QUASI-MORPHISMS AND $L^p$-METRICS ON GROUPS OF VOLUME-PRESERVING DIFFEOMORPHISMS

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Abstract. Let $M$ be a smooth compact connected oriented manifold of dimension at least two endowed with a volume form $\mu$. We show that every homogeneous quasi-morphism on the identity component $\text{Diff}_0(M, \mu)$ of the group of volume-preserving diffeomorphisms of $M$, which is induced by a quasi-morphism on the fundamental group $\pi_1(M)$, is Lipschitz with respect to the $L^p$-metric on $\text{Diff}_0(M, \mu)$. As a consequence, assuming certain conditions on $\pi_1(M)$, we construct bi-Lipschitz embeddings of finite dimensional vector spaces into $\text{Diff}_0(M, \mu)$.

1. INTRODUCTION AND MAIN RESULTS

1.A. The $L^p$-metric. Let $M$ be a compact connected and oriented Riemannian manifold and let $\text{Diff}(M, \mu)$ denote the group of smooth diffeomorphisms of $M$ acting by the identity on a neighborhood of the boundary and preserving the volume form $\mu$ induced by the metric. Unless otherwise stated we assume that $\text{Diff}(M, \mu)$ is equipped with the Whitney $C^\infty$-topology. In the present paper we study the geometry of the identity component $\text{Diff}_0(M, \mu)$ of the above group endowed with the right invariant $L^p$-metric. It is defined as follows. Let

$$L_p\{g_t\} := \int_0^1 dt \left( \int_M |\dot{g}_t(x)|^p \mu \right)^{\frac{1}{p}}$$

be the $L^p$-length of a smooth isotopy $\{g_t\}_{t \in [0,1]} \subset \text{Diff}_0(M, \mu)$, where $|\dot{g}_t(x)|$ denotes the length of the tangent vector $\dot{g}_t(x) \in T_x M$ induced by the Riemannian metric. Observe that this length is right-invariant, that is, $L_p\{g_t \circ f\} = L_p\{g_t\}$ for any $f \in \text{Diff}(M, \mu)$. It defines a non-degenerate right-invariant metric on $\text{Diff}_0(M, \mu)$ by

$$d_p(g_0, g_1) := \inf_{g_t} L_p\{g_t\},$$

where the infimum is taken over all paths from $g_0$ to $g_1$. See Arnol’d-Khesin [2] and Khesin-Wendt [20, Section 3.6] for a detailed discussion.

If $p = 2$ then the group $\text{Diff}_0(M, \mu)$ is in fact equipped with a Riemannian metric inducing the above $L^2$-length. The geodesics of this metric
are the solutions of the equations of the flow of an incompressible fluid [1], which makes the \( p = 2 \) case the most interesting. It is known that if \( M \) is a simply connected Riemannian manifold of dimension at least three then the \( L^2 \)-diameter of the group \( \text{Diff}_0(M, \mu) \) is finite [27]. On the other hand Eliashberg and Ratiu [13] proved that this diameter is infinite for surfaces and for manifolds with positive first Betti number, and whose fundamental group has a trivial center. In [8] Kedra and the author showed, that under certain conditions on the fundamental group, the diameter of the identity component of the group of volume-preserving diffeomorphisms is also infinite.

1.B. Quasi-morphisms on \( \text{Diff}_0(M, \mu) \). Quasi-morphisms are known to be a helpful tool in the study of algebraic structure of non-Abelian groups, especially the ones that admit a few or no (linearly independent) real-valued homomorphisms. Recall that a quasi-morphism on a group \( G \) is a function \( \varphi : G \to \mathbb{R} \) which satisfies the homomorphism equation up to a bounded error: there exists \( K_\varphi > 0 \) such that

\[
|\varphi(ab) - \varphi(a) - \varphi(b)| \leq K_\varphi
\]

for all \( a, b \in G \). The infimum of all such \( K_\varphi \) is called the defect of \( \varphi \) and is denoted by \( D_\varphi \). A quasi-morphism \( \varphi \) is called homogeneous if we have \( \varphi(a^m) = m\varphi(a) \) for all \( a \in G \) and \( m \in \mathbb{Z} \). Any quasi-morphism \( \varphi \) can be homogenized: setting

\[
\tilde{\varphi}(a) := \lim_{k \to +\infty} \varphi(a^k)/k
\]

we get a homogeneous (possibly trivial) quasi-morphism \( \tilde{\varphi} \).

1.B.1. Polterovich construction. Let \( m \in M \setminus \partial M \). Suppose that the center \( Z(\pi_1(M, m)) \) is trivial and the group \( \pi_1(M, m) \) admits a non-trivial homogeneous quasi-morphism \( \phi : \pi_1(M, m) \to \mathbb{R} \). For each \( x \in M \setminus \partial M \) let us choose an arbitrary geodesic path from \( x \) to \( m \). In [25] Polterovich constructed the induced non-trivial homogeneous quasi-morphism \( \tilde{\Phi} \) on \( \text{Diff}_0(M, \mu) \) as follows:

For each \( x \in M \) and an isotopy \( \{g_t\}_{t \in [0, 1]} \) between \( \text{Id} \) and \( g \) let \( g_x \) be a closed loop in \( M \) which is a concatenation of a geodesic path from \( m \) to \( x \), the path \( g_t(x) \) and a described above geodesic path from \( g(x) \) to \( m \). Denote by \( [g_x] \) the corresponding element in \( \pi_1(M, m) \) and set

\[
\Phi(g) := \int_M \tilde{\phi}([g_x])\mu \quad \tilde{\Phi}(g) := \lim_{k \to \infty} \frac{1}{k} \int_M \tilde{\phi}([g^k_x])\mu.
\]

The maps \( \Phi \) and \( \tilde{\Phi} \) are well-defined quasi-morphisms because the center \( Z(\pi_1(M, m)) \) is trivial and every diffeomorphism in \( \text{Diff}_0(M, \mu) \) is volume-preserving. In addition, the quasi-morphism \( \tilde{\Phi} \) neither depends
on the choice of a family of geodesic paths, nor on the choice of a base point $m$. For more details see [25].

1.B.2. Gambaudo-Ghys construction. Let $\mathcal{D} := \text{Diff}(\mathbb{D}^2, \partial\mathbb{D}^2, \text{area})$ be the group of smooth area-preserving diffeomorphisms of the unit disc in the Euclidean plane which equal to the identity near the boundary. The group $\mathcal{D}$ admits a unique (continuous, in the proper sense) homomorphism to the reals—the famous Calabi homomorphism (see e.g. [3, 10]). At the same time $\mathcal{D}$ is known to admit many (linearly independent) homogeneous quasi-morphisms (see e.g. [4, 6, 16]). In what follows we describe a particular geometric construction of such quasi-morphisms, essentially contained in Gambaudo-Ghys [16] and studied by the author in [7], which produces quasi-morphisms on $\mathcal{D}$ from quasi-morphisms on the pure braid groups $P_n$ on $n$ strings.

