FRAGMENTATION NORM AND RELATIVE QUASIMORPHISMS

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ABSTRACT. We prove that manifolds with complicated enough fundamental group admit measure-preserving homeomorphisms which have positive stable fragmentation norm with respect to balls of bounded measure.

1. INTRODUCTION

Homeomorphisms of a connected manifold M can be often expressed as compositions of homeomorphisms supported in sets of a given cover of M. This is known as the *fragmentation property*. Let $\mathrm{Homeo}_0(M, \mu)$ be the identity component of the group of compactly supported measure-preserving homeomorphisms of M. In this paper we are interested in groups $G \subseteq \mathrm{Homeo}_0(M, \mu)$ consisting of homeomorphisms which satisfy the fragmentation property with respect to topological balls of measure at most one. When G is a subgroup of the identity component $\mathrm{Diff}_0(M, \mu)$ of the group of compactly supported volume-preserving diffeomorphisms, we consider the fragmentation property with respect to smooth balls of volume at most one. Given $f \in G$, its *fragmentation norm* $\|f\|_{\mathrm{FRAG}}$ is defined to be the smallest n such that $f = g_1 \cdots g_n$ and each g_i is supported in a ball of measure at most one. Thus the fragmentation norm is the word norm on G associated with the generating set consisting of maps supported in balls as above. We are also interested in the stable fragmentation norm defined by $\lim_{k \to \infty} \dfrac{\|f^k\|}{k}$. The existence of an element with positive stable fragmentation norm implies that the diameter of the fragmentation norm is infinite. We say that the fragmentation norm on G is *stably unbounded* if G has an element with positive stable fragmentation norm. In general, a group G is called stably unbounded if it admits a stably unbounded conjugation invariant norm; see Section 2 for details.

The main result of the paper (Theorem 2.2) provides conditions under which the existence of an essential homogeneous quasimorphism (i.e., a quasimorphism which is not a homomorphism) on the fundamental group of M implies

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the existence of an element of G with a positive stable fragmentation norm. Then we specify this abstract result to the following concrete cases.

Standing assumption. Throughout the paper we consider manifolds M such that the evaluation map $\mathrm{ev}_x\colon \mathrm{Homeo}_0(M,\mu) \to M$, given by $\mathrm{ev}_x(f) = f(x)$ induces the trivial homomorphism on fundamental groups, where the group $\mathrm{Homeo}_0(M,\mu)$ is the identity component of the group of compactly supported measure-preserving homeomorphisms of M. The only exception is Section 4.3.1 where we consider general closed symplectic manifolds. The assumption is satisfied if, for example, M is either non-compact and connected or if the center of the fundamental group $\pi_1(M)$ is trivial.

Theorem 1.1 (Homeomorphisms). *Let M be a complete Riemannian manifold. Let μ be the measure whose value on any smooth bounded set equals to its volume. Let G be the kernel of the flux homomorphism $\mathrm{Homeo}_0(M,\mu) \to H_1(M,\mathbf{R})$. If $\pi_1(M)$ admits an essential quasimorphism then the fragmentation norm on G is stably unbounded.*

Remark 1.2. Let $[\mathrm{Homeo}_0(M,\mu), \mathrm{Homeo}_0(M,\mu)]$ be the commutator subgroup of the identity component of the group of compactly supported measure-preserving homeomorphisms of M. If $\dim M \geq 3$, then the commutator subgroup $[\mathrm{Homeo}_0(M,\mu), \mathrm{Homeo}_0(M,\mu)]$ is equal to the kernel of flux, see [11, Main Theorem]. If $\dim M = 2$, then there is only an inclusion and the equality is an open problem to the best of our knowledge. For more information about the flux homomorphism see [1, Section 3].

Theorem 1.3 (Diffeomorphisms). *Let M be a complete Riemannian manifold equipped with a volume form μ. Let G be the commutator subgroup of the identity component $\mathrm{Diff}_0(M,\mu)$ of the group of compactly supported volume-preserving diffeomorphisms of M. If $\pi_1(M)$ admits an essential quasimorphism then the fragmentation norm on G is stably unbounded.*

Theorem 1.4. *Let (M,ω) be a symplectic manifold (satisfying the standing assumption as above). If $\pi_1(M)$ admits an essential quasi-morphism then the fragmentation norm on the group $\mathrm{Ham}(M,\omega)$ of compactly supported Hamiltonian diffeomorphisms of M is stably unbounded.*

Remark 1.5. Theorem 1.4 holds true when (M,ω) is either a general open symplectic manifold or a closed symplectic manifold satisfying the standing assumption. The statement also remains true for a general closed symplectic manifold with a slightly modified fragmentation norm. We discuss the details in Section 4.3.1.

Examples. The following are simple examples of manifolds for which the stable unboundedness of the fragmentation norm is a new result.

