfy conditions of Theorem 1.1.

Let $q: \pi_1(\Sigma) \rightarrow \mathbf{R}$ be a quasimorphism. It defines a quasimorphism $\varphi: P_n(\Sigma) \rightarrow \mathbf{R}$ as follows. Let

$$i_*: P_n(\Sigma) \rightarrow (\pi_1(\Sigma))^n$$

be the homomorphism induced by the inclusion $i: C_n(\Sigma) \rightarrow (\Sigma)^n$. We define $\varphi = q \circ p_j \circ i_*$, where $p_j: (\pi_1(\Sigma))^n \rightarrow \pi_1(\Sigma)$ is the projection on the j -th factor. It is a well-known fact that the kernel of the homomorphism i_* equals to the normal closure H_n of $P_n(\Delta)$ in the group $P_n(\Sigma)$ [23, Theorem 1], see also [4]. Thus every such φ vanishes on H_n .

Let us consider the case when $\Sigma = T$ is a torus. In this case the group H_n equals to the commutator subgroup $[P_n(T), P_n(T)]$, see [4]. Hence there are infinitely many linearly independent quasimorphisms on $P_n(T)$, that vanish on $P_n(\Delta)$, which are obviously different from each such φ , since in this case φ is a homomorphism. Let us discuss in more details the case $n = 2$. The commutator subgroup $[P_2(T), P_2(T)] \cong \mathbf{F}_2 = \langle a, b \rangle$ and $P_2(\Delta) \cong \mathbf{Z}$ is generated by the commutator $[a, b]$. Let $\mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ be the subspace of quasimorphisms on \mathbf{F}_2 invariant under the action generated by inverting generators. Then every homogeneous quasimorphism $\phi \in \mathbf{Q}(\mathbf{F}_2; \mathbf{Z}/2 \times \mathbf{Z}/2)$ vanishes on $[a, b]$ and hence on $P_2(\Delta)$. Moreover, each such ϕ extends to $B_2(T)$, see [11, Proposition 2.8], hence by (3) in Theorem 1.1 Ψ is non-trivial. In particular, this gives an elementary construction of unbounded quasimorphisms on $\text{Homeo}_0(T, \mu)$. Note that in this case, one needs to take $n > 1$, since for $n = 1$ we have $B_1(T) = Z(B_1(T)) \cong \mathbf{Z}^2$.

Let us discuss the remaining case when Σ is a closed hyperbolic surface. We would like to point out that there are infinitely many linearly independent quasimorphisms on $P_n(\Sigma)$, that vanish on $P_n(\Delta)$, which are different from each φ described above. Note that, there are infinitely many linearly independent quasimorphisms on $P_n(\Sigma)$ that vanish on $P_n(\Delta)$, but do not vanish on H_n . Indeed, by [2, Theorem 12] it is enough to show that the group H_n is a non-reducible subgroup of MCG_n . This can be shown as follows: Let C be a multicurve (a simplex in the curve complex). If it intersects non-trivially $\Delta \setminus z$, there is an element in $P_n(\Delta)$ which does not preserve C . Otherwise, if C lies outside of Δ , then since $P_n(\Sigma)$ is non-reducible, there exists $\alpha \in P_n(\Sigma)$ so that $\alpha(C) \neq C$. Note that one can choose α such that $\alpha(C)$ intersects $\Delta \setminus z$ non-trivially. Hence there exists $\beta \in P_n(\Delta)$ such that $\beta\alpha(C) \neq \alpha(C)$. It follows that $\alpha^{-1}\beta\alpha \in H_n$ and $\alpha^{-1}\beta\alpha(C) \neq C$. Thus H_n is a non-reducible subgroup of MCG_n .

Remark 1.3. Semi-boundedness of 2-cocycles was introduced by Gal-Kędra [19] where it was used to prove undistortedness of symplectic diffeomorphisms for certain symplectic manifolds. It was further generalized to the concept of p -boundedness in [20], which also introduced p -bounded cohomology of groups. Functions with semi-bounded differentials were recently used in [14] to study the L^p -geometry of the space of contractible loops on surfaces.

C^0 -continuity. Let μ be the measure associated with a volume form on M . Given a homogeneous quasimorphism $\varphi: P_n(M) \rightarrow \mathbf{R}$ vanishing on the centre of $P_n(M)$, and in case M is a surface vanishing on $P_n(\Delta)$, the formula (1.1) well defines a quasimorphism $\Psi: \text{Diff}_0(M, \mu) \rightarrow \mathbf{R}$.

Corollary 1.4. *The homogenization $\bar{\Psi}: \text{Diff}_0(M, \mu) \rightarrow \mathbf{R}$ has a unique C^0 -continuous extension to $\text{Homeo}_0(M, \mu)$ which is a homogeneous quasimorphism.*

Entropy. Let Σ be a closed orientable surface of positive genus. For certain quasimorphisms $\varphi: B_n(\Sigma) \rightarrow \mathbf{R}$ the restriction of Ψ to $\text{Diff}_0(\Sigma, \mu)$ is bounded on the set Ent_0 of diffeomorphisms of zero entropy [12]. Moreover, these quasimorphisms come from Bestvina-Fujiwara construction and thus satisfy conditions of Theorem 1.1, see [12].

It follows from Corollary 1.4 that the extension of the homogenization of Ψ ,

$$\tilde{\Psi}: \text{Homeo}_0(\Sigma, \mu) \rightarrow \mathbf{R}$$

vanishes on the C^0 -closure of Ent_0 in $\text{Homeo}_0(\Sigma, \mu)$ and on all its conjugates.

Consider the set

$$S = \bigcup_{f \in \text{Homeo}_0(\Sigma, \mu)} f^{-1} \cdot \overline{\text{Ent}_0}^{\text{Homeo}_0} \cdot f,$$

which is the normal closure of the C^0 -closure in $\text{Homeo}_0(\Sigma, \mu)$ of the set of zero entropy measure preserving diffeomorphisms. Let $G_S \leq \text{Homeo}_0(\Sigma, \mu)$ be the subgroup generated by S . Note that G_S is a large subgroup. For example, by simplicity of the kernel of the Flux homomorphism on $\text{Diff}_0(\Sigma, \mu)$ [1], it is easily seen that $\text{Diff}_0(\Sigma, \mu) \leq G_S$, see e.g. [12]. On the other hand, by [16, Theorem 1.10], the commutator group $[\text{Homeo}_0(\Sigma, \mu), \text{Homeo}_0(\Sigma, \mu)]$ is simple. This implies that

$$[\text{Homeo}_0(\Sigma, \mu), \text{Homeo}_0(\Sigma, \mu)] \leq G_S.$$

Since $\tilde{\Psi}$ does not vanish on $\text{Diff}(\Sigma, \mu)$, we have the following consequence of Corollary 1.4.