Denote by $X_n$ the space of all ordered $n$-tuples of distinct points in $\mathbb{D}^2$. Let us fix a base point $\overline{x} = (z_1, \ldots, z_n) \in X_n$ and let $\overline{x} = (x_1, \ldots, x_n)$ be any other point in $X_n$. Take $g \in \mathcal{D}$ and any path $g_t$, $0 \leq t \leq 1$, in $\mathcal{D}$ between $Id$ and $g$. Connect $\overline{x}$ to $\overline{g}$ by a straight line in $(\mathbb{D}^2)^n$, then act on $\overline{x}$ with the path $g_t$, and then connect $g(x_i)$ to $z_i$ by straight lines in $(\mathbb{D}^2)^n$. We get a loop in $(\mathbb{D}^2)^n$. More specifically it looks as follows. Connect $z_i$ to $x_i$ by straight lines $l_{1,i} : [0, \frac{1}{3}] \to \mathbb{D}^2$ in the disc, then act with the path $g_{3t-1} : [\frac{1}{3}, \frac{2}{3}] \to \mathbb{D}^2$ on each $x_i$, and then connect $g(x_i)$ to $z_i$ by straight lines $l_{2,i} : [\frac{2}{3}, 1] \to \mathbb{D}^2$ in the disc, for all $1 \leq i \leq n$. It is easy to show that for almost all $n$-tuples of different points $x_1, \ldots, x_n$ in the disc the concatenations of the paths $l_{1,i} : [0, \frac{1}{3}] \to \mathbb{D}^2$, $g_{3t-1} : [\frac{1}{3}, \frac{2}{3}] \to \mathbb{D}^2$ and $l_{2,i} : [\frac{2}{3}, 1] \to \mathbb{D}^2$, $i = 1, \ldots, n$, yield a loop in $X_n$. The homotopy type of this loop is an element in $P_n$ (here $P_n$ is identified with the fundamental group $\pi_1(X_n, \overline{x})$). This element is independent of the choice of $g_t$, because $\mathcal{D}$ is contractible (see e.g. [15, 28]), it will be denoted by $\gamma(g; \overline{x})$. Let $\varphi_n$ be a homogeneous quasi-morphism on $P_n$. Denote $d\overline{x} := dx_1 \cdots dx_n$ and set

$$\Phi_n(g) := \int_{X_n} \varphi_n(\gamma(g; \overline{x}))d\overline{x}$$

$$\bar{\Phi}_n(g) := \lim_{k \to +\infty} \Phi_n(g^k)/k$$

The function $\Phi_n$ is a well-defined quasi-morphism on $\mathcal{D}$ and the function $\bar{\Phi}_n$ is a well-defined homogeneous quasi-morphism on $\mathcal{D}$, see [7].

Remark. The group $P_2$ is infinite cyclic, hence every homogeneous quasi-morphism $\varphi_2 : P_2 \to \mathbb{R}$ is a homomorphism. The celebrated theorem of Banyaga [3] states that the kernel of the Calabi homomorphism $C : \mathcal{D} \to \mathbb{R}$ is a simple group. It follows that $\bar{\Phi}_2(g) = K_{\varphi_2} \cdot C(g)$ for every $g \in \mathcal{D}$, where $K_{\varphi_2}$ is a real constant independent of $g$. 
1.C. **Main results.** A map $\psi: (X_1,d_1) \to (X_2,d_2)$ between metric spaces is called *Lipschitz* if there exists a constant $A \geq 0$ such that

$$d_2(\psi(x),\psi(y)) \leq A \cdot d_1(x,y).$$

The following theorems are our main technical results. They are proven in Section 2.

**Theorem 1.** Let $M$ be a compact connected and oriented Riemannian manifold of dimension at least two, such that $Z(\pi_1(M,m))$ is trivial. Let $\phi: \pi_1(M,m) \to \mathbb{R}$ be a homogeneous quasi-morphism. Then the induced homogeneous quasi-morphism

$$\tilde{\phi}: \text{Diff}_0(M,\mu) \to \mathbb{R}$$

is Lipschitz with respect to the $L^p$-metric on the group $\text{Diff}_0(M,\mu)$.

**Remark.** Theorem 1 shows that the diameter of $(\text{Diff}_0(M,\mu),d_p)$ is infinite if $Z(\pi_1(M,m))$ is trivial and $\pi_1(M,m)$ admits a non-trivial homogeneous quasi-morphism.

**Theorem 2.** Let $\tilde{\varphi}_n$ be a homogeneous quasi-morphism on $P_n$. Then the induced homogeneous quasi-morphism

$$\tilde{\varphi}_n: \mathcal{D} \to \mathbb{R}$$

is Lipschitz with respect to the $L^p$-metric on the group $\mathcal{D}$.

**Remark.** It follows from [7, 16] that for every $n \geq 3$ there exists a homogeneous quasi-morphism $\tilde{\varphi}_n: \mathcal{D} \to \mathbb{R}$ such that it does not vanish on the kernel of the Calabi homomorphism $C: \mathcal{D} \to \mathbb{R}$. Hence Theorem 2 gives another proof of the Theorem of Eliashberg and Ratiu [13], which states that the diameter of $(\text{Ker}(C),d_p)$ is infinite (see also [18]).

1.D. **Applications.** Let $(G,\| \cdot \|_G)$ and $(G',\| \cdot \|_{G'})$ be two normed semigroups. A function $f: G \to G'$ is a *bi-Lipschitz embedding* if it is an injective homomorphism, and there exists a constant $A \geq 1$ such that

$$A^{-1}\|g\|_G \leq \|f(g)\|_{G'} \leq A\|g\|_G.$$

Note that if $G$ and $G'$ are groups, then the norms $\| \cdot \|_G$ and $\| \cdot \|_{G'}$ define right-invariant metrics on the groups $G$ and $G'$ in a natural way, i.e., $d_G(g,h) := \|gh^{-1}\|_G$ and $d_{G'}(g',h') := \|g'h'^{-1}\|_{G'}$ for all $g,h \in G$ and $g',h' \in G'$. In this case by definition every bi-Lipschitz embedding is a quasi-isometric embedding.