Example 1.6. Let $M = \mathbf{R}^3 \setminus (L_1 \cup L_2)$, where L_i are disjoint lines. Then the commutator subgroup of $\mathrm{Diff}_0(M, \mu)$ has stably unbounded fragmentation norm, and in particular is stably unbounded. $\diamondsuit$

Example 1.7. Let $M = \mathbf{R}^4 \setminus (P_1 \cup P_2)$, where P_i are disjoint planes. Suppose that M is equipped with the standard symplectic form ω induced from $\mathbf{R}^4$. Then $\mathrm{Ham}(M, \omega)$ has stably unbounded fragmentation norm. $\diamondsuit$

Example 1.8. Let M be the Klein Bottle with a point removed equipped with a measure from Theorem 1.1. Then for $k > 0$ the group of compactly supported measure-preserving homeomorphisms generated by maps supported in balls of area at most k has stably unbounded fragmentation norm. $\diamondsuit$

Remark 1.9. Lanzat [16] and Monzner-Vichery-Zapolsky [18] showed that the fragmentation norm on groups of Hamiltonian diffeomorphisms of certain non-compact symplectic manifolds is stably unbounded. Stable unboundedness of the Hofer norm on the group $\mathrm{Ham}(\mathbf{R}^2 \setminus \{x, y\})$ was proved by Polterovich-Sieburg [20]. Stable unboundedness of the fragmentation norm on the group $\mathrm{Ham}(\mathbf{R}^2 \setminus \{x, y\})$ follows from the results from Monzner-Vichery-Zapolsky [18], as well as from our Theorem 1.4. For M compact Theorem 1.3 was proven by Polterovich, see [19, Section 3.7].

Remark 1.10. Unboundedness of a group is usually proven with the use of unbounded quasimorphisms. When M is not of finite measure it is not known whether its groups of transformations admit unbounded quasimorphisms. Our proofs involve a construction of relative quasimorphisms on groups in question. We would like to note that several constructions of relative quasimorphisms in symplectic geometry appeared in [9, 18, 15].

2. PRELIMINARIES

2.1. **Definitions.** Let G be a group. A function $\nu \colon G \to [0, \infty)$ is called a *conjugation-invariant norm* if it satisfies the following conditions for all $g, h \in G$:

  (1) $\nu(g) = 0$ if and only if $g = 1_G$

  (2) $\nu(g) = \nu(g^{-1})$

(3) $\nu(gh) \leq \nu(g) + \nu(h)$

(4) $\nu(ghg^{-1}) = \nu(h)$.

Let $\psi \colon G \to \mathbf{R}$ be a function. The *stabilization* of ψ is a function $\overline{\psi} \colon G \to \mathbf{R}$ defined by

$$\overline{\psi}(g) = \lim_{n \to \infty} \frac{\psi(g^n)}{n},$$

provided that the above limit exists for all $g \in G$. Note that for a norm ν its stabilization always exists because ν is a non negative subadditive function.

A norm ν is called *stably unbounded* if there exists $g \in G$ with positive stabilization: $\overline{\nu}(g) > 0$. A group G is called (stably) *unbounded* if it admits a (stably) unbounded conjugation-invariant norm. For more information about these notions see [7].

A function $\psi \colon G \to \mathbf{R}$ is called a *quasimorphism* if there exists a real number $C \geq 0$ such that

$$|\psi(gh) - \psi(g) - \psi(h)| \leq C$$

for all $g, h \in G$. The infimum of such C's is called the *defect* of ψ and is denoted by D_ψ. A quasimorphism ψ is called *homogeneous* if

$$\psi(g^k) = k\psi(g)$$

for all $k \in \mathbf{Z}$ and all $g \in G$. The stabilization of a quasimorphism exists and is a homogeneous quasimorphism [8]. A homogeneous quasimorphism is called *essential* if it is not a homomorphism.

A function $\psi \colon G \to \mathbf{R}$ is called a *relative quasimorphism* with respect to a conjugation-invariant norm ν if there exists a positive constant C such that for all $g, h \in G$

$$|\psi(gh) - \psi(g) - \psi(h)| \leq C \min\{\nu(g), \nu(h)\}.$$

For more information about quasimorphisms and their connections to different branches of mathematics, see [8].

2.2. **The setup and assumptions.**

(1) Let M be a smooth complete Riemannian manifold. Let $\mathcal{B}$ be the set of all subsets $B \subset M$ which are homeomorphic to the n-dimensional Euclidean unit open ball

$$B^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$$

and of measure at most one:

$$\mathcal{B} = \{B \subset M \mid B \cong B^n \text{ and } \mu(B) \leq 1\}.$$

Notice that the group of all measure-preserving homeomorphisms of M acts on the set $\mathcal{B}$. Let $\mathcal{B}'$ be the set of all subsets $B \subset M$ which are diffeomorphic to the n-dimensional Euclidean unit open ball $B^n \subset \mathbf{R}^n$ and of volume at

most one. Analogously, the group of all volume-preserving diffeomorphisms of M acts on the set $\mathcal{B}'$.

