ON THE ENTROPY NORM ON $\mathrm{Ham}(S^2)$

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Abstract. In this note we prove that for each positive integer m there exists a bi-Lipschitz embedding $\mathbf{Z}^m \to \mathrm{Ham}(S^2)$, where $\mathrm{Ham}(S^2)$ is equipped with the entropy metric. In particular, the same result holds when the entropy metric is substituted with the autonomous metric.

1. Introduction

Let S^2 be the standard 2-sphere and $\mathrm{Ham}(S^2)$ the group of Hamiltonian diffeomorphisms of S^2. There exist several unbounded bi-invariant metrics on $\mathrm{Ham}(S^2)$. The most notable are the Hofer metric, the autonomous metric and the entropy metric, see e.g. [3, 5, 10]. The following question due to Kapovich and Polterovich is widely open and seems to be quite difficult: "Is $\mathrm{Ham}(S^2)$ equipped with Hofer metric quasi-isometric to $\mathbf{R}$?"

Let $\mathrm{Ent}(S^2) \subset \mathrm{Ham}(S^2)$ be the set of topological entropy-zero diffeomorphisms. This set is conjugation invariant and it generates $\mathrm{Ham}(S^2)$, since this group is simple [1]. In other words, each diffeomorphism in $\mathrm{Ham}(S^2)$ is a finite product of entropy-zero diffeomorphisms. One may ask for a minimal decomposition and this question leads to the concept of the entropy norm which we define by

$$\|f\|_{\mathrm{Ent}} := \min\{k \in \mathbf{N} \mid f = h_1 \cdots h_k,\, h_i \in \mathrm{Ent}(S^2)\}.$$

It is the word norm associated with the generating set $\mathrm{Ent}(S^2)$. The entropy norm is conjugation-invariant since $\mathrm{Ent}(S^2)$ is. The associated bi-invariant metric is denoted by $\mathbf{d}_{\mathrm{Ent}}$.

In this note we answer in negative the Kapovich-Polterovich question with respect to the entropy metric and hence with respect to the autonomous metric. The main result of this paper is the following

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Theorem 1. *For each $m \in \mathbf{N}$ there exists a bi-Lipschitz embedding*

$$\mathbf{Z}^m \hookrightarrow (\mathrm{Ham}(S^2), \mathbf{d}_{\mathrm{Ent}}),$$

where $\mathbf{Z}^m$ is endowed with the l^1-metric.

In particular, the above result implies that $\mathrm{Ham}(S^2)$ equipped with either the entropy or the autonomous metric is not a hyperbolic space.

Remark 1.1. The above theorem for $\mathrm{Ham}(D^2)$, where D^2 is a unit disc in the Euclidean plane, was recently proved in [5]. Also, the above theorem for the Hofer metric and $\mathrm{Ham}(S_g)$, where S_g is a closed hyperbolic surface, was proved by Py in [11]. For analogous results on other metrics on diffeomorphism groups of surfaces see e.g. [6].

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2. Preliminaries

In this section we recall the notion of a quasimorphism and describe the Gambaudo-Ghys construction. Throughout the paper the area of S^2 is normalized to be one.

2.A. **Quasimorphisms.** Recall that a function ψ from a group G to the reals is called a quasimorphism if there exists a constant $D \geq 0$ such that $|\psi(g) - \psi(gh) + \psi(h)| < D$ for all $g, h \in G$. Minimal such D is called the defect of ψ and denoted by D_ψ. A quasimorphism ψ is homogeneous if $\psi(g^n) = n\psi(g)$ for all $n \in \mathbf{Z}$ and $g \in G$. Quasimorphism ψ can be homogenized by setting

$$\overline{\psi}(g) := \lim_{p\to\infty} \frac{\psi(g^p)}{p}.$$

The vector space of homogeneous quasimorphisms on G is denoted by $Q(G)$. For more information about quasimorphisms and their connections to different branches of mathematics, see [7].