Corollary 1.5. *The diameter of the bi-invariant metric on G_S associated with the generating set S is infinite.* \square

This corollary says that the set S is *small*, in the sense that for any positive integer $m \in \mathbf{N}$ there are elements of $\text{Homeo}_0(\Sigma, \mu)$ which cannot be presented as product of up to m elements from S . It follows from [33] that S is contained in a closed subset with empty interior, so it is topologically small. The above corollary says that it is also algebraically small. Since entropy is not C^0 -continuous, the relation between S and the set of measure preserving homeomorphisms of zero entropy is unclear.

It is also unclear how large the group G_S really is. Indeed, it contains all the elements which are currently known not to lie in $[\text{Homeo}_0(\Sigma, \mu), \text{Homeo}_0(\Sigma, \mu)]$. However, the abelianization $\text{Homeo}_0(\Sigma, \mu)/[\text{Homeo}_0(\Sigma, \mu), \text{Homeo}_0(\Sigma, \mu)]$ is not yet completely determined and it is unclear what subgroup G_S defines in it.

Question 1. *Is it true that $G_S = \text{Homeo}_0(\Sigma, \mu)$?*

Distortion. Recall that an element g of a finitely generated group Γ is undistorted if

$$\lim_{k \rightarrow \infty} \frac{|g^k|}{k} > 0,$$

where $|g|$ denotes the word norm with respect to any finite generating set of Γ . An element g of an arbitrary group G is undistorted if it is undistorted in every finitely generated subgroup $\Gamma \leq G$ containing it. Understanding distortion in groups of homeomorphisms is motivated by the topological version of the Zimmer conjecture.

Existence of functions on a group G whose differential is semi-bounded can be used to prove undistortedness of elements of G . The following observation almost immediately follows from [20, Proposition 4.2].

Theorem 1.6. *Let $\psi: G \rightarrow \mathbf{R}$ be a function such that $\delta\psi$ is semi-bounded. If*

$$\limsup_{k \rightarrow \infty} \frac{|\psi(f^k)|}{k} > 0$$

then f is undistorted.

Proof. It follows from Lemma 2.1 that $|f|_\psi = \sup_g |\psi(g) - \psi(fg)|$ is a pseudo-norm with respect to which ψ is Lipschitz which implies that

$$(1.2) \quad \limsup_{k \rightarrow \infty} \frac{|f^k|_\psi}{k} > 0.$$

Let $\Gamma \subseteq G$ be a finitely generated subgroup containing f and let $C = \max\{|s|_\psi \mid s \in S\}$, where $S \subseteq \Gamma$ is a finite generating set. Then

$$|f|_\psi = |s_1 \dots s_m|_\psi \leq |s_1|_\psi + \dots + |s_m|_\psi \leq C|f|_S$$

which, together with (1.2) implies that f is undistorted in Γ and hence in G . \square

Corollary 1.7. *Let $f \in \text{Homeo}_0(M)$ and let Ψ be defined by (1.1). If*

$$\limsup_{k \rightarrow \infty} \frac{|\Psi(f^k)|}{k} > 0$$

then f is undistorted in $\text{Homeo}_0(M)$. □

Verifying the hypothesis of the above corollary may not be easy in a concrete case. However, we tackle this problem by redefining Ψ in Theorem 1.1 as follows. Let point $z \in C_n(M)$ be a base-point and let $\Psi_z: \text{Homeo}_0(M) \rightarrow \mathbf{R}$ be given by

$$\Psi_z(f) = \varphi(\gamma(f, z)),$$

where $\varphi: P_n(M) \rightarrow \mathbf{R}$ is a homogeneous quasimorphism vanishing on $P_n(\Delta)$ and on the centre of $P_n(M)$. We show in Section 4 that the cocycle $\delta\Psi_z$ is semi-bounded (Proposition 4.1).

Corollary 1.8. *Let $f \in \text{Homeo}_0(M)$. If $z \in M$ is a fixed point of f such that $\varphi(\gamma(f, z)) > 0$ then f is undistorted in $\text{Homeo}_0(M)$.*

Proof. Since z is a fixed point of f , we get that $\gamma(f^k, z) = \gamma(f, z)^k$. It follows that

$$\Psi_z(f^k) = \varphi(\gamma(f^k, z)) = k\Psi_z(f) > 0$$

and the statement follows from Theorem 1.6. □

Example 1.9. Let f be a homeomorphism of a closed oriented hyperbolic manifold. Suppose $z \in M$ is a fixed point of f such that

$$\gamma(f, z) \in \pi_1(M, z)$$

is non-trivial. Then f is undistorted. Indeed, in this case there exists a homogeneous quasimorphism non-vanishing on $\gamma(f, z)$ (see e.g. [8]) and the statement follows from Corollary 1.7. ◇

1-bounded cohomology. Since Ψ is semi-bounded it is natural to ask whether it represents a non-trivial class in 1-bounded cohomology. The next corollary states that this is indeed the case.

Corollary 1.10. *Let M be a closed orientable surface of positive genus, or a higher dimensional manifold whose fundamental group modulo its centre admits infinitely many linearly independent homogeneous quasimorphisms. Then*

$$\dim H_{(1)}^2(\text{Homeo}_0(M), \mathbf{R}) = \infty.$$

Proof. The hypothesis implies that the construction (1.1) yields infinitely many linearly independent quasimorphisms on $\text{Diff}_0(M, \text{vol})$, according to [7, 11]. In particular $H_b^2(\text{Diff}_0(M, \text{vol}); \mathbf{R})$ is infinite dimensional. Since the comparison map $H_b^2(G; \mathbf{R}) \rightarrow H_{(1)}^2(G; \mathbf{R})$ is injective for any group [20], we obtain that the image of homomorphism $H_{(1)}^2(\text{Homeo}_0(M), \mathbf{R}) \rightarrow H_{(1)}^2(\text{Diff}_0(M, \text{vol}); \mathbf{R})$ induced by the inclusion, contains an infinite dimensional part of the image of the comparison map $H_b^2(\text{Diff}_0(M, \text{vol}); \mathbf{R}) \rightarrow H_{(1)}^2(\text{Diff}_0(M, \text{vol}); \mathbf{R})$, which proves the statement. □

Remark 1.11. The above statement for surfaces of positive genus also follows from the result of Bowden, Hensel and Webb [5, Theorem 1.4] who showed that $H_b^2(\text{Homeo}_0(\Sigma), \mathbf{R})$ is infinite dimensional. Since the comparison homomorphism $H_b^*(G, \mathbf{R}) \rightarrow H_{(1)}^*(G, \mathbf{R})$ is injective, their result implies the above. However, we conjecture that for surfaces our 1-bounded classes are not represented by bounded cocycles. In other words, they don't come from quasimorphisms.