Recall that the word norm on a group $\Gamma$ generated by a symmetric finite set $S \subset \Gamma$ is defined by

$$|\gamma|_S := \min\{k \in \mathbb{N} \mid \gamma = s_1 \ldots s_k \text{ where } s_i \in S\}.$$
The word metric is defined by \( d_S(\gamma_1, \gamma_2) := |\gamma_1(\gamma_2)^{-1}|_S \). It is right-invariant and it depends on the choice of a finite generating set up to a bi-Lipschitz equivalence [9, Example 8.17].

In [8] Kedra and the author showed, that under certain conditions on the fundamental group \( \pi_1(M, m) \), the group \((\text{Diff}_0(M, \mu), d_p)\) contains quasi-isometrically embedded finitely generated free Abelian group of an arbitrary finite rank. The following theorem generalizes this result for a wide class of compact Riemannian manifolds.

**Theorem 3.1.** Let \( M \) be a compact connected and oriented Riemannian manifold of dimension at least 3, such that \( Z(\pi_1(M, m)) \) is trivial. Let \( n \in \mathbb{N} \) and \( \{[\gamma_i]\}_{i=1}^n \) in \( \pi_1(M, m) \). Suppose that \( \pi_1(M, m) \) admits a family of homogeneous quasi-morphisms \( \{\tilde{\phi}_i\}_{i=1}^n \), such that \( \tilde{\phi}_i([\gamma_j]) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. Then \((\text{Diff}_0(M, \mu), d_p)\) contains bi-Lipschitz embedded \( \mathbb{R}^n \).

2. Let \( \Sigma_g \) be a closed orientable surface of genus \( g \geq 2 \). Then the group \((\text{Diff}_0(\Sigma_g, \mu), d_p)\) contains bi-Lipschitz embedded \( \mathbb{R}^g \).

Let \( M \) be a closed negatively curved Riemannian manifold. Then \( \pi_1(M, m) \) has a trivial center and is word-hyperbolic. It follows from [14, Proposition 3.6] that for each \( n \in \mathbb{N} \) there exist words \( \{[\gamma_i]\}_{i=1}^n \) in \( \pi_1(M, m) \) and a family of homogeneous quasi-morphisms \( \{\phi_i\}_{i=1}^n \), such that \( \phi_i([\gamma_j]) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. As an immediate corollary we have

**Corollary.** Let \( M \) be a closed negatively curved Riemannian manifold of dimension at least 3. Then \((\text{Diff}_0(M, \mu), d_p)\) contains bi-Lipschitz embedded vector space of an arbitrary finite dimension.

If \( (M, \omega) \) is a symplectic manifold, then the group \( \text{Diff}_0(M, \mu) \) in all the results above can be replaced either by the group \( \text{Symp}_0(M, \omega) \) of symplectic diffeomorphisms isotopic to the identity, or by the group \( \text{Ham}(M, \omega) \) of Hamiltonian diffeomorphisms, see Remark 2.1 in the proof of Theorem 3. In case of the group \( \text{Ham}(M, \omega) \) the assumption on the triviality of \( Z(\pi_1(M, m)) \) may be dropped, i.e., Polterovich quasi-morphisms are well-defined on \( \text{Ham}(M, \omega) \). This follows from the fact that the map \( ev : \text{Ham}(M, \omega) \rightarrow M \), where \( ev(g) = g(m) \), induces a trivial map on \( \pi_1(\text{Ham}(M, \omega), \text{Id}) \), see [24]. The same proof as the proof of Theorem 3 (part 2) proves the following

**Corollary.** The group \((\text{Ham}(\Sigma_g, \omega), d_p)\) contains bi-Lipschitz embedded \( \mathbb{R}^g \) for each \( g \geq 1 \). In particular, the diameter of \((\text{Ham}(\Sigma_g, \omega), d_p)\) is infinite for all \( g \geq 1 \).

**Remark.** The group \( \text{Ham}(M, \omega) \) may be equipped with the famous Hofer metric [19, 21]. Similar results to ours with respect to the Hofer
metric were obtained by Py and Usher. In [26] Py showed that for \( g \geq 2 \) the group \( \text{Ham}(\Sigma_g, \omega) \) contains bi-Lipschitz embedded copy of an arbitrary finitely generated free Abelian group. Recently Usher [29] generalized this result and showed that for a wide class of symplectic manifolds \( \ell^\infty \) bi-Lipschitz embeds into \( \text{Ham}(M, \omega) \).

Recall that \( \mathcal{D} := \text{Diff}(D^2, \partial D^2, \text{area}) \) and \( C : \mathcal{D} \to \mathbb{R} \) is the Calabi homomorphism. In [5] Benaim and Gambaudo showed that the group \( (\text{Ker}(C), d_2) \) contains quasi-isometrically embedded finitely generated free Abelian group of an arbitrary rank. The following theorem generalizes the result above.

**Theorem 4.** For each \( n \) the group \( (\mathcal{D}, d_p) \) contains bi-Lipschitz embedded \( R^n \). Moreover, this statement holds for the group \( (\text{Ker}(C), d_p) \).

In [11] Crisp and Wiest generalized the results of [5] and proved that the group \( (\mathcal{D}, d_2) \) contains quasi-isometrically embedded planar right-angled Artin groups. To the best knowledge of the author no similar results are known for infinitely generated groups.

Let \( Z^\infty \) be a lattice in \( \ell^1 \), i.e., \( Z^\infty \) consists of all infinite sequences of integers, such that for each sequence \( (n_1, n_2, \ldots) \in Z^\infty \) there exists \( d \in \mathbb{N} \) such that \( n_i = 0 \) for each \( i > d \). It follows that the metric on \( Z^\infty \) is the word metric with respect to the infinite set \( \{ \pm e_i \}_{i=1}^\infty \), where \( \pm e_i = (0, \ldots, 0, \pm 1, 0, \ldots) \) and \( 1 \) is placed in the \( i \)-th entry. The following question was posed to the author by M. Sapir.

**Question** (M. Sapir). Does \( Z^\infty \) quasi-isometrically embed into \( (\mathcal{D}, d_p) \)?

Let \( R^\infty_+ \) denote the following positive normed semigroup in \( \ell^1 \): the semigroup \( R^\infty_+ \) consists of sequences \( (v_1, \ldots, v_k, \ldots) \), where \( v_i \geq 0 \) and there exists \( N > 0 \) such that \( v_i = 0 \) for each \( i \geq N \). The following result is related to the question above and is proven in Section 2. It gives an example of a bi-Lipschitz embedding of an infinitely generated semigroup into \( (\mathcal{D}, d_p) \).

