ON THE ENTROPY NORM ON $\text{Ham}(S^2)$

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Abstract. In this note we prove that for each positive integer $m$ there exists a bi-Lipschitz embedding $\mathbb{Z}^m \to \text{Ham}(S^2)$, where $\text{Ham}(S^2)$ is equipped with the entropy metric. In particular, the same result holds when the entropy metric is substituted with the autonomous metric.

1. Introduction

Let $S^2$ be the standard 2-sphere and $\text{Ham}(S^2)$ the group of Hamiltonian diffeomorphisms of $S^2$. There exist several unbounded bi-invariant metrics on $\text{Ham}(S^2)$. The most notable are the Hofer metric, the autonomous metric and the entropy metric, see e.g. [3, 5, 10]. The following question due to Kapovich and Polterovich is widely open and seems to be quite difficult: "Is $\text{Ham}(S^2)$ equipped with Hofer metric quasi-isometric to $\mathbb{R}$?"

Let $\text{Ent}(S^2) \subset \text{Ham}(S^2)$ be the set of topological entropy-zero diffeomorphisms. This set is conjugation invariant and it generates $\text{Ham}(S^2)$, since this group is simple [1]. In other words, each diffeomorphism in $\text{Ham}(S^2)$ is a finite product of entropy-zero diffeomorphisms. One may ask for a minimal decomposition and this question leads to the concept of the entropy norm which we define by

$$\|f\|_{\text{Ent}} := \min\{k \in \mathbb{N} \mid f = h_1 \cdots h_k, \ h_i \in \text{Ent}(S^2)\}.$$ 

It is the word norm associated with the generating set $\text{Ent}(S^2)$. The entropy norm is conjugation-invariant since $\text{Ent}(S^2)$ is. The associated bi-invariant metric is denoted by $d_{\text{Ent}}$.

In this note we answer in negative the Kapovich-Polterovich question with respect to the entropy metric and hence with respect to the autonomous metric. The main result of this paper is the following

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Theorem 1. For each \( m \in \mathbb{N} \) there exists a bi-Lipschitz embedding\[ Z^m \hookrightarrow (\text{Ham}(S^2), d_{\text{Ent}}), \]
where \( Z^m \) is endowed with the \( l^1 \)-metric.

In particular, the above result implies that \( \text{Ham}(S^2) \) equipped with either the entropy or the autonomous metric is not a hyperbolic space.

Remark 1.1. The above theorem for \( \text{Ham}(D^2) \), where \( D^2 \) is a unit disc in the Euclidean plane, was recently proved in [5]. Also, the above theorem for the Hofer metric and \( \text{Ham}(S_g) \), where \( S_g \) is a closed hyperbolic surface, was proved by Py in [11]. For analogous results on other metrics on diffeomorphism groups of surfaces see e.g. [6].

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2. Preliminaries

In this section we recall the notion of a quasimorphism and describe the Gambaudo-Ghys construction. Throughout the paper the area of \( S^2 \) is normalized to be one.

2.A. Quasimorphisms. Recall that a function \( \psi \) from a group \( G \) to the reals is called a quasimorphism if there exists a constant \( D \geq 0 \) such that \( |\psi(g) - \psi(gh) + \psi(h)| < D \) for all \( g, h \in G \). Minimal such \( D \) is called the defect of \( \psi \) and denoted by \( D_\psi \). A quasimorphism \( \psi \) is homogeneous if \( \psi(g^n) = n \psi(g) \) for all \( n \in \mathbb{Z} \) and \( g \in G \). Quasimorphism \( \psi \) can be homogenized by setting \[ \overline{\psi}(g) := \lim_{p \to \infty} \frac{\psi(g^p)}{p}. \]

The vector space of homogeneous quasimorphisms on \( G \) is denoted by \( Q(G) \). For more information about quasimorphisms and their connections to different branches of mathematics, see [7].
2.B. Gambaudo-Ghys construction. In their influential paper [8] Gambaudo and Ghys constructed quasimorphisms on \( \text{Ham}(S) \), where \( S \) is either a 2-disc or a 2-sphere, from quasimorphisms on pure braid group \( P_n \) or spherical pure braid group \( P_n(S^2) \) respectively (see also [5]). The first named author generalized their construction to other surfaces [3]. Let us recall the construction.