2.B. **Gambaudo-Ghys construction.** In their influential paper [8] Gambaudo and Ghys constructed quasimorphisms on $\mathrm{Ham}(S)$, where S is either a 2-disc or a 2-sphere, from quasimorphisms on pure braid group P_n or spherical pure braid group $P_n(S^2)$ respectively (see also [5]). The first named author generalized their construction to other surfaces [3]. Let us recall the construction.

Let $\{f_t\}$ be an isotopy in $\mathrm{Ham}(S^2)$ from the identity to $f \in \mathrm{Ham}(S^2)$ and let $z \in S^2$ be a basepoint. For $y \in S^2$ we define a loop $\gamma_{y,z} \colon [0,1] \to S^2$ by

$$\gamma_{y,z}(t) := \begin{cases} \alpha_{3t}(z) & \text{for } t \in \left[0, \frac{1}{3}\right] \\ f_{3t-1}(y) & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ \beta_{3t-2}(f(y)) & \text{for } t \in \left[\frac{2}{3}, 1\right], \end{cases}$$

where $\{\alpha_t\}$ is the shortest path on S^2 from z to y, and $\{\beta_t\}$ is the shortest path on S^2 from $f(y)$ to z.

Let $\mathrm{X}_n(S^2)$ be the configuration space of all ordered n-tuples of pairwise distinct points in S^2. Its fundamental group $\pi_1(\mathrm{X}_n(S^2))$ is identified with the spherical pure braid group $P_n(S^2)$. Let $z = (z_1, \ldots, z_n)$ in $\mathrm{X}_n(S^2)$ be a base point. For almost every $x = (x_1, \ldots, x_n) \in \mathrm{X}_n(S^2)$ the n-tuple of loops $(\gamma_{x_1,z_1}, \ldots, \gamma_{x_n,z_n})$ is a based loop in the configuration space $\mathrm{X}_n(S^2)$. Let $\gamma(f,x) \in P_n(S^2) = \pi_1(\mathrm{X}_n(S^2), z)$ be the element represented by this loop, and let $\varphi\colon P_n(S^2) \to \mathbf{R}$ be a homogeneous quasimorphism. Since $\pi_1(\mathrm{Ham}(S^2))$ is isomorphic to $\mathbf{Z}_2$, the number $\varphi(\gamma(f,x))$ does not depend on the choice of the isotopy $\{f_t\}$. Define the quasimorphism $\Phi_n \colon \mathrm{Ham}(S^2) \to \mathbf{R}$ and its homogenization $\overline{\Phi}_n \colon \mathrm{Ham}(S^2) \to \mathbf{R}$ by

$$\Phi_n(f) := \int\limits_{\mathrm{X}_n(S^2)} \varphi(\gamma(f;x))dx \qquad \overline{\Phi}_n(f) := \lim_{p\to+\infty} \frac{\Phi_n(f^p)}{p}. \tag{2.1}$$

Remark 2.1. The assertion that both the above functions are well defined quasimorphisms is proved in [8]. Using the family of signature quasimorphisms on the group $P_n(S^2)$ Gambaudo-Ghys showed that $\dim(Q(\mathrm{Ham}(S^2))) = \infty$. This fact was also proved in [5].

3. Proof of the main result

Proposition 3.1. *Let $m, n \in \mathbf{N}$ such that $n \geq 4$. Then there exist $f \in \mathrm{Ham}(S^2)$ supported in an embedded disc $D_m \subset S^2$ such that* $\mathrm{area}(D_m) < \frac{1}{m}$*, and a quasimorphism $\overline{\Phi}_n$ as above such that $\overline{\Phi}_n(f) \neq 0$.*

Proof. Let $m, n \in \mathbf{N}$ such that $n \geq 4$, and let $\mathbf{X}_n(S^2)$ be the configuration space of all unordered n-tuples of pairwise distinct points in S^2. Recall that the Birman map:

$$\mathrm{Push}\colon B_n(S^2) \to \mathrm{MCG}(S^2, n),$$

where $B_n(S^2) = \pi_1(\mathbf{X}_n(S^2), z)$ is the spherical braid group on n strings and $\mathrm{MCG}(S^2, n)$ is the mapping class group of the n-punctured sphere, is defined as follows: let $\alpha(t)$, $t \in [0,1]$, be a loop in $\mathbf{X}_n(S^2)$ based at z and $h_t \in \mathrm{Diff}(S^2)$ an isotopy such that $h_t(z) = \alpha(t)$. We define $\mathrm{Push}(\alpha) := [h_1]$ where α is the braid represented by the loop $\alpha(t)$. The braid α is called *reducible* if $\mathrm{Push}(\alpha)$ is a reducible mapping class.