**Theorem 5.** The semigroup \( R^\infty_+ \) bi-Lipschitz embeds into \( (\mathcal{D}, d_p) \). Moreover, \( R^\infty_+ \) bi-Lipschitz embeds into \( (\text{Ker}(C), d_p) \).

## 2. Proofs

The full braid group \( B_n \) on \( n \) strings is abstractly defined via the following presentation:

\[
B_n := \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle.
\]

For a braid \( \gamma \in B_n \) denote by \( l(\gamma) \) the length of \( \gamma \) with respect to the set \( \{ \sigma_i \}_{i=1}^{n-1} \). For each \( g \in \text{Diff}_0(M, \mu) \) we denote by \( \|g\|_p := d_p(\text{Id}, g) \).
2.A. Proof of Theorem 1. Let $S$ be a finite generating set for the group $\pi_1(M,m)$, and denote by $\Pi_M: M_\bullet \to M$ the universal Riemannian covering of $M$. This means that the metric on $M_\bullet$ is induced from the Riemannian metric on $M$. The corresponding distance will be denoted by $d_\bullet$.

It is enough to show that $\Phi$ is a large scale Lipschitz map. This means that we have to show that there exist constants $A, B \geq 0$ independent of $g$ such that

$$A \cdot \|g\|_p + B \geq |\Phi(g)|.$$  

Let $g \in \text{Diff}_0(M,\mu)$ and $\{g_t\}_{t \in [0,1]} \in \text{Diff}_0(M,\mu)$ be an isotopy from the identity to $g$. It follows from the Hölder inequality that $\|g\|_p \geq C_p \cdot \|g\|_1$, where $C_p$ is some positive constant independent of $g$. Hence it is enough to prove the statement for $p = 1$.

For any homogeneous quasi-morphism $\tilde{\phi}: \pi_1(M,m) \to \mathbb{R}$ we have

$$|\tilde{\phi}(\alpha)| \leq \left(D_{\tilde{\phi}} + \max_{s \in S} |\tilde{\phi}(s)| \right) \|\alpha\|_S. \tag{4}$$

It follows that

$$|\Phi(g)| \leq K \int_M \|[g_x]\|_S \mu, \tag{5}$$

where $K = D_{\tilde{\phi}} + \max_{s \in S} |\tilde{\phi}(s)|$.

Recall that the loop $g_x$ is a concatenation of a geodesic path from $m$ to $x$, the path $\{\tilde{g}_t(x)\}$ and a geodesic path from $g(x)$ to $m$. Let $m_\bullet \in \Pi_M^{-1}(m)$ and let $\{g_{x,t}(m_\bullet)\}$ be the lift of the loop $g_x$ starting at the point $m_\bullet$. The manifold $M$ is compact, hence by the Švarc-Milnor lemma [9, 23], the inclusion of the orbit of $m_\bullet$ with respect to the deck transformation group $\pi_1(M,m)$ defines a quasi-isometry

$$\pi_1(M,m) \overset{q.i.}{\simeq} (M_\bullet, d_\bullet).$$

In particular, it means that there exist positive constants $A', B'$, such that

$$d_\bullet(m_\bullet, g_{x,1}(m_\bullet)) \geq A' \|[g_x]\|_S - B'. \tag{6}$$

Denote by $\text{diam}(M)$ the diameter of $M$. We also have the estimate

$$d_\bullet(m_\bullet, g_{x,1}(m_\bullet)) \leq 2 \text{diam}(M) + \int_0^1 |\dot{g}_t(x)| \, dt. \tag{7}$$

Combining inequalities (4), (5), (6) and (7) we get that

$$|\Phi(g)| \leq K(A')^{-1} \left( \left( \int_0^1 dt \int_M |\dot{g}_t(x)| \mu \right) + \text{vol}(M)(2 \text{diam}(M) + B') \right)$$

$$= K(A')^{-1} L_1(\{g_t\}) + K(A')^{-1} \cdot \text{vol}(M)(2 \text{diam}(M) + B').$$
Since the above inequalities hold for any isotopy \( \{g_t\}_{t \in [0,1]} \) between the identity and \( g \), we obtain that
\[
|\Phi(g)| \leq A \cdot \|g\|_p + B,
\]
where \( A = C_p \cdot K(A')^{-1} \) and \( B = K(A')^{-1} \cdot \text{vol}(M)(2 \text{diam}(M) + B') \)
and this concludes the proof. \( \square \)

2.B. Proof of Theorem 2. Let \( n \geq 2 \). It is enough to show that the non-homogeneous quasi-morphism \( \Phi_n : \mathcal{D} \to \mathbb{R} \) is large scale Lipschitz, i.e., there exist two constants \( A, B \geq 0 \), such that for every \( g \in \mathcal{D} \)
\[
|\Phi_n(g)| \leq A \|g\|_p + B.
\]

Let \( g \in \mathcal{D} \). For an isotopy \( \{g_t\} \in \mathcal{D} \) between \( \text{Id} \) and \( g \), any \( x \in X_n \) and \( 1 \leq i, j \leq n, \ i \neq j \) let \( l_{i,j} : [0,1] \to S^1 \), such that
\[
l_{i,j}(t) := \frac{g_t(x_i) - g_t(x_j)}{\|g_t(x_i) - g_t(x_j)\|} \quad \text{and} \quad L_{i,j}(x) := \frac{1}{2\pi} \int_0^1 \left\| \frac{\partial}{\partial t}(l_{i,j}(t)) \right\| dt,
\]
where \( \|\cdot\| \) is the Euclidean norm. Note that \( L_{i,j}(x) \) is the length of the path \( l_{i,j}(t) \) divided by \( 2\pi \). It follows that \( L_{i,j}(x) + 4 \) is an upper bound for the number of times the string \( i \) turns around the string \( j \) in the positive direction plus the number of times the string \( i \) turns around the string \( j \) in the negative direction in the braid \( \gamma(g;x) \). Recall that a representative of the braid \( \gamma(g;x) \) is build using any isotopy \( \{g_t\} \in \mathcal{D} \) between \( \text{Id} \) and \( g \). It follows that the number of crossings in any such representative is less then or equal to \( \sum_{i<j} 2 \left( L_{i,j}(x) + 4 \right) \). By definition the number of crossings in any such representative of the braid \( \gamma(g;x) \) is bigger than the length of the braid \( \gamma(g;x) \). Thus we get the following inequality
\[
(8) \quad \sum_{i<j} 2 \left( L_{i,j}(x) + 4 \right) \geq l(\gamma(g;x)),
\]
where \( l(\gamma(g;x)) \) is the word length of the braid \( \gamma(g;x) \). Take any finite generating set \( S \) of \( P_n \). Note that for any homogeneous quasi-morphism \( \tilde{\varphi}_n : P_n \to \mathbb{R} \) one has
\[
(9) \quad |\tilde{\varphi}_n(\gamma)| \leq \left( D_{\tilde{\varphi}_n} + \max_{s \in S} |\tilde{\varphi}_n(s)| \right) l_S(\gamma),
\]
where \( l_S(\gamma) \) is the length of a word \( \gamma \) with respect to \( S \), and \( D_{\tilde{\varphi}_n} \) is the defect of \( \tilde{\varphi}_n \). Recall that the pure braid group \( P_n \) is a normal subgroup of finite index in \( B_n \). It follows from [12, Corollary 24] that there exist two positive constants \( K_{1,S} \) and \( K_{2,S} \) independent of \( \gamma \), such that
\[
l_S(\gamma) \leq K_{1,S} \cdot l(\gamma) + K_{2,S}.
\]
It follows from (9) that
\begin{equation}
|\tilde{\varphi}_n(\gamma(g; \vec{x}))| \leq N_1 I(\gamma(g; \vec{x})) + N_2,
\end{equation}
where \(N_1 = K_1, s(D_{\tilde{\varphi}_n} + \max_{s \in S} |\tilde{\varphi}_n(s)|)\) and \(N_2 = K_2, s(D_{\tilde{\varphi}_n} + \max_{s \in S} |\tilde{\varphi}_n(s)|)\).