Standard cohomology. The formula (1.1) always defines a morphism between bounded cochains

$$C_b^*(P_n(\Sigma), \mathbf{R}) \rightarrow C_b^*(\text{Homeo}_0(\Sigma, \mu), \mathbf{R}),$$

which, according to [13], descends to a homomorphism

$$\mathcal{G}: H_b^*(P_n(\Sigma), \mathbf{R}) \rightarrow H_b^*(\text{Homeo}_0(\Sigma, \mu), \mathbf{R}).$$

We thus have a commutative diagram

$$\begin{array}{ccccc} H_b^2(P_n(\Sigma); \mathbf{R}) & \xrightarrow{\mathcal{G}} & H_b^2(\text{Homeo}_0(\Sigma, \mu), \mathbf{R}) & \xrightarrow{c} & H^2(\text{Homeo}_0(\Sigma, \mu), \mathbf{R}) \\ \delta \uparrow & & \delta \uparrow & \nearrow 0 & \\ Q(P_n(\Sigma)) & \dashrightarrow & Q(\text{Homeo}_0(\Sigma, \mu)) & & \end{array}$$

where c is the comparison homomorphism, the dashed arrow is not defined, and Q denotes the space of quasimorphisms on a group in question. However, when homeomorphisms are replaced by diffeomorphisms this arrow is well defined. In particular, for any quasimorphism φ the class

$$c(\mathcal{G}[\delta\varphi]) = 0$$

in $H^2(\text{Diff}_0(\Sigma, \mu))$. For homeomorphism we get classes $\mathcal{G}[\delta\varphi]$ which are potentially nontrivial in $H^2(\text{Homeo}_0(\Sigma, \mu), \mathbf{R})$.

Question 2. *Does there exist $\varphi: P_n(\Sigma) \rightarrow \mathbf{R}$ such that the class $c(\mathcal{G}[\delta\varphi])$ is non-trivial in the second cohomology group $H^2(\text{Homeo}_0(\Sigma, \mu), \mathbf{R})$?*

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2. PRELIMINARIES

Quasimorphisms. Let G be a group. A function $\varphi: G \rightarrow \mathbf{R}$ is called a quasimorphism if

$$D_\varphi := \sup_{g,h} |\delta\varphi(g,h)| = \sup_{g,h} |\varphi(h) - \varphi(gh) + \varphi(g)| < \infty.$$

In other words, $\delta\varphi$ is a bounded 2-cocycle on G with values in the trivial module \mathbf{R} . The constant D_φ is called the defect of φ . A function $\psi: G \rightarrow \mathbf{R}$ is called homogeneous if

$$\psi(g^n) = n\psi(g)$$

for every $g \in G$ and every $n \in \mathbf{Z}$. A quasimorphism φ can be homogenized by defining

$$\bar{\varphi}(g) = \lim_{n \rightarrow \infty} \frac{\varphi(g^n)}{n}.$$

The above limit exists due to Fekete's Lemma and, moreover, $\sup_g |\bar{\varphi}(g) - \varphi(g)| \leq D_\varphi$. A homogeneous quasimorphism is constant on conjugacy classes. The differential $\delta\varphi$ of a quasimorphism represents a class of the second bounded cohomology $H_b^2(G, \mathbf{R})$. If the homogenization of φ is not a homomorphism then this class is non-trivial. Proofs of the above facts are standard and can be found for example in [15].

Semi-bounded cohomology. A k -cochain $c: G^k \rightarrow \mathbf{R}$ is called p -bounded, where $p \leq k$, if for fixed $g_1, \dots, g_{k-p} \in G$ the function

$$(g_{k-p+1}, \dots, g_k) \mapsto c(g_1, \dots, g_k)$$

is bounded. It follows that p -bounded cochains form subcomplexes and their homology, denoted by $H_{(p)}^*(G, \mathbf{R})$, is called p -bounded (real) cohomology of G . In particular, k -bounded cohomology is the bounded cohomology $H_{(k)}^k(G, \mathbf{R}) = H_b^k(G, \mathbf{R})$ and 0-bounded k -th cohomology is the ordinary one: $H_{(0)}^k(G, \mathbf{R}) = H^k(G, \mathbf{R})$. Since $(p+1)$ -bounded cochains are p -bounded, there are comparison homomorphisms induced by inclusion of complexes

$$H_b^k(G, \mathbf{R}) \rightarrow H_{(p+1)}^k(G, \mathbf{R}) \rightarrow H_{(p)}^k(G, \mathbf{R}) \rightarrow H^k(G, \mathbf{R}).$$

We refer to [20] for a general discussion and here we only discuss 1-bounded 2-coboundaries. So, let $\psi: G \rightarrow \mathbf{R}$ be a function such that its coboundary is 1-bounded:

$$D(f) := \sup_g |\psi(g) - \psi(fg) + \psi(f)| < \infty.$$

Lemma 2.1. *Let $\delta\psi$ be a 1-bounded cocycle. The formula*

$$|f|_\psi = \sup_g |\psi(g) - \psi(fg)|$$

defines a pseudo-norm on G . Moreover, ψ is 1-Lipschitz with respect to the pseudo-metric defined by $d_\psi(f, g) = |fg^{-1}|_\psi$.