(2) Let G be a subgroup of $\mathrm{Homeo}_0(M, \mu)$. We assume that G has the *fragmentation property* with respect to the family $\mathcal{B}$ (respectively the family $\mathcal{B}'$ if it is a subgroup of $\mathrm{Diff}_0(M, \mu)$). This means that for every $f \in G$ there exist $g_1, \cdots, g_n \in G$ such that

(1) $f = g_1 \cdots g_n$ and

(2) $\mathrm{supp}(g_i) \subset B_{g_i} \in \mathcal{B}$ (respectively $\mathcal{B}'$).

(3) The *fragmentation norm* on G associated with $\mathcal{B}$ (respectively $\mathcal{B}'$) is defined by
$$\|f\|_{\mathrm{FRAG}} = \min\{n \in \mathbf{N} \mid f = g_1 \cdots g_n\},$$
where g_i's are as in the previous item. Notice that it is a conjugation invariant norm.

(4) Let $z \in M$ be a base-point. For each $x \in M$ let $\gamma_x \colon [0, 1] \to M$ be a shortest geodesic path from z to x. Such a geodesic is unique for all x away from the cut-locus (which is a set of measure zero [21, Lemma III.4.4]) and on this set we also have that γ_x depend continuously on x. Moreover, the paths γ_x are of bounded lengths on compact subsets, i.e., for each compact subset $K \subset M$ there exists a constant $C_K > 0$ such that $\ell(\gamma_x) \leq C_K$ for every $x \in K$. Here $\ell(\gamma)$ denotes the Riemannian length of a path γ.

(5) Let μ be a Lebesgue measure on M, i.e., a measure whose value on any smooth bounded set is equal to its volume, and let $G \subseteq \mathrm{Homeo}_0(M, \mu)$ be a subgroup of the identity component of the group of compactly-supported measure-preserving homeomorphisms of M.

(6) Let $\mathrm{ev}_z \colon G \to M$ be the *evaluation* map defined by $\mathrm{ev}_z(f) = f(z)$. We say that G has *trivial evaluation* if the homomorphism
$$\mathrm{ev}_{z*} \colon \pi_1(G, 1) \to \pi_1(M, z)$$
is trivial. For example, this is the case if $\pi_1(M, z)$ has trivial center because the image of ev_{z*} lies in the center of $\pi_1(M, z)$, see e.g. [13]. Observe that the above triviality condition is independent of the choice of the basepoint. It follows that if M is non-compact then the above condition is always satisfied.

(7) Let $\psi \colon \pi_1(M, z) \to \mathbf{R}$ be a nontrivial homogeneous quasimorphism. Let $f \in G$, and let $\{f_t\}$ be an isotopy from the identity to f. We take $f_t \in \mathrm{Homeo}_0(M, \mu)$. Now we define $\Psi \colon G \to \mathbf{R}$ by
$$\Psi(f) = \int_M \psi([f_x])\mu,$$
where f_x is a loop represented by the concatenation $\gamma_x \cdot \{f_t(x)\} \cdot \overline{\gamma_{f(x)}}$. Note that if $\mathrm{Homeo}_0(M, \mu)$ has trivial evaluation then $[f_x]$ does not depend on the isotopy $\{f_t\}$. Hence in this case $[f_x]$ also does not depend on the isotopy

$\{f_t\}$. Thus in order to show that $\Psi(f)$ is well-defined it is enough to show that $\Psi(f) < \infty$ is for each f. Note that this construction appeared before in case M is compact and $G = \mathrm{Diff}_0(M, \mu)$ in [19].

Proposition 2.1. *The function* $\Psi \colon G \to \mathbf{R}$ *is well-defined.*

Proof. Let $f \in G$. It is enough to show that the set $\{[f_x]\}_{x \in M}$ is finite. Let $\{f_t\} \in \mathrm{Homeo}_0(M, \mu)$ be an isotopy from the identity to f. The union of the supports $\bigcup_{t \in [0,1]} \mathrm{supp}(f_t)$ is a compact subset of M. Recall that M admits a complete Riemannian metric. Hence there exists $r > 0$ such that the geodesic ball $B_r(z)$ of radius r centered at z contains $\bigcup_{t \in [0,1]} \mathrm{supp}(f_t)$. Note that for each $x \in M \setminus B_r(z)$ the element $[f_x]$ is trivial in $\pi_1(M, z)$. Hence it is enough to show that the set $\{[f_x]\}_{x \in B_r(z)}$ is finite in $\pi_1(B_r(z), z)$.