We denote by $Q_{\mathrm{BF}}(B_n(S^2))$ the space of homogeneous quasimorphisms on $B_n(S^2)$ which vanish on reducible braids. It follows from the celebrated paper by Bestvina and Fujiwara [2] that the space $Q_{\mathrm{BF}}(B_n(S^2))$ is infinite dimensional, see [5, Section 4.A.]. Let $\varphi \in Q_{\mathrm{BF}}(B_n(S^2))$ and $g \in \mathrm{Ham}(S^2)$. Observe that if $x = (x_1, \ldots, x_n) \in \mathrm{X}_n(S^2)$ is such that there are $1 \leq i < j \leq n$ so that x_i and x_j lie outside of the support of g, then the braid $\gamma(g, x)$ is reducible and hence

$$\varphi(\gamma(g, x)) = 0. \tag{3.1}$$

Let $\iota_n\colon P_n(S^2) \to B_n(S^2)$ be the standard inclusion. In [9] Ishida proved that the composition map

$$\mathcal{G}_n \circ \iota_n^*\colon Q(B_n(S^2)) \to Q(\mathrm{Ham}(S^2))$$

is injective (see also [5, Section 2.D.]), where

$$\mathcal{G}_n\colon Q(P_n(S^2)) \to Q(\mathrm{Ham}(S^2))$$

is the map defined by $\mathcal{G}_n(\varphi)(f) = \overline{\Phi}_n(f)$, see equation (2.1). In particular, the restriction

$$\mathcal{G}_n \circ \iota_n^*\colon Q_{\mathrm{BF}}(B_n(S^2)) \to Q(\mathrm{Ham}(S^2))$$

is injective.

Let $\psi \in Q_{\mathrm{BF}}(B_n(S^2))$ a non-trivial quasimorphism. It follows from the paper of Ishida [9] (see also [5, Section 2.D.]) that there exists an embedded disc $D_a \subset S^2$ of area a (it could be very close to one) and $f_a \in \mathrm{Ham}(S^2)$ such that the support of f_a is contained in D_a and

$\overline{\Phi}_n(f_a) \neq 0$. It follows from equation (3.1) that

$$\overline{\Phi}_n(f_a) = \lim_{p\to\infty}\left(\int\limits_{\mathrm{X}_n(D_a)} \frac{\psi(\gamma(f^p;x))}{p}\,dx + n(1-a)\int\limits_{\mathrm{X}_{n-1}(D_a)} \frac{\psi(\gamma(f^p;x))}{p}\,dx\right).$$

Set $A := \int\limits_{\mathrm{X}_n(D_a)} \lim\limits_{p\to\infty} \dfrac{\psi(\gamma(f^p;x))}{p}\,dx$ and $B := \int\limits_{\mathrm{X}_{n-1}(D_a)} \lim\limits_{p\to\infty} \dfrac{\psi(\gamma(f^p;x))}{p}$.
Thus by the result of Ishida we have

$$\overline{\Phi}_n(f_a) = A + n(1-a)B \neq 0. \tag{3.2}$$

Moreover, by the construction of Ishida, if we shrink the area of D_a by ε then we get a disc $D_{\varepsilon a}$ of area εa and $f_{\varepsilon a} \in \mathrm{Ham}(S^2)$ such that the support of $f_{\varepsilon a}$ is contained in $D_{\varepsilon a}$, and

$$\overline{\Phi}_n(f_{\varepsilon a}) = \varepsilon^n A + n\varepsilon^{n-1}(1-\varepsilon a)B = \varepsilon^{n-1}(\varepsilon A + n(1-\varepsilon a)B). \tag{3.3}$$