Proof. Both the symmetry and triangle inequality is straightforward:

$$|f^{-1}|_\psi = \sup_g |\psi(g) - \psi(f^{-1}g)| = \sup_g |\psi(f^{-1}g) - \psi(f \cdot f^{-1}g)| = |f|_\psi.$$

$$\begin{aligned}
|fg|_\psi &= \sup_h |\psi(h) - \psi(fgh)| \\
&\leq \sup_h |\psi(h) - \psi(gh)| + \sup_h |\psi(gh) - \psi(fgh)| = |f|_\psi + |g|_\psi.
\end{aligned}$$

As well as the Lipschitz property:

$$|\psi(f) - \psi(g)| = |\psi(g) - \psi(fg^{-1} \cdot g)| \leq |fg^{-1}|_\psi = d_\psi(f, g),$$

and the proof follows. \square

The main construction. Let d be an auxiliary Riemannian metric on M . It induces the supremum metric d_0 on $\text{Homeo}_0(M)$ by

$$d_0(f, g) := \sup_{x \in M} d(f(x), g(x)).$$

Since M is a compact connected smooth manifold it has a CW-decomposition with one top dimensional cell. Let $\Delta \subseteq M$ be the interior of that cell which is diffeomorphic to a ball in \mathbf{R}^n and it is of full measure in M . Choose a fundamental domain in the universal cover of M so that the lift of the inclusion $\Delta \subseteq M$ is contained in it. Let $z = (z_1, \dots, z_n) \in C_n(M)$ be the base-point, such that $z_i \in \Delta$ for each $i = 1, \dots, n$. Given two points $x, y \in C_n(\Delta)$, let $\ell_{y,x}$ be a path from x to y consisting of n unit speed geodesics from x_i to y_i with respect to the Euclidean metric on Δ . This Euclidean metric is used to define geodesics $\ell_{y,x}$ and has nothing to do with the auxiliary metric chosen in the beginning of this section.

In case M is a surface and $n > 1$, then the geodesics between x_i and y_i may collide, and then $\ell_{x,y}$ is not defined. However, this happens only for a measure zero set, thus does not affect the integral from the definition of Ψ . For more details see [21, Section 3.2]. By convention, we read concatenation of paths from right to left. That is, $\ell_{z,y} * \ell_{y,x}$ is a path from x to y and then to z .

Let $f \in \text{Homeo}_0(M)$ and let $x \in C_n(\Delta)$ be a point such that $f(x) \in C_n(\Delta)$. Let a braid $\gamma(f, x)$ in $P_n(M, z)$ be represented by

$$(2.1) \quad \gamma(f, x) = [\ell_{z, f(x)} * \{f_t(x)\} * \ell_{x,z}],$$

where $\{f_t\}$ is an isotopy from the identity to f .

The above braids are defined for points $x \in M$ from a set of full measure.

If $P_n(M)$ has no centre then the braid $\gamma(f, x)$ does not depend on the choice of an isotopy $\{f_t\}_{t=0}^1$ (see, e.g., [10, Section 2.2 (6),(7)]). Otherwise, it does depend and that is why we assume that the homogeneous quasimorphism φ from Theorem 1.1 vanishes on the centre and we need the following observation.

Lemma 2.2. *Let $\varphi: G \rightarrow \mathbf{R}$ be a homogeneous quasimorphism vanishing on a normal subgroup $H \leq G$. Then*

$$\varphi(gh) = \varphi(g)$$

for all $g \in G$ and $h \in H$.

Proof. Observe that for any two elements $g, h \in G$ we have $(gh)^k = g^k h^k c_1 \dots c_k$, where c_i are conjugates of commutators of h [15, Section 2.2.4]. It follows that

$$\begin{aligned} |\varphi(gh) - \varphi(g)| &= \frac{1}{k} |\varphi((gh)^k) - \varphi(g^k)| \\ &= \frac{1}{k} |\varphi(g^k h^k c_1 \dots c_k) - \varphi(g^k)| \\ &\leq \frac{1}{k} (|\varphi(g^k) + \varphi(h^k c_1 \dots c_k) - \varphi(g^k)| + D_\varphi) = \frac{D_\varphi}{k}. \end{aligned}$$

The last equality follows since H is normal and hence $h^k c_1 \dots c_k \in H$. Since $k \in \mathbf{N}$ is arbitrary the computation proves the statement. \square

Let $f, g \in \text{Homeo}_0(M)$ and let $x \in C_n(\Delta)$ be such that $g(x), fg(x) \in C_n(\Delta)$. Choosing the isotopy $\{(fg)_t\}$ to be the concatenation $\{f_t \circ g\} * \{g_t\}$ proves the following crucial fact.

Lemma 2.3. *The function $\gamma: \text{Homeo}_0(M) \times C_n(\Delta) \rightarrow P_n(M)$ is a cocycle. That is,*

$$\gamma(fg, x) = \gamma(f, g(x))\gamma(g, x)$$

for every $f, g \in \text{Homeo}_0(M)$ and almost every $x \in C_n(\Delta)$. \square

3. PROOFS

Proof of Theorem 1.1. If $n = 1$, it is known that $\varphi(\gamma(f, x))$ is bounded as a function of x , thus the integral (1.1) is well defined [10]. Assume therefore that $M = \Sigma$ is a surface and n is arbitrary. Let $P_n(\Delta) = \pi_1(C_n(\Delta))$. The inclusion $\Delta \subseteq \Sigma$ induces an inclusion $P_n(\Delta) \leq P_n(\Sigma)$, see [23, Theorem 1]. Let $f \in \text{Homeo}_0(\Sigma)$ and let $\varphi: P_n(\Sigma) \rightarrow \mathbf{R}$ be a quasimorphism which vanishes on $P_n(\Delta)$ and on the centre of $P_n(\Sigma)$. Let $\text{sys}(\Sigma, d)$ denote the systole of Σ . Suppose that f is C^0 -small, in the sense, that $d_0(f, \text{Id}) < \epsilon$ for ϵ chosen such that there exists an isotopy f_t , connecting the identity to f , such that

$$d_0(f_t, \text{Id}) < \frac{1}{2n} \text{sys}(\Sigma, d)$$

for all t . Existence of such an ϵ follows from local contractibility of $\text{Homeo}_0(\Sigma)$ [17, Corollary 1.1]. For every x , we shall construct a decomposition $\gamma(f, x) = \alpha \cdot \beta$, where $\beta \in P_n(\Delta)$ and $\alpha \in P_n(\Sigma)$ is a braid from a certain fixed finite set.

Let $x = (x_1, \dots, x_n)$. Assume first, that the evaluation $\{f_t(x_i)\}$ of an isotopy $\{f_t\}$ is contained in Δ for all $i = 1, \dots, n$. Then $\gamma(f, x) \in P_n(\Delta)$. In this case $\beta = \gamma(f, x)$ and $\alpha = e$. A simplified picture is presented in Figure 3.1. The points z, x and $f(x)$ should be seen as n -tuples, the path connecting z and x (blue) denotes the union of geodesics connecting z_i and x_i , the path connecting $f(x)$ and z (brown) denotes the union of geodesics connecting $f(x_i)$ and z_i , and the path between x and $f(x)$ (green) denotes the union of images of the isotopy $\{f_t\}$ evaluated at x_i 's which is possibly braided.