The ball $B_r(z)$ is compact, so we cover it with finite number of balls B_i, where each $B_i \in \mathcal{B}$. Then f can be written as a product of measure-preserving homeomorphisms h_i such that the support of h_i lies in B_i. Since M is a smooth manifold, for each i there exits a smooth ball B_i', such that it is ϵ-close to B_i and such that it is ϵ-homotopic to B_i, see smooth approximation theorem [5, Theorem 2.11.8]. Note that $[f_x]$ satisfies a cocycle condition. It means that

$$[f_x] = [(h_1 \circ \ldots \circ h_n)_x] = [(h_1)_{(h_2 \circ \ldots \circ h_n)(x)}] \ldots [(h_n)_x].$$

Let r' be such that $B_{r'}(z)$ contains all balls B_i, B_i' and the the images of ϵ-homotopies between them. Hence it is enough to prove that the set $\{([(h_i)_x]\}_{x \in B_i}$ is finite in $\pi_1(B_{r'}(z), z)$.

The ball B_i' is smooth, thus it has finite diameter d_i. The group of measure-preserving homeomorphisms of a ball is connected. Every path inside B_i can be free ϵ-homotoped to a path in B_i' and hence to a path whose Riemannian length is less than the diameter d_i. Thus $[(h_i)_x]$ can be represented by a path whose Riemannian length is less than $d_i + 2(r_i + \epsilon)$, where r_i is a radius of a geodesic ball $B_{r_i}(z)$ which contains B_i. By the Milnor-Schwarz lemma [6] the word length of $[(h_i)_x]$ is bounded in $\pi_1(B_{r'}(z), z)$ and we are done. $\square$

2.3. **The main technical result.** Before stating the theorem we summarize the assumptions we need.

Assumptions:

- M is an n-dimensional complete Riemannian manifold.

- μ is a Lebesgue measure on M.

- $\mathcal{B}$ is the set of topological balls in M of measure at most 1.

- $\mathcal{B}'$ is the set of smooth balls in M of measure at most 1.

- $\mathrm{Homeo}_0(M, \mu)$ admits a trivial evaluation.

- $G \subseteq \mathrm{Homeo}_0(M, \mu)$, G is not a subgroup of $\mathrm{Diff}_0(M)$, G has fragmentation property with respect to $\mathcal{B}$.

- $G \subseteq \mathrm{Diff}_0(M, \mu)$, G has fragmentation property with respect to $\mathcal{B}'$.

Theorem 2.2. *Let M, μ, $\mathcal{B}$, $\mathcal{B}'$ and G be as above. If there exists a homogeneous quasimorphism $\psi \colon \pi_1(M) \to \mathbf{R}$ such that the homogenization $\overline{\Psi}$ is nonzero then the fragmentation norm on G is stably unbounded.*

In order to apply Theorem 2.2 to a concrete group G we need to verify that the function $\overline{\Psi} \colon G \to \mathbf{R}$ is nonzero. It is done with the use of various *push* maps. These applications are presented in Section 4.

3. Proof of Theorem 2.2

Lemma 3.1. *The function $\Psi \colon G \to \mathbf{R}$ is a relative quasimorphism with respect to the fragmentation norm.*

Proof. Let $f, g \in G$ and let $\{f_t\}$ and $\{g_t\}$ be isotopies from the identity to $f = f_1$, and to $g = g_1$ respectively. Recall that we can take both isotopies f_t and g_t in $\mathrm{Homeo}_0(M, \mu)$ and not in G. Denote by $\{f_t\} * \{g_t\}$ the concatenation of $\{g_t\}$ and $\{f_t \circ g\}$, so that it is an isotopy from the identity to fg. In what follows these isotopies are used to represent $[f_{g(x)}]$, $[g_x]$ and $[(fg)_x]$ respectively. Let us discuss several cases:

(1) If $x \notin \bigcup_{t \in [0,1]} \mathrm{supp}(g_t)$, then $[(fg)_x] = [f_{g(x)}]$ and $[g_x]$ is the identity element of $\pi_1(M, z)$. It follows that for all such x we have
$$\psi([(fg)_x]) - \psi([f_{g(x)}]) - \psi([g_x]) = 0.$$

(2) If $x \notin \bigcup_{t \in [0,1]} g^{-1}(\mathrm{supp}(f_t))$, then $[(fg)_x] = [g_x]$ and $[f_{g(x)}]$ is the identity element of $\pi_1(M, z)$. It follows that for all such x we have
$$\psi([(fg)_x]) - \psi([f_{g(x)}]) - \psi([g_x]) = 0.$$