Let \( \{f_t\} \) be an isotopy in \( \text{Ham}(S^2) \) from the identity to \( f \in \text{Ham}(S^2) \) and let \( z \in S^2 \) be a basepoint. For \( y \in S^2 \) we define a loop \( \gamma_{y,z} : [0, 1] \to S^2 \) by

\[
\gamma_{y,z}(t) := \begin{cases} 
\alpha_{3t}(z) & \text{for } t \in \left[0, \frac{1}{3}\right], \\
 f_{3t-1}(y) & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
 \beta_{3t-2}(f(y)) & \text{for } t \in \left[\frac{2}{3}, 1\right],
\end{cases}
\]

where \( \{\alpha_t\} \) is the shortest path on \( S^2 \) from \( z \) to \( y \), and \( \{\beta_t\} \) is the shortest path on \( S^2 \) from \( f(y) \) to \( z \).

Let \( X_n(S^2) \) be the configuration space of all ordered \( n \)-tuples of pairwise distinct points in \( S^2 \). Its fundamental group \( \pi_1(X_n(S^2)) \) is identified with the spherical pure braid group \( P_n(S^2) \). Let \( z = (z_1, \ldots, z_n) \) in \( X_n(S^2) \) be a base point. For almost every \( x = (x_1, \ldots, x_n) \in X_n(S^2) \) the \( n \)-tuple of loops \( (\gamma_{x_1,z_1}, \ldots, \gamma_{x_n,z_n}) \) is a based loop in the configuration space \( X_n(S^2) \). Let \( \gamma(f, x) \in P_n(S^2) = \pi_1(X_n(S^2), z) \) be the element represented by this loop, and let \( \varphi : P_n(S^2) \to \mathbb{R} \) be a homogeneous quasimorphism. Since \( \pi_1(\text{Ham}(S^2)) \) is isomorphic to \( \mathbb{Z}_2 \), the number \( \varphi(\gamma(f, x)) \) does not depend on the choice of the isotopy \( \{f_t\} \).

Define the quasimorphism \( \Phi_n : \text{Ham}(S^2) \to \mathbb{R} \) and its homogenization \( \overline{\Phi}_n : \text{Ham}(S^2) \to \mathbb{R} \) by

\[
\Phi_n(f) := \int_{X_n(S^2)} \varphi(\gamma(f, x))dx \quad \overline{\Phi}_n(f) := \lim_{p \to +\infty} \frac{\Phi_n(f^p)}{p}.
\]

Remark 2.1. The assertion that both the above functions are well defined quasimorphisms is proved in [8]. Using the family of signature quasimorphisms on the group \( P_n(S^2) \) Gambaudo-Ghys showed that \( \dim(Q(\text{Ham}(S^2))) = \infty \). This fact was also proved in [5].

3. Proof of the main result

Proposition 3.1. Let \( m, n \in \mathbb{N} \) such that \( n \geq 4 \). Then there exist \( f \in \text{Ham}(S^2) \) supported in an embedded disc \( D_m \subset S^2 \) such that \( \text{area}(D_m) < \frac{1}{m} \), and a quasimorphism \( \overline{\Phi}_n \) as above such that \( \overline{\Phi}_n(f) \neq 0 \).
Proof. Let $m, n \in \mathbb{N}$ such that $n \geq 4$, and let $X_n(S^2)$ be the configuration space of all unordered $n$-tuples of pairwise distinct points in $S^2$. Recall that the Birman map:

$$\text{Push}: B_n(S^2) \to \text{MCG}(S^2, n),$$

where $B_n(S^2) = \pi_1(X_n(S^2), z)$ is the spherical braid group on $n$ strings and $\text{MCG}(S^2, n)$ is the mapping class group of the $n$-punctured sphere, is defined as follows: let $\alpha(t), t \in [0, 1]$, be a loop in $X_n(S^2)$ based at $z$ and $h_t \in \text{Diff}(S^2)$ an isotopy such that $h_t(z) = \alpha(t)$. We define $\text{Push}(\alpha) := [h_1]$ where $\alpha$ is the braid represented by the loop $\alpha(t)$. The braid $\alpha$ is called reducible if $\text{Push}(\alpha)$ is a reducible mapping class.

We denote by $Q_{BF}(B_n(S^2))$ the space of homogeneous quasimorphisms on $B_n(S^2)$ which vanish on reducible braids. It follows from the celebrated paper by Bestvina and Fujiwara [2] that the space $Q_{BF}(B_n(S^2))$ is infinite dimensional, see [5, Section 4.A.]. Let $\varphi \in Q_{BF}(B_n(S^2))$ and $g \in \text{Ham}(S^2)$. Observe that if $x = (x_1, \ldots, x_n) \in X_n(S^2)$ is such that there are $1 \leq i < j \leq n$ so that $x_i$ and $x_j$ lie outside of the support of $g$, then the braid $\gamma(g, x)$ is reducible and hence

$$(3.1) \quad \varphi(\gamma(g, x)) = 0.$$  

Let $\iota_n: P_n(S^2) \to B_n(S^2)$ be the standard inclusion. In [9] Ishida proved that the composition map

$$\mathcal{G}_n \circ \iota_n^*: Q(P_n(S^2)) \to Q(\text{Ham}(S^2))$$

is injective (see also [5, Section 2.D.]), where

$$\mathcal{G}_n: Q(P_n(S^2)) \to Q(\text{Ham}(S^2))$$

is the map defined by $\mathcal{G}_n(\varphi)(f) = \Phi_n(f)$, see equation (2.1). In particular, the restriction

$$\mathcal{G}_n \circ \iota_n^*: Q_{BF}(B_n(S^2)) \to Q(\text{Ham}(S^2))$$

is injective.

Let $\psi \in Q_{BF}(B_n(S^2))$ a non-trivial quasimorphism. It follows from the paper of Ishida [9] (see also [5, Section 2.D.]) that there exists an embedded disc $D_a \subset S^2$ of area $a$ (it could be very close to one) and $f_a \in \text{Ham}(S^2)$ such that the support of $f_a$ is contained in $D_a$ and
$\Phi_n(f_a) \neq 0$. It follows from equation (3.1) that

$$
\Phi_n(f_a) = \lim_{p \to \infty} \left( \int_{X_n(D_a)} \frac{\psi(\gamma(f^p; x))}{p} dx + n(1 - a) \int_{X_{n-1}(D_a)} \frac{\psi(\gamma(f^p; x))}{p} dx \right).
$$

Set $A := \int_{X_n(D_a)} \lim_{p \to \infty} \frac{\psi(\gamma(f^p; x))}{p} dx$ and $B := \int_{X_{n-1}(D_a)} \lim_{p \to \infty} \frac{\psi(\gamma(f^p; x))}{p} dx$.