In general, since f is C^0 -small, we can choose an isotopy f_t , such that for every i the path $\{f_t(x_i)\}$ is contained in a ball B_i centered in x_i , and of radius smaller than $\frac{1}{2n} \text{sys}(\Sigma, d)$. Recall that

$$(3.1) \quad \gamma(f, x) = [\ell_{z, f(x)} * \{f_t(x)\} * \ell_{x, z}],$$

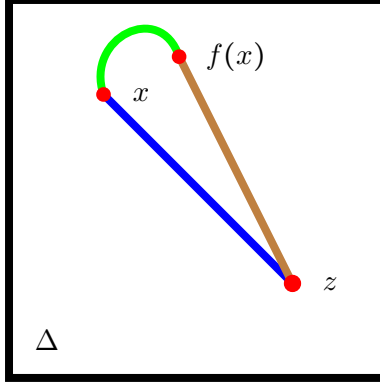


FIGURE 3.1. Isotopy inside Δ .

see Figure 3.2. Our goal is to move $\{f_t(x)\}$ (the green part) inside $C_n(\Delta)$ and modify $\ell_{z,f(x)}$ (the brown part) in a controlled way. To this end, for each i we choose a point $y_i \in \Delta \cap B_i$ such that the unit speed geodesic segment $\sigma_{y_i, f(x_i)}$ connecting $f(x_i)$ to y_i lies entirely in B_i (in particular, we may take $y_i = x_i$). Note that some of these geodesic segments might intersect in the configuration space, but it happens only for measure zero set of points $x \in C_n(\Delta)$. Let $y = \{y_1, \dots, y_n\}$, and let $\ell_{y,f(x)}$ denote the path in $C_n(\Sigma)$ consisting of segments $\sigma_{y_i, f(x_i)}$. We have

$$(3.2) \quad \gamma(f, x) = [\ell_{z,f(x)} * \ell_{f(x),y} * \ell_{y,f(x)} * \{f_t(x)\} * \ell_{x,z}],$$

where $\ell_{f(x),y}$ denotes the time-reverse of $\ell_{y,f(x)}$.

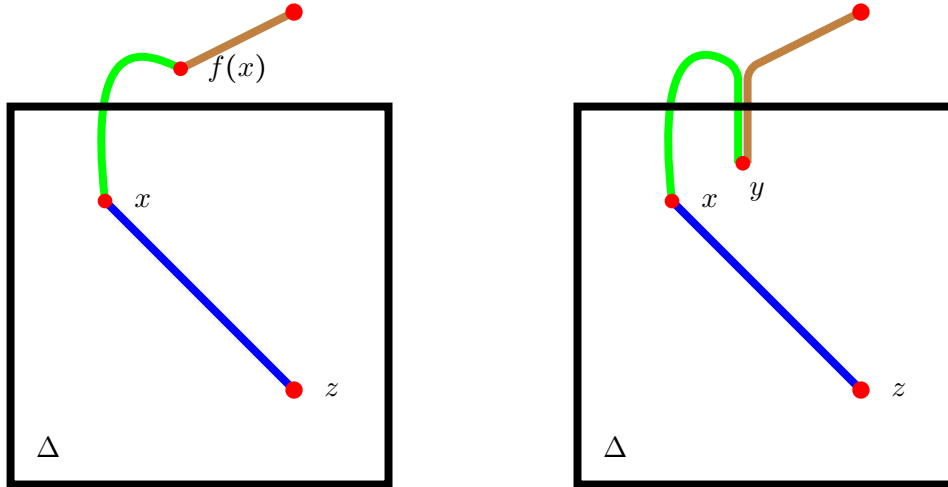


FIGURE 3.2. Back-and-forth.

The path $\ell_{y,f(x)} * \{f_t(x)\}$, in green in Figure 3.2, is contained in $C_n(\cup B_i)$. Moreover, since each B_i has radius less than $\frac{1}{2n} \text{sys}(\Sigma, d)$, every connected component of $\cup B_i$ has diameter less than $\text{sys}(\Sigma)$ and is therefore contractible in Σ . Consequently, we can homotope $\ell_{y,f(x)} * \{f_t(x)\}$ inside $C_n(\Delta)$,

keeping the endpoints fixed:

$$(3.3) \quad \gamma(f, x) = [\ell_{z,f(x)} * \ell_{f(x),y} * s * \ell_{x,z}],$$

where s denotes $\ell_{y,f(x)} * \{f_t(x)\}$ after being pushed into $C_n(\Delta)$, see Figure 3.3. The final step is to connect the endpoint of s to z , and return back. We have:

$$(3.4) \quad \gamma(f, x) = [\ell_{z,f(x)} * \ell_{f(x),y} * \ell_{y,z} * \ell_{z,y} * s * \ell_{x,z}].$$

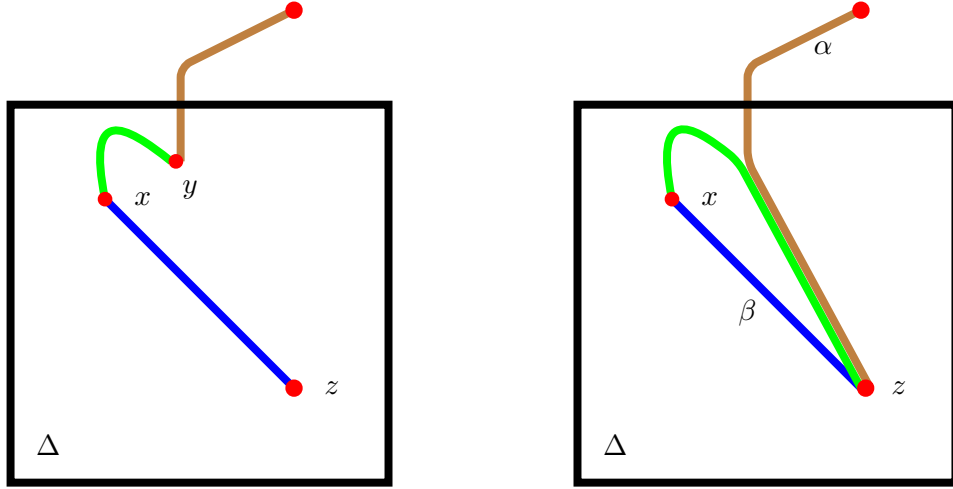


FIGURE 3.3. Push into Δ and back-and-forth again.