Thus for every
$$x \notin \left(\bigcup_{t \in [0,1]} \mathrm{supp}(g_t) \right) \bigcap \left(\bigcup_{t \in [0,1]} g^{-1}(\mathrm{supp}(f_t)) \right)$$
we have that
$$\psi([(fg)_x]) - \psi([f_{g(x)}]) - \psi([g_x]) = 0.$$

Note that by result of Fathi [11, Proposition 3.8] the group $\mathrm{Homeo}(B, \mu)$ is connected where B is a topological ball. Let $\|g\|_{\mathrm{FRAG}} = k$ and $\|f\|_{\mathrm{FRAG}} = m$. It follows that $g = g_1 \circ \ldots \circ g_k$ where each g_i is supported in a ball $B_{i,g}$, and $f = f_1 \circ \ldots \circ f_m$ where each f_i is supported in a ball $B_{i,f}$. Let $g_{i,t}$ and $f_{i,t}$ be the isotopies from the identity to g_i and f_i respectively, such

that they are supported in $B_{i,g}$ and $B_{i,f}$ respectively. These isotopies lie in $\mathrm{Homeo}_0(M, \mu)$. We choose g_t to be a concatenation of isotopies $g_{i,t}$, and f_t to be a concatenation of isotopies $f_{i,t}$ respectively. Hence

$$\mu\Big(\bigcup_{t\in[0,1]}\mathrm{supp}(g_t)\Big) \leq \|g\|_{\mathrm{FRAG}} \quad\text{and}\quad \mu\Big(\bigcup_{t\in[0,1]}g^{-1}(\mathrm{supp}(f_t))\Big) \leq \|f\|_{\mathrm{FRAG}}.$$

Let $U = \Big(\bigcup_{t\in[0,1]}\mathrm{supp}(g_t)\Big)\bigcap\Big(\bigcup_{t\in[0,1]}g^{-1}(\mathrm{supp}(f_t))\Big)$. The two inequalities above imply that

$$\mu(U) \leq \min\{\|f\|_{\mathrm{FRAG}}, \|g\|_{\mathrm{FRAG}}\}$$

and we obtain that

$$\begin{aligned}
\big|\Psi(fg) - \Psi(f) - \Psi(g)\big| &\leq \int_M \Big|\psi([(fg)_x]) - \psi([f_{g(x)}]) - \psi([g_x])\Big|\mu \\
&\leq \int_U D_\psi\,\mu \\
&\leq D_\psi \min\{\|f\|_{\mathrm{FRAG}}, \|g\|_{\mathrm{FRAG}}\},
\end{aligned}$$

where the first inequality follows from the fact that $[(fg)_x] = [f_{g(x)}][g_x]$ and D_ψ is the defect of the quasimorphism ψ. This shows that Ψ is a relative quasi-morphism with respect to the fragmentation norm. $\qquad\square$

Lemma 3.2. *Let $\Psi\colon G \to \mathbf{R}$ be the function defined in Section 2.2 (4). Its homogenization $\overline{\Psi}\colon G \to \mathbf{R}$ is well defined and invariant under conjugations.*

Proof. If M is a closed manifold then Polterovich proved that $\overline{\Psi}$ is a homogeneous quasi-morphism [19, Section 3.7]. In particular, it is invariant under conjugation. Since $\overline{\Psi}$ is evaluated on a compactly supported homeomorphism, there exists a finite measure subset of M containing this support and the base-point and an argument of Polterovich implies the statement. $\qquad\square$

Lemma 3.3. *The homogenization $\overline{\Psi}$ is Lipschitz with respect to the fragmentation norm. More precisely,*

$$\big|\overline{\Psi}(f)\big| \leq 3\,D_\psi\,\|f\|_{\mathrm{FRAG}},$$

for every $f \in G$.

Proof. Let $f \in G$ be such that $\|f\|_{\mathrm{Frag}} = n$. This means that there exist $g_1, \ldots, g_n \in G$ such that $f = g_1 \cdots g_n$ and each g_i is supported in a ball of measure at most one. Then for $n > 1$ we have

$$\begin{aligned}
\big|\Psi(f) - \Psi(g_1) - \ldots - \Psi(g_n)\big| &\leq \sum_{i=1}^{n-1}\Big|\Psi(g_1\cdots g_{i+1}) - \Psi(g_1\cdots g_i) - \Psi(g_{i+1})\Big| \\
&\leq \sum_{i=1}^{n-1} D_\psi < nD_\psi,
\end{aligned}$$

where the second inequality comes from the chain of inequalities at the end of the proof of Lemma 3.1. It follows from the adaptation of the proof of Lemma 2.21 in [8] to our case that for each $g \in G$ one has

$$|\overline{\Psi}(g) - \Psi(g)| \leq D_\psi \, \mu(\mathrm{supp}(g)) \leq D_\psi \|g\|_{\mathrm{Frag}}.$$