Note that if $B = 0$, then by equation (3.2) we get that $A \neq 0$, and hence by equation (3.3) we get for each ε that $\overline{\Phi}_n(f_{\varepsilon a}) \neq 0$ and the proof follows. If $B \neq 0$, than

$$\lim_{\varepsilon\to 0}(\varepsilon A + n(1-\varepsilon a)B) = nB \neq 0.$$

It follows that there exists ε such that $\varepsilon a < \frac{1}{m}$ and $\overline{\Phi}_n(f_{\varepsilon a}) \neq 0$. We set $D_m := D_{\varepsilon a}$, $f := f_{\varepsilon a}$ and the proof follows. □

Let us continue the proof of the theorem. It follows from [5, Theorem 1] that the subspace

$$\mathcal{G}_n \circ \iota_n^*(Q_{\mathrm{BF}}(B_n(S^2))) \subset Q(\mathrm{Ham}(S^2))$$

is infinite dimensional for $n \geq 4$ and that every quasimorphism in this space vanishes on the set of entropy-zero diffeomorphisms. It follows from [4, Lemma 3.10] and Proposition 3.1 that there exists a family of quasimorphisms $\{\overline{\Phi}_{n,i}\}_{i=1}^m \in \mathcal{G}_n \circ \iota_n^*(Q_{\mathrm{BF}}(B_n(S^2)))$ and a family of diffeomorphisms $\{f_{n,i}\}_{i=1}^m$ in $\mathrm{Ham}(S^2)$ such that the support of each $f_{n,i}$ is contained in a disc D_m from Proposition 3.1, and $\overline{\Phi}_{n,i}(f_{n,j}) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Since $\mathrm{area}(D_m) < \frac{1}{m}$ there exists a family of diffeomorphisms $\{h_i\}_{i=1}^m$ in $\mathrm{Ham}(S^2)$ such that $h_i \circ f_{n,i} \circ h_i^{-1}$ and $h_j \circ f_{n,j} \circ h_j^{-1}$ have disjoint supports for $i \neq j$. Denote by $\hat{f}_i := h_i \circ f_{n,i} \circ h_i^{-1}$ and let

$$J\colon \mathbf{Z}^m \to \mathrm{Ham}(S^2),$$

where $J(k_1,\dots,k_m)=\hat{f}_1^{k_1}\dots\hat{f}_m^{k_m}$. It is clear that this map is a monomorphism. We prove that it is bi-Lipschitz. Since all $\hat{f}_i$ commute with each other and $\overline{\Phi}_{n,i}(\hat{f}_j)=\delta_{ij}$, we obtain

$$\|\hat{f}_1^{k_1}\circ\ldots\circ\hat{f}_m^{k_m}\|_{\mathrm{Ent}}\geq\frac{|\overline{\Phi}_{n,i}(\hat{f}_1^{k_1}\circ\ldots\circ\hat{f}_m^{k_m})|}{D_{\overline{\Phi}_{n,i}}}=\frac{|k_i|}{D_{\overline{\Phi}_{n,i}}},$$

where $D_{\overline{\Phi}_{n,i}}$ is the defect of the quasimorphism $\overline{\Phi}_{n,i}$. We denote by $\mathfrak{D}_m:=\max_i D_{\overline{\Phi}_{n,i}}$ and obtain the following inequality

$$\|\hat{f}_1^{k_1}\circ\ldots\circ\hat{f}_m^{k_m}\|_{\mathrm{Ent}}\geq(m\cdot\mathfrak{D}_m)^{-1}\sum_{i=1}^{m}|k_i|\,.$$

Denote by $\mathfrak{M}_J:=\max_i\|\hat{f}_i\|_{\mathrm{Ent}}$. Now we have the following inequality

$$\|\hat{f}_1^{k_1}\circ\ldots\circ\hat{f}_m^{k_m}\|_{\mathrm{Ent}}\leq\sum_{i=1}^{m}|k_i|\cdot\|\hat{f}_i\|_{\mathrm{Ent}}\leq\mathfrak{M}_J\cdot\sum_{i=1}^{m}|k_i|\,.$$

Last two inequalities conclude the proof of the theorem. □

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