We define β to be the paths $\ell_{z,y} * s * \ell_{x,z}$ obtained by the concatenation of the blue and green paths, and the brown path $\ell_{z,f(x)} * \ell_{f(x),y} * \ell_{y,z}$ comprise the braid α . By Lemma 3.1 proven below, φ is bounded on all such braids α .

Thus, since the quasimorphism φ vanishes on $P_n(\Delta)$, we obtain

$$(3.5) \quad |\varphi(\gamma(f, x))| = |\varphi(\beta\alpha)| \leq |\varphi(\alpha)| + D_\varphi,$$

and hence the function $x \mapsto |\varphi(\gamma(f, x))|$ is bounded, and so the integral

$$\Psi(f) = \int_{C_n(\Sigma)} \varphi(\gamma(f, x)) dx$$

is well defined, i.e. $\varphi(\gamma(f, x))$ is a L^1 -function, for a C^0 -small f .

Let $f \in \text{Homeo}_0(\Sigma)$ be an arbitrary homeomorphism and let $\{f_t\}$ be an isotopy from the identity to f , where $t \in [0, 1]$. Let $0 = t_0 < t_1 < t_2 < \dots < t_m = 1$ be a partition of the interval $[0, 1]$ such that

$$d_0(f_{t_i}^{-1} f_{t_{i+1}}, \text{Id}) < \epsilon,$$

where ϵ is chosen as in the beginning of the proof, i.e., $f_{t_i}^{-1} f_{t_{i+1}}$ are C^0 -small for all i . Of course,

$$f = f_{t_0} \circ f_{t_0}^{-1} f_{t_1} \circ f_{t_1}^{-1} f_{t_2} \circ \dots \circ f_{t_{m-1}}^{-1} f_{t_m} = g_1 \dots g_m,$$

where $g_i = f_{t_{i-1}}^{-1} f_{t_i}$. That is, f is a product of C^0 -small homeomorphisms. We get

$$\begin{aligned}\varphi(\gamma(f, x)) &= \varphi(\gamma(g_1 \dots g_m, x)) \\ &= \varphi(\gamma(g_1, g_2 \dots g_m x) \gamma(g_2, g_3 \dots g_m x) \dots \gamma(g_m, x)) \\ &= \varphi(\gamma(g_1, g_2 \dots g_m x)) + \varphi(\gamma(g_2, g_3 \dots g_m x)) + \dots + \varphi(\gamma(g_m, x)) + d_1 + \dots + d_{m-1},\end{aligned}$$

where $|d_i| \leq D_\varphi$. This implies that $\varphi(\gamma(f, x))$ attains finitely many values for almost every $x \in \Delta$. As before, it follows that $\Psi(f)$ is well defined.

Conversely, assume that the integral (1.1) is well defined for all $f \in \text{Homeo}_0(\Sigma)$. Suppose that $\varphi: P_n(\Sigma) \rightarrow \mathbf{R}$ does not vanish on a braid $\beta \in P_n(\Delta)$. Let $f_1 \in \text{Homeo}_0(\Sigma)$ be a homeomorphism supported in a ball $B_1 \subset \Delta$ of area

$$0 < a < \frac{1}{10} \mu(\Delta)$$

such that

$$\gamma(f, x) = \beta^{10}$$

for $x \in B_1$ in the set of area $b > 0$. Let f_2 be a homeomorphism supported in a ball $B_2 \subset \Delta$ disjoint from B_1 and of area $a/2$ such that

$$\gamma(f, x) = \beta^{10^2}$$

for $x \in B_2$ in the set of area $b/2$. By iterating this construction and taking the limiting homeomorphism $f = f_1 \circ f_2 \circ f_3 \dots$ we get that the integral (1.1) diverges which contradicts our hypothesis. This shows the main statement of Theorem 1.1. We now prove the additional statements.

(1) Let $A(f) = \max\{\varphi(\gamma(f, x)) \mid x \in C_n(M)\}$. It is a well defined number because $\varphi(\gamma(f, x))$ is bounded as explained in the first part of the proof. The following computations shows that $\delta\Psi$ is semi-bounded.

$$\begin{aligned}|\delta\Psi(f, g)| &= |\Psi(g) - \Psi(fg) + \Psi(f)| \\ &\leq \int_{C_n(M)} |\varphi(\gamma(g, x)) - \varphi(\gamma(fg, x)) + \varphi(\gamma(f, x))| dx \\ &\leq \int_{C_n(M)} |\varphi(\gamma(g, x)) - \varphi(\gamma(f, gx) \gamma(g, x)) + \varphi(\gamma(f, x))| dx \\ &\leq \int_{C_n(M)} (|\varphi(\gamma(g, x)) - \varphi(\gamma(f, gx)) - \varphi(\gamma(g, x)) + \varphi(\gamma(f, x))| + D_\varphi) dx \\ &\leq \int_{C_n(M)} (|\varphi(\gamma(f, gx)) - \varphi(\gamma(f, x))| + D_\varphi) dx \\ &\leq \int_{C_n(M)} (2A(f) + D_\varphi) dx < \infty.\end{aligned}$$

(2) The quasimorphism property for the restriction of Ψ to area preserving homeomorphism is a standard computation as in [6]. The C^0 -continuity of the homogenization $\bar{\Psi}$ follows from a theorem of Shtern [32, Theorem 1], which states that a homogeneous quasimorphism on a topological group

is continuous if and only if it is bounded on a neighborhood of the identity. Indeed, if f is C^0 -small then

$$|\varphi(\gamma(f, x))| = |\varphi(\beta\alpha)| \leq |\varphi(\alpha)| + D_\varphi,$$

which is bounded independently of f . Hence $\Psi(f)$ is bounded on a neighborhood of the identity. Since

$$\sup_f |\overline{\Psi}(f) - \Psi(f)| \leq D_\Psi,$$

we get that $\overline{\Psi}$ is also bounded on a neighborhood of the identity and its continuity follows from Shtern's theorem.