This leads to the following inequalities

$$|\overline{\Psi}(f) - \overline{\Psi}(g_1) - \ldots - \overline{\Psi}(g_n)|$$

$$\leq |\overline{\Psi}(f) - \Psi(f)| + \sum_{i=1}^{n} |\overline{\Psi}(g_i) - \Psi(g_i)| + |\Psi(f) - \Psi(g_1) - \ldots - \Psi(g_n)|$$

$$\leq D_\psi \|f\|_{\mathrm{Frag}} + \sum_{i=1}^{n} D_\psi + nD_\psi = 3nD_\psi.$$

Recall that each g_i is supported in a ball. By result of Fathi [11, Proposition 3.8] the group $\mathrm{Homeo}(B, \mu)$ is connected where B is a topological ball. It follows that there exists a measure preserving isotopy $g_{t,i}$ from the identity to g_i which is supported in a ball. If this ball is smooth, then for each $x \in M$ element $[(g_i^p)_x] \in \pi_1(M, z)$ can be represented by a path whose Riemannian length is bounded by some constant independent of p, see proof of Proposition 2.1. If this ball is not smooth, then there exists a smooth ball which is ϵ-homotopic to this ball, see [5, Theorem 2.11.8], and hence the same statement holds by an argument from the proof of Proposition 2.1. It follows from the Milnor-Schwarz lemma that for each i

$$(1) \qquad \overline{\Psi}(g_i) = \int_M \lim_{p \to \infty} \frac{\psi([(g_i^p)_x])}{p} \mu = 0.$$

Combining (1) with previous inequalities we get

$$(2) \qquad |\overline{\Psi}(f)| \leq 3nD_\psi = 3D_\psi \|f\|_{\mathrm{FRAG}}$$

which shows that $\overline{\Psi}$ is Lipschitz with respect to the fragmentation norm. $\square$

Proof of Theorem 2.2. Recall that the hypothesis says that there exists a non-trivial homogeneous quasimorphism $\psi \colon \pi_1(M) \to \mathbf{R}$ such that the relative quasimorphism $\overline{\Psi}$ is nontrivial. Take $f \in G$ such that $\overline{\Psi}(f) \neq 0$.

Now (2) yields

$$\lim_{k \to \infty} \frac{\|f^k\|_{\mathrm{Frag}}}{k} \geq \lim_{k \to \infty} \frac{|\overline{\Psi}(f^k)|}{3kD_\psi} = \lim_{k \to \infty} \frac{k|\overline{\Psi}(f)|}{3kD_\psi} = \frac{|\overline{\Psi}(f)|}{3D_\psi} > 0,$$

which proves that the fragmentation norm is stably unbounded. $\square$

4. Applications of Theorem 2.2

4.1. The commutator subgroup of the group of homeomorphisms.
Consider $\mathbf{S}^1 \times \mathbf{D}^{n-1}$, where $\mathbf{D}^{n-1}$ is the closed $(n-1)$-dimensional Euclidean disc of radius $1 + \epsilon$, where $\epsilon > 0$ is an arbitrarily small number. Let $\varphi_s \colon \mathbf{D}^{n-1} \to \mathbf{R}$, for $s \in [0,1]$ be a family of smooth functions supported away from the boundary such that it is equal to s on the disc of radius 1. Let $f_s \colon \mathbf{S}^1 \times \mathbf{D}^{n-1} \to \mathbf{S}^1 \times \mathbf{D}^{n-1}$ be defined by $f_s(t, z) = (t + \varphi_s(z), z)$; here $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. It is straightforward to verify that f_s preserves the standard product Lebesgue measure. Let $\gamma \colon \mathbf{S}^1 \to M$ be an embedded loop and let $\widehat{\gamma} \colon \mathbf{S}^1 \times \mathbf{D}^{n-1} \to M$ be a measure preserving embedding such that $\widehat{\gamma}(t,0) = \gamma(t)$. Here we need to assume that with respect to the product measure the disk $\mathbf{D}^{n-1}$ has sufficiently small measure. We define the associated push-map $f_{\widehat{\gamma}} \colon M \to M$ by

$$f_{\widehat{\gamma}}(x) = \begin{cases} \widehat{\gamma} \circ f_1 \circ \widehat{\gamma}^{-1}(x) & \text{for } x \in \widehat{\gamma}\left(\mathbf{S}^1 \times \mathbf{D}^{n-1}\right) \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

Changing the parameter s defines an isotopy from the identity to $f_{\widehat{\gamma}}$ through measure preserving homeomorphisms. More precisely, the isotopy is defined by

$$f_{\widehat{\gamma},s}(x) = \begin{cases} \widehat{\gamma} \circ f_s \circ \widehat{\gamma}^{-1}(x) & \text{for } x \in \widehat{\gamma}\left(\mathbf{S}^1 \times \mathbf{D}^{n-1}\right) \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

A similar construction appears in Fathi [11].