Thus by the result of Ishida we have

$$
(3.2) \quad \Phi_n(f_a) = A + n(1 - a)B \neq 0.
$$

Moreover, by the construction of Ishida, if we shrink the area of $D_a$ by $\varepsilon$ then we get a disc $D_{\varepsilon a}$ of area $\varepsilon a$ and $f_{\varepsilon a} \in \text{Ham}(S^2)$ such that the support of $f_{\varepsilon a}$ is contained in $D_{\varepsilon a}$, and

$$
(3.3) \quad \Phi_n(f_{\varepsilon a}) = \varepsilon^n A + n\varepsilon^{n-1}(1 - \varepsilon a)B = \varepsilon^{n-1}(\varepsilon A + n(1 - \varepsilon a)B).
$$

Note that if $B = 0$, then by equation (3.2) we get that $A \neq 0$, and hence by equation (3.3) we get for each $\varepsilon$ that $\Phi_n(f_{\varepsilon a}) \neq 0$ and the proof follows. If $B \neq 0$, then

$$
\lim_{\varepsilon \to 0} (\varepsilon A + n(1 - \varepsilon a)B) = nB \neq 0.
$$

It follows that there exists $\varepsilon$ such that $\varepsilon a < \frac{1}{m}$ and $\Phi_n(f_{\varepsilon a}) \neq 0$. We set $D_m := D_{\varepsilon a}$, $f := f_{\varepsilon a}$ and the proof follows. \(\Box\)

Let us continue the proof of the theorem. It follows from [5, Theorem 1] that the subspace

$$
\mathcal{G}_n \circ \iota^*_n(Q_{BF}(B_n(S^2))) \subset Q(\text{Ham}(S^2))
$$

is infinite dimensional for $n \geq 4$ and that every quasimorphism in this space vanishes on the set of entropy-zero diffeomorphisms. It follows from [4, Lemma 3.10] and Proposition 3.1 that there exists a family of quasimorphisms $\{\Phi_{n,i}\}_{i=1}^m \subset \mathcal{G}_n \circ \iota^*_n(Q_{BF}(B_n(S^2)))$ and a family of diffeomorphisms $\{f_{n,i}\}_{i=1}^m \subset \text{Ham}(S^2)$ such that the support of each $f_{n,i}$ is contained in a disc $D_m$ from Proposition 3.1, and $\Phi_{n,i}(f_{n,j}) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta.

Since $\text{area}(D_m) < \frac{1}{m}$ there exists a family of diffeomorphisms $\{h_i\}_{i=1}^m$ in $\text{Ham}(S^2)$ such that $h_i \circ f_{n,i} \circ h_i^{-1}$ and $h_j \circ f_{n,j} \circ h_j^{-1}$ have disjoint supports for $i \neq j$. Denote by $\hat{f}_i := h_i \circ f_{n,i} \circ h_i^{-1}$ and let

$$
J: \mathbb{Z}^m \to \text{Ham}(S^2),
$$
where \( J(k_1, \ldots, k_m) = \hat{f}_1^{k_1} \ldots \hat{f}_m^{k_m} \). It is clear that this map is a monomorphism. We prove that it is bi-Lipschitz. Since all \( \hat{f}_i \) commute with each other and \( \Phi_{n,i}(\hat{f}_j) = \delta_{ij} \), we obtain

\[
\| \hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m} \|_{\text{Ent}} \geq \left| \frac{\Phi_{n,i}(\hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m})}{D_{\Phi_{n,i}}} \right| = \left| k_i \right| \frac{D_{\Phi_{n,i}}}{D_{\Phi_{n,i}}},
\]

where \( D_{\Phi_{n,i}} \) is the defect of the quasimorphism \( \Phi_{n,i} \). We denote by \( D_m := \max_i D_{\Phi_{n,i}} \) and obtain the following inequality

\[
\| \hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m} \|_{\text{Ent}} \geq (m \cdot D_m)^{-1} \sum_{i=1}^{m} |k_i|.
\]

Denote by \( M_J := \max_i \| \hat{f}_i \|_{\text{Ent}} \). Now we have the following inequality

\[
\| \hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m} \|_{\text{Ent}} \leq \sum_{i=1}^{m} |k_i| \cdot \| \hat{f}_i \|_{\text{Ent}} \leq M_J \sum_{i=1}^{m} |k_i|.
\]

Last two inequalities conclude the proof of the theorem. \( \square \)

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