Inequalities (8) and (10) yield the following inequality:
\begin{equation}
|\tilde{\varphi}_n(\gamma(g; \vec{x}))| \leq 2N_1 \left( \sum_{i<j}^n L_{i,j}(\vec{x}) + 4 \right) + N_2.
\end{equation}

It follows that
\begin{equation}
|\Phi_n(g)| \leq N_3 \left( \sum_{i<j}^n L_{i,j}(\vec{x})dx_i dx_j \right) + B,
\end{equation}
where \(N_3 = 2N_1 \cdot \text{vol}((\mathbb{D}^2)^{n-2})\) and \(B = (4N_1 (n-1)n + N_2) \text{vol}(\mathbb{D}^2)^n)\).

It follows from the definition of \(L_{i,j}\) that
\begin{equation}
\sum_{i<j}^n \int_{\mathbb{D}^2 \times \mathbb{D}^2} L_{i,j}(\vec{x})dx_i dx_j = \frac{(n-1)n}{4\pi} \int_0^1 \int_{\mathbb{D}^2 \times \mathbb{D}^2} \left\| \frac{\partial}{\partial t} \left( \frac{g_t(x) - g_t(y)}{\|g_t(x) - g_t(y)\|} \right) \right\| dxdydt.
\end{equation}

Cauchy-Schwartz inequality yields
\begin{equation}
\int_0^1 \int_{\mathbb{D}^2 \times \mathbb{D}^2} \left\| \frac{\partial}{\partial t} \left( \frac{g_t(x) - g_t(y)}{\|g_t(x) - g_t(y)\|} \right) \right\| dxdydt \leq \int_0^1 \int_{\mathbb{D}^2 \times \mathbb{D}^2} \frac{4\|g_t(x)\|}{\|g_t(x) - g_t(y)\|} dxdydt.
\end{equation}

By using the polar coordinates we conclude that for each \(x \in \mathbb{D}^2\)
\begin{equation}
\int_{\mathbb{D}^2} \frac{1}{\|x - y\|} dy \leq 4\pi.
\end{equation}

Using the above inequality and the fact that the isotopy \(g_t\) is area-preserving we have
\begin{equation}
\int_0^1 \int_{\mathbb{D}^2 \times \mathbb{D}^2} \frac{\|\dot{g_t}(x)\|}{\|g_t(x) - g_t(y)\|} dxdy dt \leq 4\pi \int_0^1 \int_{\mathbb{D}^2} \|\dot{g_t}(x)\| dxdt.
\end{equation}

We combine inequalities (11), (12), (13), (14) and get
\begin{equation}
|\Phi_n(g)| \leq A \int_0^1 \int_{\mathbb{D}^2} \|\dot{g_t}(x)\| dxdt + B := A\mathcal{L}_1 \{g_t\} + B,
\end{equation}
where \(A = 4N_3 (n-1)n\). Since the above inequality holds for any isotopy \(\{g_t\}_{t \in [0,1]}\) between the identity and \(g\), we obtain that
\begin{equation}
|\Phi_n(g)| \leq A\|g\|_1 + B.
\end{equation}

The above inequality concludes the proof of the theorem in case \(p = 1\).
Let $p > 1$. It follows from Hölder inequality that there exists a positive constant $C_p$ such that $\|g\|_1 \leq C_p \|g\|_p$, and the proof follows. \hfill \square

2.C. **Proof of Theorem 3.** Suppose that the dimension of $M$ is at least 3. Hence there exists a family of disjoint simple closed curves $\{[\gamma_i']\}_{i=1}^n$ such that the curve $\gamma_i'$ is free loop homotopic to the curve $\gamma_i$. It follows from the tubular neighborhood theorem that for each $\gamma_i'$ there exists $r_i > 0$, the standard $(n-1)$-dimensional ball $B_{r_i}^{n-1} \subset \mathbb{R}^n$ of radius $r_i > 0$, and a volume-preserving embedding

$$\text{emb}_i : B_{r_i}^{n-1} \times S^1 \hookrightarrow M,$$

such that $\text{emb}_i |_{\{0\} \times S^1}$ is the curve $\gamma_i'$ and $r_i = d_M(\gamma_i', \text{emb}_i |_{\partial B_{r_i}^{n-1} \times S^1})$. Moreover, if $i \neq j$ then the images of $\text{emb}_i$ and $\text{emb}_j$ are disjoint. The volume form on the product $B_{r_i}^{n-1} \times S^1$ is the standard Euclidean volume and $d_M$ denotes the distance on $M$ induced by the Riemannian metric.

Let $r = \min_{1 \leq i \leq k} r_i$. Then $\text{emb}_i : B_{r}^{n-1} \times S^1 \hookrightarrow M$ is volume-preserving for each $i$ and $\text{emb}_i |_{\{0\} \times S^1} = \gamma_i'$. It is straightforward to construct a smooth isotopy of volume-preserving diffeomorphisms

$$g_t : B_{r}^{n-1} \times S^1 \rightarrow B_{r'}^{n-1} \times S^1$$

with $g_0 = \text{Id}$ such that:

- For each $t \in \mathbb{R}$ the diffeomorphism $g_t$ equals to the identity in the neighborhood of $\partial B_{r}^{n-1} \times S^1$, and the time-one map $g_t$ is equal to the identity on $B_{r'}^{n-1} \times S^1$, where $0 < r' < r$.