Let $\mathbf{D}_1 \subset \mathbf{D}^{n-1}$ denote the $n-1$ dimensional disc of radius 1 centered at zero. It follows that for each $x \in \widehat{\gamma}\left(\mathbf{S}^1 \times \mathbf{D}_1\right)$ we get $[(f_{\widehat{\gamma}}^p)_x] = \gamma^p$, where we view γ as an element of $\pi_1(M, z)$. For each $x \in \widehat{\gamma}\left(\mathbf{S}^1 \times (\mathbf{D}^{n-1} \setminus \mathbf{D}_1)\right)$ we get

$$[(f_{\widehat{\gamma}}^p)_x] = \alpha_{f,x,p} \cdot \gamma^{p_{f,x}} \cdot \alpha'_{f,x,p},$$

where $|p_{f,x}| < |p|$, and the word length of $\alpha_{f,x,p}$ and $\alpha'_{f,x,p}$ is bounded by a constant which is independent of p, see e.g. [4, Section 2.D.1]. This yields

$$\left|\overline{\Psi}(f_{\widehat{\gamma}}) - \psi(\gamma)\mu(\mathrm{supp}(f_{\widehat{\gamma}}))\right| \leq |\psi(\gamma)|\mu\left(\mathbf{S}^1 \times (\mathbf{D}^{n-1} \setminus \mathbf{D}_1)\right)$$

It follows that we may choose the constant ϵ suitably small so that the value $\overline{\Psi}(f_{\widehat{\gamma}})$ is arbitrarily close to $\psi(\gamma)\mu(\mathrm{supp}(f_{\widehat{\gamma}}))$.

Let G be the kernel of the flux homomorphism. It has the fragmentation property by [11, Theorem A.6.5]. Theorem 1.1 is a consequence of the following result.

Proposition 4.1. *If $\psi \colon \pi_1(M) \to \mathbf{R}$ is an essential quasimorphism then the fragmentation norm on G is stably unbounded.*

Proof. For simplicity we denote elements of the fundamental group $\pi_1(M, z)$ and their representing loops by the same Greek letters. Since ψ is an essential

quasimorphism, there exist $\alpha, \beta \in \pi_1(M)$ such that
$$\left|\psi(\alpha) - \psi(\alpha\beta) + \psi(\beta)\right| = a > 0.$$
Let $\alpha, \beta \colon \mathbf{S}^1 \to M$ be embedded based loops representing the above elements of the fundamental group. If $\dim M \geq 3$ then such loops exists for all elements of the fundamental group for dimensional reasons. If $\dim M = 2$ then if $\pi_1(M)$ admits an essential quasimorphism then $\pi_1(M)$ is either non-abelian free or the surface group of higher genus. In this case $\pi_1(M)$ has abundance of essential quasimorphisms [10] and we can choose ψ and embedded loops α, β which satisfy the above requirement.

Consider the push maps $f_{\widehat{\alpha}}, f_{\widehat{\beta}}$ such that their support have equal measures and $\overline{\Psi}(f_{\widehat{\alpha}})$ is arbitrarily close to $\psi(\alpha)\mu(\mathrm{supp}(f_{\widehat{\alpha}}))$ and similarly for $f_{\widehat{\beta}}$.

Claim 4.2. *With the above notation we have that*
$$|\overline{\Psi}(f_{\widehat{\alpha}}) - \overline{\Psi}(f_{\widehat{\alpha}}f_{\widehat{\beta}}) + \overline{\Psi}(f_{\widehat{\beta}})| \geq a\mu(\mathrm{supp}(f_{\widehat{\alpha}}) \cap \mathrm{supp}(f_{\widehat{\beta}})) - \delta > 0,$$
where $\delta > 0$ is a constant which can be made arbitrarily small by a suitable choice of the push maps.

The proof of the above claim is straightforward and relies on the observation that the subsets of the supports of the push maps where they vary between the identity and the rotation can be made arbitrarily small. It is similar to the proof of Lemma 3.1 and similar arguments are presented in [2, 14].

It follows that the relative quasimorphism $\overline{\Psi} \colon \mathrm{Homeo}_0(M, \mu) \to \mathbf{R}$ is nonzero and it is not a homomorphism. Hence it must be nontrivial on the commutator subgroup $[\mathrm{Homeo}_0(M, \mu), \mathrm{Homeo}_0(M, \mu)]$ because a relative quasimorphism is a genuine quasimorphism when restricted to homeomorphisms supported on any fixed subset of finite measure. Nontriviality of a quasimorphism which is not a homomorphism on the commutator subgroup is straightforward. Since $[\mathrm{Homeo}_0(M, \mu), \mathrm{Homeo}_0(M, \mu)] < G$, the statement follows from a direct application of Theorem 2.2. $\qquad\square$

Remark 4.3. Let $\dim M = 2$. Then $[\mathrm{Homeo}_0(M, \mu), \mathrm{Homeo}_0(M, \mu)] < G$, and it is not known whether it admits fragmentation property. However, it admits an induced fragmentation norm from G. The proof of Proposition 4.1 shows that this norm is stably unbounded.