(3) Assume $n = 1$ and let γ be a braid such that $\overline{\varphi}(\gamma) \neq 0$. Let f be a point pushing homeomorphism f along the loop γ such that $\gamma(f, x) \in \{\gamma, e\}$, where e is the trivial loop, for all x away from a set of arbitrary small measure. This gives us $\overline{\Psi}(f) \neq 0$ (see, e.g., [12, Theorem 2.5]). If $n > 1$ and φ extends to the full braid group $B_n(\Sigma)$ then the unboundedness of Ψ is proven in [7] extending an argument of Ishida [25]. \square

We finish this section with the technical lemma used in the proof of Theorem 1.1.

Lemma 3.1. *Let Σ be a closed surface of positive genus and let φ be a quasimorphism vanishing on the centre of $P_n(\Sigma)$. Fix an integer $N \geq 1$. Then φ is bounded on the set of braids $\alpha \in P_n(\Sigma)$ whose every strand is a concatenation of at most N geodesic segments, each one of length at most $\text{diam}(\Sigma)$.*

Proof. Let $Z = \{z_1, \dots, z_n\}$ and put $S = \Sigma \setminus Z$. Consider the point-pushing homomorphism

$$h: P_n(\Sigma) \longrightarrow \text{PMod}(S),$$

where $\text{PMod}(S)$ is the pure mapping class group of S . By Birman exact sequence (see, e.g., [3] and [18]), the kernel of h is the centre of $P_n(\Sigma)$. Since φ vanishes on the centre, it is enough to show that the set of point-pushing homeomorphisms $h(\alpha)$, where α is as in the lemma, is finite.

Choose a finite system $\mathcal{A} = \{a_1, \dots, a_m\}$ of essential simple closed curves and properly embedded arcs in S , with endpoints at the punctures allowed, such that every component of $S \setminus \mathcal{A}$ is a disc and \mathcal{A} satisfies the assumptions of [18, Proposition 2.8]. It means that by definition \mathcal{A} fills S . By the Alexander method, an element h of $\text{PMod}(S)$ is determined, up to finite ambiguity depending only on \mathcal{A} , by the isotopy classes of the curves and arcs $h(a_1), \dots, h(a_m)$. We show that, for the point-pushing classes $h_\alpha = h(\alpha)$ arising from the braids α , each $h_\alpha(a_j)$ belongs to a finite set of isotopy classes. Fix $a_j \in \mathcal{A}$. The image $h_\alpha(a_j)$ is obtained from a_j by pushing the punctures along the strands of α . An example of $h_\alpha(a_j)$ is shown in Figure 3.4. Away from a small regular neighborhood of the strands the arc or curve is unchanged. Inside this neighborhood, every change is caused by one of the finitely many local events that corresponds to crossings of strands with a_j and crossings between strands. Since the strands of α are sums of at most N short geodesic segments, there exists a bound on the number of such crossings, and it does not depend on α . Thus there is a constant C , such that $i(h_\alpha(a_j), \mathcal{A}) \leq C$ for every j and for every braid α as in the lemma.

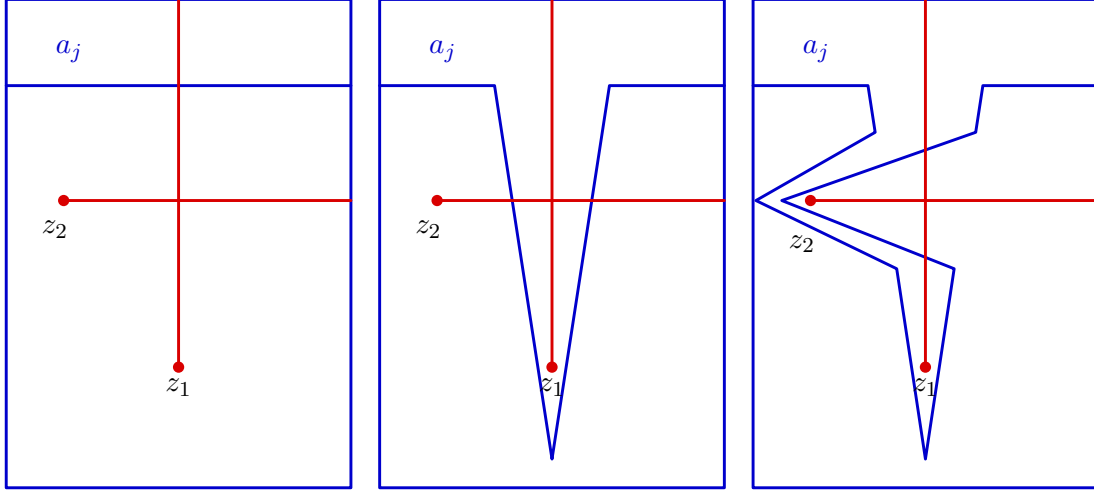


FIGURE 3.4. An example of $h_\alpha(a_j)$. The two strands have one crossing and the strand that ends at z_1 crosses first.

Finally, since \mathcal{A} fills S , there are only finitely many isotopy classes of essential simple closed curves and essential arcs whose geometric intersection number with \mathcal{A} is at most C . To see this, cut S along \mathcal{A} . A curve or arc in minimal position with \mathcal{A} is then encoded by a finite word recording the sequence of sides of the complementary discs which it crosses; the length of this word is bounded by C , and the alphabet is finite. Thus each $h_\alpha(a_j)$ has only finitely many possible isotopy classes. It follows that the tuple

$$(h_\alpha(a_1), \dots, h_\alpha(a_m))$$

has only finitely many possible values. By the Alexander method, only finitely many mapping classes $h_\alpha \in \text{PMod}(S)$ can arise. Since φ vanishes on the centre, the set of possible values of $\varphi(\alpha)$ is finite. \square

Remark 3.2. Note that for a closed hyperbolic surface Σ there are finitely many braids in $P_n(\Sigma)$ whose every strand is a concatenation of at most N geodesic segments, each one of length at most $\text{diam}(\Sigma)$. Indeed, if Σ is a closed hyperbolic surface, then $P_n(\Sigma)$ has no centre, the push-point homomorphism is injective, and the claim follows from Lemma 3.1.

4. DISTORTION

Let $z \in C_n(M)$ be the base-point and let $\Psi_z: \text{Homeo}_0(M) \rightarrow \mathbf{R}$ be the composition

$$\Psi_z(f) = \varphi(\gamma(f, z)),$$

where $\varphi: P_n(M) \rightarrow \mathbf{R}$ is a homogeneous quasimorphism vanishing on the centre of $P_n(M)$ (recall that if the dimension of M is greater than 2 we assume $n = 1$). If M is a surface we additionally assume that φ vanishes on $P_n(\Delta)$. Recall that under this assumptions the function $x \mapsto |\varphi(\gamma(f, x))|$ is bounded, according to (3.5).