- Each diffeomorphism $g_t$ preserves the foliation of $B_{r}^{n-1} \times S^1$ by the circles $\{x\} \times S^1$. Each diffeomorphism $g_t$ preserves the orientation for $t \geq 0$. In addition, for every $x \in B_{r}^{n-1}$ the restriction $g_t : \{x\} \times S^1 \rightarrow \{x\} \times S^1$ is the rotation by $2\pi t$.

- For all $s, t \in \mathbb{R}$ we have $g_{t+s} = g_t \circ g_s$.

**Remark 2.1.** Notice that if $M$ is a symplectic manifold then the above isotopies can be constructed to be Hamiltonian.

We identify $B_{r}^{n-1} \times S^1$ with its image with respect to the embedding $\text{emb}_i$. Then we extend an isotopy $g_t$ by the identity on $M \setminus (B_{r}^{n-1} \times S^1)$ obtaining smooth isotopies $g_{t,i} \in \text{Diff}_0(M, \mu)$. Using the fact that $\tilde{\phi}_t([\gamma_j]) = \delta_{ij}$ we get

$$\tilde{\Phi}_t(g_{t,i}) := \lim_{k \to \infty} \frac{1}{k} \int_M \tilde{\phi}_t([((g_{t,i})_x)] \mu \geq \text{vol}(B_{r}^{n-1} \times S^1) > 0,$$
where the first inequality follows from the fact that \( \lim_{k \to \infty} \widetilde{\alpha}_t([(g^k_{1,i})_x]) \) is zero for \( x \in M \setminus (B_r^{n-1} \times S^1) \), non-negative for \( x \in (B_r^{n-1} \times S^1) \) and equals to 1 for \( x \in (B_r^{n-1} \times S^1) \). Similarly, in case when \( j \neq i \) we have

\[
(16) \quad \widetilde{\Phi}_t(g_{i,j}) := \lim_{k \to \infty} \frac{1}{k} \int_M \widetilde{\alpha}_t([(g^k_{1,i})_x]) \mu = 0.
\]

We define a homomorphism \( \Psi : \mathbb{R}^n \to \text{Diff}_0(M, \mu) \) by setting

\[
\Psi(v) := g_{v_1,1} \circ \ldots \circ g_{v_n,n},
\]

where \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \). Diffeomorphisms \( g_{v_1,i} \) and \( g_{v_1,j} \) have disjoint supports for \( i \neq j \) and commute for all \( 1 \leq i, j \leq n \). It follows that \( \Psi \) is a monomorphism, and in addition \( \widetilde{\Phi}_t(g_{v_1,i}) = v_t \widetilde{\Phi}_t(g_{1,i}) \) for all \( 1 \leq i \leq n \), because \( \widetilde{\Phi}_t \) is a continuous homogeneous quasi-morphism.

We claim that \( \Psi \) is a bi-Lipschitz embedding. Indeed, by Theorem 1 and by (15) and (16) the following inequalities hold for each \( 1 \leq i \leq n \):

\[
\|g_{v_1,1} \circ \ldots \circ g_{v_n,n}\|_p \geq A^{-1} \left| \widetilde{\Phi}_t(g_{v_1,1} \circ \ldots \circ g_{v_n,n}) \right| = A^{-1} \cdot |v_i| \cdot \left| \widetilde{\Phi}_t(g_{1,i}) \right|,
\]

where \( A \) is the maximum over the Lipschitz constants of the functions \( \widetilde{\Phi}_t : \text{Diff}_0(M, \mu) \to \mathbb{R} \). It follows that

\[
\|g_{v_1,1} \circ \ldots \circ g_{v_n,n}\|_p \geq \left( (n \cdot A)^{-1} \min \left| \widetilde{\Phi}_t(g_{1,i}) \right| \right) \|v\|,
\]

where \( \|v\| \) is the \( l^1 \)-norm of \( v \) in \( \mathbb{R}^n \). On the other hand

\[
\|g_{v_1,1} \circ \ldots \circ g_{v_n,n}\|_p \leq \sum_{i=1}^n \mathcal{L}_p(g_{v_i,i}) = \sum_{i=1}^n |v_i| \mathcal{L}_p(g_{1,i}) \leq \max_i \mathcal{L}_p(g_{1,i}) \|v\|,
\]

where \( \{g_{1,i}\} \) is the isotopy between Id and \( g_{1,i} \) and \( t \in [0,1] \).

Now we deal with the case when \( M = \Sigma_g \) and \( g \geq 2 \). Let us take a basis \( \{[\alpha_i], [\beta_i]\}_{i=1}^g \) of \( H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^g \) such that all \( \alpha_i, \beta_i \) are simple closed curves, the oriented intersection number \( \#([\alpha_i] \cap [\beta_i]) = 1 \), the intersection \( \alpha_i \cap \beta_i \) consists of 1 point, \( \alpha_i \cap \beta_j = \emptyset \), \( \alpha_i \cap \alpha_j = \emptyset \) and \( \beta_i \cap \beta_j = \emptyset \) for all different \( 1 \leq i, j \leq g \). Denote by \( \rho_i : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{R} \) the homomorphism which sends \( [\alpha_i] \) to 1 and all other elements to zero.

Let \( \hat{\alpha}_i : \pi_1(\Sigma_g, m) \to \mathbb{R} \) be the composition of \( \rho_i \) with the projection homomorphism \( \pi_1(\Sigma_g, m) \to H_1(\Sigma_g, \mathbb{Z}) \).

The same proof as in the previous step shows that there exists a family of smooth isotopies \( g_{t,i} \) in \( \text{Diff}_0(\Sigma_g, \mu) \) such that

- For each \( t \in \mathbb{R} \) diffeomorphisms \( g_{t,i} \) have disjoint supports for \( 1 \leq i \leq g \).

- For all \( s, t \in \mathbb{R} \) and \( 1 \leq i \leq g \) we have \( g_{t+s,i} = g_{t,i} \circ g_{s,i} \).
For each induced homomorphism $\Phi_i: \text{Diff}_0(\Sigma_g, \mu) \to \mathbb{R}$ we have $\Phi_i(g_{1,i}) > 0$ and $\Phi_i(g_{1,j}) = 0$ for all $1 \leq i, j \leq n$.

Let $\Psi: \mathbb{R}^n \to \text{Diff}_0(\Sigma_g, \mu)$ be a homomorphism such that

$$
\Psi(v) := g_{v,1} \circ \ldots \circ g_{v,g},
$$

where $v = (v_1, \ldots, v_g)$. The same proof as in the previous step shows that $\Psi$ is a bi-Lipschitz embedding.

2.D. Proof of Theorem 4. Let $\widetilde{\text{sign}}_n: \mathbb{P}_n \to \mathbb{R}$ be the homogeneous quasi-morphism defined by signature link invariant $\text{sign}$ [22] of links in $\mathbb{R}^3$. For a precise definition and properties of $\widetilde{\text{sign}}_n$ see [7, 16].

Denote by $\widetilde{\text{Sign}}_n$ the induced homogeneous quasi-morphism on $\mathcal{D}$. Let $H: \mathbb{D}^2 \to \mathbb{R}$ be a smooth function, such that $H(x) = h(\|x\|^2)$, where $h = 0$ in the neighborhood of 0 and 1, and $\int H(x)dx = 0$. Denote by $g_t$ the flow generated by $H$. Then $g_t \in \text{Ker}(C)$, because $C(g_t) = 2t \int H(x)dx$ (see [17]). In [16] it was shown that there exists $K_n > 0$, such that

$$
\widetilde{\text{Sign}}_n(g_t) = K_n \cdot t \int_0^1 y^{n-2}h(y)dy.
$$

Let $n \in \mathbb{N}$. It is straight forward to construct a family of functions $\{H_i\}_{i=1}^n$, where $H_i(x) = h_i(\|x\|^2)$, such that

- Each Hamiltonian flow $g_{t,i}$ generated by $H_i$ lies in $\text{Ker}(C)$.
- Diffeomorphisms $g_{t,i}$ and $g_{s,j}$ commute for $s, t \in \mathbb{R}$, $1 \leq i, j \leq n$.
- The matrix

  $$
  \begin{pmatrix}
  \widetilde{\text{Sign}}_3(g_{1,1}) & \cdots & \widetilde{\text{Sign}}_3(g_{1,n}) \\
  \vdots & \ddots & \vdots \\
  \widetilde{\text{Sign}}_{n+2}(g_{1,1}) & \cdots & \widetilde{\text{Sign}}_{n+2}(g_{1,n})
  \end{pmatrix}
  $$

  is non-singular.

It follows that there exists a family $\{\widetilde{\Phi}_i\}_{i=1}^n$ of homogeneous quasi-morphisms on $\mathcal{D}$, such that $\widetilde{\Phi}_i$ is a linear combination of $\widetilde{\text{Sign}}_n$'s and

$$
\widetilde{\Phi}_i(g_{t,j}) = \begin{cases} 
  t & \text{if } i = j \\
  0 & \text{if } i \neq j
\end{cases}.
$$

Let $\Psi: \mathbb{R}^n \to \mathcal{D}$ be a map, such that $\Psi(v) := g_{v,1} \circ \ldots \circ g_{v,n}$ and $v = (v_1, \ldots, v_n)$. It follows from the construction of $\{g_{v,i}\}_{i=1}^n$ that $\Psi$ is a monomorphism. The same proof as in Theorem 3 shows that there exists $A_n > 0$, such that

$$
\|g_{v,1} \circ \ldots \circ g_{v,n}\|_p \leq A_n\|v\|.
$$
Let $1 \leq i \leq n$. All diffeomorphisms $g_{v_1,1}, \ldots, g_{v_n,n}$ pair-wise commute. Hence by Theorem 2 and equality (17) we have

$$\|g_{v_1,1} \circ \ldots \circ g_{v_n,n}\|_p \geq A^{-1} \left| \Phi_i(g_{v_1,1} \circ \ldots \circ g_{v_n,n}) \right| = A^{-1} \cdot |v_i| \left| \Phi_i(g_{1,i}) \right|,$$

where $A$ is the maximum over the Lipschitz constants of the functions $\Phi_i : \mathcal{D} \to \mathbb{R}$. It follows that

$$\|g_{v_1,1} \circ \ldots \circ g_{v_n,n}\|_p \geq \left( (n \cdot A)^{-1} \min_i \left| \Phi_i(g_{1,i}) \right| \right) \|v\|,$$

and the proof follows. $\square$

2.E. Proof of Theorem 5.

**Lemma 2.2.** Suppose that there exists a homogeneous quasi-morphism $\Phi : \mathbb{R}_+^\infty \to \mathbb{R}$ and a family of diffeomorphisms $\{g_{t,i}\}_{i=1}^\infty$ in $\text{Ker}(C)$ for each $t \in \mathbb{R}$ such that

- The map $\mathbb{R}_+^\infty \to \text{Ker}(C)$, $(v_1, \ldots, v_k, 0 \ldots) \to g_{v_1,1} \circ \ldots \circ g_{v_1,k}$, is an injective homomorphism for each $k$;
- the numbers $\left\{ \Phi(g_{1,i}) \right\}_{i=1}^\infty$ have the same sign for all $i \in \mathbb{N}$;
- $\inf_i \left| \Phi(g_{1,i}) \right| \geq M_1$, $\sup_i L_p \{g_{t,i}\} \leq M_2$, where $\{g_{t,i}\}_{i \in [0,1]}$, and $M_1 > 0$ and $M_2 > 0$.

Let $\Psi : \mathbb{R}_+^\infty \to \text{Ker}(C)$ where $\Psi(v_1, \ldots, v_k, 0 \ldots) = g_{v_1,1} \circ \ldots \circ g_{v_1,k}$. Then $\Psi$ is a bi-Lipschitz embedding.

**Proof.** Let $(v_1, \ldots, v_k, 0 \ldots) \in \mathbb{R}_+^\infty$. It follows from the triangle inequality and the hypothesis that

$$\|g_{v_1,1} \circ \ldots \circ g_{v_k,k}\|_p \leq \max_{1 \leq i \leq k} \sum_{i=1}^k v_i \leq \sum_{i=1}^k \sum_{i=1}^k v_i.$$

By Theorem 2 there exists a positive constant $A$ such that

$$\|g_{v_1,1} \circ \ldots \circ g_{v_k,k}\|_p \geq A^{-1} \left| \Phi \left( g_{v_1,1} \circ \ldots \circ g_{v_k,k} \right) \right|.$$ 

Note that $\Phi \left( g_{v_1,1} \circ \ldots \circ g_{v_k,k} \right) = \sum_{i=1}^k v_i \Phi(g_{v_i,i})$ because diffeomorphisms $\{g_{v_i,i}\}$ and $\{g_{v_j,j}\}$ pair-wise commute. By hypothesis the numbers $\Phi(g_{1,i})$ have the same sign for all $i \in \mathbb{N}$. Hence

$$\|g_{v_1,1} \circ \ldots \circ g_{v_k,k}\|_p \geq A^{-1} \min_{1 \leq i \leq k} \left| \Phi(g_{1,i}) \right| \sum_{i=1}^k v_i \geq M_1 \cdot A^{-1} \sum_{i=1}^k v_i.$$