Proposition 4.1. *The differential $\delta\Psi_z$ is a semi-bounded cocycle.*

Proof. Let $B_f = \sup_x |\varphi(\gamma(f, x))|$. For $f, g \in \text{Homeo}_0(M)$ we have

$$\begin{aligned} |\delta\Psi_z(f, g)| &= |\varphi(\gamma(g, z)) - \varphi(\gamma(fg, z)) + \varphi(\gamma(f, z))| \\ &\leq |\varphi(\gamma(g, z)) - \varphi(\gamma(f, g(z))) - \varphi(\gamma(g, z)) + \varphi(\gamma(f, z))| + D_\varphi \\ &= |\varphi(\gamma(f, g(z))) - \varphi(\gamma(f, z))| + D_\varphi \\ &= \begin{cases} |\varphi(\gamma(f, g(z))) - \varphi(\gamma(f, z))| + D_\varphi & \text{if } g(z) \neq z \\ D_\varphi & \text{if } g(z) = z \end{cases} \\ &\leq 2B_f + D_\varphi. \end{aligned}$$

This shows that $\delta\Psi$ is 1-bounded. □

As explained in the introduction, if $\limsup_{k \rightarrow \infty} \frac{|\Psi_z(f^k)|}{k} > 0$ then f is undistorted in $\text{Homeo}_0(M)$, according to Theorem 1.6. The following proposition is a direct application and its proof is straightforward.

Proposition 4.2. *Let $f \in \text{Homeo}_0(M)$ and $x \in M$ be such that*

$$(4.1) \quad \gamma(f^{n_k}, x) = \gamma(f, x)^{n_k}$$

for an increasing sequence $\{n_k\}_{k \in \mathbf{N}}$ of positive integers. If φ is a homogeneous quasimorphism on $P_n(M)$ satisfying the condition of Theorem 1.1 and $\varphi(\gamma(f, x)) > 0$ then f is undistorted in $\text{Homeo}_0(M)$. □

Example 4.3. The hypothesis (4.1) of the above proposition is immediate if $x \in M$ is a fixed point of f . If not then there exists a homeomorphism $s: M \rightarrow M$ and its isotopy $\{s_t\}$ to the identity such that $t \mapsto s_t(f(x))$ is equal to $\ell_{x, f(x)}$. It follows that x is a fixed point of sf and

$$\gamma(sf, x) = \gamma(f, x).$$

Moreover,

$$\gamma((sf)^k, x) = \gamma(sf, x)^k = \gamma(f, x)^k.$$

If $\varphi(\gamma(f, x)) > 0$ then sf is undistorted, according to Proposition 4.2. ◇

The above example shows that the dynamics of distorted homeomorphisms is often very restricted, see also [29]. Let us discuss a family of examples. We say that a group G is quasi-residually real if for every non-trivial element $g \in G$ there exists a homogeneous quasimorphism $\varphi: G \rightarrow \mathbf{R}$ such that $\varphi(g) \neq 0$. Examples of such groups include right-angled Artin groups, pure braid groups and many hyperbolic groups [8].

Corollary 4.4. *Let M be a closed manifold with quasi-residually real fundamental group $\pi_1(M)$ with trivial centre. Let $f \in \text{Homeo}_0(M)$ be a distorted element. Then every fixed point $x \in M$ of f is contractible. That is, the loop $\{f_t(x)\}$ is contractible for every choice of isotopy from the identity to f . In particular, given a fixed point x of f , there is an isotopy from the identity to f fixing x .*

Proof. Let $\text{Homeo}_0(M, x) = \{g \in \text{Homeo}_0(M) \mid g(x) = x\}$. Consider the evaluation fibration

$$\text{Homeo}_0(M, x) \rightarrow \text{Homeo}_0(M) \xrightarrow{\text{ev}} \pi_1(M),$$

where $\text{ev}(g) = g(x)$. It follows from a theorem of Gottlieb [24, Theorem I.4] that the image of the homomorphism induced by the evaluation on the fundamental group is contained in the center of $\pi_1(M, x)$. Since this center is trivial and the group $\text{Homeo}_0(M)$ is connected, the connecting homomorphism

$$\partial: \pi_1(M, x) \rightarrow \pi_0(\text{Homeo}_0(M, x))$$

is an isomorphism of groups. Since f is distorted and $\pi_1(M)$ is quasi-residually real, it follows from Proposition 4.2 that $\varphi(\gamma(f, x)) = 0$ for every homogeneous quasimorphism on $\pi_1(M)$. Since the latter group is quasi-residually real, it follows that the element $\gamma(f, x)$ is trivial. This implies that f belongs to the connected component of the identity of $\text{Homeo}_0(M, x)$ and hence there is an isotopy from the identity to f fixing x . \square

Remark 4.5. If we drop the hypothesis about trivial centre of $\pi_1(M)$ then the element $\gamma(f, x)$ is not well defined and it depends on the choice of isotopy $\{f_t\}$ from the identity to f . However, the conclusion in this case is that the loop $\{f_t(x)\}$ represents a central element of $\pi_1(M)$.

A homeomorphism f of a manifold M is called *recurrent* if there exists an increasing sequence $\{n_k\}_{k=1}^\infty$ of natural numbers such that

$$\lim_{k \rightarrow \infty} d_0(f^{n_k}, \text{Id}) = 0.$$

In other words, arbitrary large powers of f are arbitrarily close to the identity. For example, if f is periodic then it is obviously recurrent. Another example is an irrational rotation of the circle. The only recurrent diffeomorphisms of surfaces of higher genus are periodic, according to [27].

Proposition 4.6. *Let $f \in \text{Homeo}_0(M)$ be a recurrent homeomorphism of a closed manifold. Suppose that $\pi_1(M, z)$ has trivial centre. If $z \in M$ is a fixed point of f , then $\gamma(f, z) \in \pi_1(M, z)$ is torsion.*

Proof. Since z is a fixed point of f , we have $\gamma(f^k, z) = \gamma(f, z)^k$. On the other hand, since f is recurrent, there exists $k_0 \in \mathbf{N}$ such that f^{k_0} is C^0 -close to the identity. Hence $\gamma(f^{k_0}, z) = 1$. \square

Milton proved that every recurrent *diffeomorphism* of a closed manifold M is distorted [28]. The recurrence of a diffeomorphism is meant here with respect to the C^∞ -topology.

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