Inequalities (18) and (19) conclude the proof of the lemma. $\square$
Let us finish the proof of the theorem by constructing a family of diffeomorphisms \( \{g_{t,i}\}_{i=1}^{\infty} \in \text{Ker}(C) \) as in Lemma 2.2. Let \( \{h_s\}_{s \in [\frac{1}{4}, 1]} \) be a family of \( C^\infty \) functions from the interval \([0, 1]\) to \( \mathbb{R} \) such that:

- the support of \( h_s \) is \([\frac{1}{2} - s, \frac{1}{2} + s]\);
- \( h_s|_{(\frac{1}{2} - s, \frac{1}{2})} \) is positive and \( h_s|_{(\frac{1}{2}, \frac{1}{2} + s)} \) is negative;
- If \( s \neq s' \) then \( h_s|_{[\frac{3}{8}, \frac{5}{8}]} \equiv h_{s'}|_{[\frac{3}{8}, \frac{5}{8}]} \). If \( s > s' \) and \( y \in (\frac{3}{8}, \frac{1}{2}) \) then \( h_s(y) \geq h_{s'}(y) \), if \( s > s' \) and \( y \in (\frac{1}{2}, \frac{3}{8}) \) then \( h_s(y) \leq h_{s'}(y) \);
- For each \( s \in [\frac{1}{4}, \frac{1}{3}] \) and \( p \in \mathbb{N} \) the integral \( \int_0^1 h_s(y)dy = 0 \), the integral \( \int_0^1 yh_s(y)dy < 0 \), and both functions \( s \rightarrow \int_0^1 yh_s(y)dy \) and \( s \rightarrow \int_0^1 \left| \frac{\partial}{\partial y} h_s(y) \right|^p dy \) are continuous.

An example of such functions is shown in Figure 1.

Let us pick an arbitrary discrete set \( \{s_i\}_{i=1}^{\infty} \in (\frac{1}{4}, \frac{1}{3}) \). For each \( i \) we define a function \( G_i: \mathbb{D}^2 \rightarrow \mathbb{R} \) such that \( G_i(x) = h_{s_i}(\|x\|^2) \). We denote by \( g_{t,i} \) the flow generated by the Hamiltonian \( G_i \). Recall that \( \text{sign}_n: \mathbb{P}_n \rightarrow \mathbb{R} \) is the homogeneous quasi-morphism defined by classical signature link invariant sign, and \( \text{Sign}_n \) is the induced homogeneous quasi-morphism on \( \mathbb{D} \). It is left to show that the family \( \{g_{t,i}\}_{i=1}^{\infty} \) satisfies conditions of Lemma 2.2 for each \( t \in \mathbb{R} \).

1. Each diffeomorphism \( g_{t,i} \) lies in \( \text{Ker} C \) because \( C(g_{t,i}) = 2t \int_{\mathbb{D}^2} G_i(x)dx \) (see [17]), and \( \int_{\mathbb{D}^2} G_i(x)dx = \pi \int_0^1 h_{s_i}(y)dy = 0 \). A straightforward
computation shows that in polar coordinates we have
\[ g_{t,i}(r,\theta) = \left( r, \theta + 2t \frac{\partial}{\partial(r^2)} h_{s_i}(r^2) \right). \]

Hence the diffeomorphisms \( g_{t,i} \) and \( g_{t',j} \) commute for \( t, t' \in \mathbb{R} \) and for \( i, j \in \mathbb{N} \). It follows that the map \( \mathbb{R}_+ \to \text{Ker}(C) \) defined by \( (v_1, \ldots, v_k, 0, \ldots) \to g_{t,1} \circ \cdots \circ g_{t,k} \), is an injective homomorphism.

2. Let \( n = 3 \). There exists a positive constant \( K_3 \) such that
\[ \text{Sign}_3(g_{t,i}) = K_3 \cdot t \int_0^1 (y + 1) h_{s_i}(y) dy \]
for each \( i \), see [7, 16]. Recall that \( g_{t,i} \in \text{Ker} C \), hence
\[ \text{Sign}_3(g_{t,i}) = K_3 \cdot t \int_0^1 y h_{s_i}(y) dy. \]
By construction of functions \( h_{s_i} \) we have \( \int_0^1 y h_{s_i}(y) dy < 0 \) for each \( i \). Hence the numbers \( \left\{ \text{Sign}_3(g_{t,i}) \right\}_{i=1}^{\infty} \) have the negative sign for all \( i \).

3. Recall that for each \( s \in \left[ \frac{1}{4}, \frac{1}{3} \right] \) the function \( s \mapsto \int_0^1 y h_{s}(y) dy \) is continuous and \( \int_0^1 y h_{s}(y) dy < 0 \). It follows that there exists a constant \( M_1 > 0 \) such that \( \left| \int_0^1 y h_{s}(y) dy \right| \geq M_1 K_3^{-1} \) for each \( s \). Hence
\[ \inf_i \left| \text{Sign}_3(g_{1,i}) \right| \geq M_1. \]

4. Let \( p \in \mathbb{N} \). For each \( i \) we have
\[ \|(g_{1,i})\|_p \leq \mathcal{L}_p \{g_{t,i}\} = \int_0^1 dt \left( \int_D \left\| \dot{g}_{s,i}(x) \right\|^p dx \right)^{\frac{1}{p}} \]
\[ = 2\pi \frac{1}{p} \left( \int_0^1 y^\frac{p}{2} \left| \frac{\partial}{\partial y} h_{s_i}(y) \right|^p dy \right)^{\frac{1}{p}}. \]
Recall that for each \( s \in \left[ \frac{1}{4}, \frac{1}{3} \right] \) the function \( s \mapsto \int_0^1 y^\frac{p}{2} \left| \frac{\partial}{\partial y} h_{s}(y) \right|^p dy \) is continuous. It follows that there exists a constant \( M_2 > 0 \) such that
\[ \left( 2p \pi \int_0^1 y^\frac{p}{2} \left| \frac{\partial}{\partial y} h_{s}(y) \right|^p dy \right)^{\frac{1}{p}} \leq M_2 \]
for each \( s \). Hence
\[ \sup_i \mathcal{L}_p \{g_{t,i}\} \leq M_2. \]
This concludes the proof of the theorem. \( \square \)
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