4.2. **The commutator subgroup of** $\mathrm{Diff}_0(M, \mu)$**.** The fragmentation property is due to Thurston; see Banyaga [1, Lemma 5.1.2]. The proof of Theorem 1.3 is analogous to the above proof for homeomorphisms in the sense that the push map is obtained from the same maps
$$f_s \colon \mathbf{S}^1 \times \mathbf{D}^{n-1} \to \mathbf{S}^1 \times \mathbf{D}^{n-1}$$
which are transplanted to M via differentiable maps. Then the application of Proposition 4.1 is the same.

4.3. **The group of Hamiltonian diffeomorphisms.** The fragmentation property is due to Banyaga [1, page 110]. Here the strategy is the same but the construction of the push map needs more care. This is done as follows.

Proof of Theorem 1.4. Let $T = \mathbf{S}^1 \times [-1, 1] \times \mathbf{D}^{2n-2}$ be equipped with the product of an area form on the annulus and the standard symplectic form on the disc. The coordinate on $\mathbf{S}^1$ is denoted by x, on $[-1, 1]$ by y and a point in the disc by z. So the symplectic form is $dx \wedge dy + \omega_0$. Let $\varphi \colon \mathbf{D}^{2n-2} \to \mathbf{R}$ be a non-negative function supported in the interior of the disc and equal to 1 on a disc of radius arbitrarily close to the radius of $\mathbf{D}^{2n-2}$. Let $f \colon [-1, 1] \to \mathbf{R}$ be a smooth function supported in the interior of the interval $[-1, 1]$. Let $H \colon T \to \mathbf{R}$ be defined by $H(x, y, z) = yf(y)\varphi(z)$. Then

$$dH = \varphi(f + yf')dy + yf\,d\varphi$$

and the corresponding Hamiltonian vector field is given by

$$X_H(x, y, z) = \varphi(z)(f(y) + yf'(y))\partial_x + Z(y, z),$$

where Z is a suitable vector field on the disc which depends on y. If

$$f(y) = \begin{cases} e^{-\frac{1}{1-y^2}} & y \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

is the standard bell-shaped function then the equation $f(y) + yf'(y) = 0$ has two solutions $\pm y_0 = \pm\sqrt{2 - \sqrt{3}}$ in the interval $[-1, 1]$ and we have that $f(y) + yf'(y) > 0$ for $y \in (-y_0, y_0)$. We restrict the vector field X_H to the subset $\mathbf{S}^1 \times (-y_0, y_0) \times \mathbf{D}^{2n-2}$ and extend it by zero to the rest of T. Notice that this vector field is symplectic but not Hamiltonian and it points in the non-negative direction of ∂_x.

The rest of the proof is the same as in the case of homeomorphisms. That is, we choose the classes $\alpha, \beta \in \pi_1(M)$ for which

$$\psi(\alpha\beta) \neq \psi(\alpha) + \psi(\beta).$$

Then we choose their embedded representatives and choose their small tubular neighborhoods symplectically diffeomorphic to T with possibly rescaled summands of the symplectic form ([17, Exercise 3.37]. We transplant the above symplectic flows to create $f_{\widehat{\alpha}}, f_{\widehat{\beta}} \in \mathrm{Symp}_0(M, \omega)$ and the same argument shows that $\overline{\Psi}$ is nontrivial and that it is not a homomorphism and hence it is nontrivial on the commutator subgroup of $\mathrm{Symp}_0(M, \omega)$ which is the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. $\square$

4.3.1. *The case of a closed symplectic manifold (M, ω).* Recall that if the induced evaluation map on $\pi_1(\mathrm{Homeo}_0(M, \mu), 1)$ is trivial then Theorem 1.4 holds. Let us discuss the general case without assuming the above triviality.

It follows from the proof of Arnold's conjecture that the induced evaluation map on $\pi_1(\mathrm{Ham}(M,\omega),1)$ is trivial, see [17, Exercise 11.28]; see also discussion in [12] about the proof of Arnold's conjecture in full generality. Thus we can modify the construction of the map

$$\Psi \colon \mathrm{Ham}(M,\omega) \to \mathbf{R}$$

defined in (**7**) by taking isotopies only in $\mathrm{Ham}(M,\omega)$. Note that this map is a quasimorphism, and was defined by Polterovich in [19]. Its homogenization $\overline{\Psi}$ is an essential quasimorphism whenever $\psi \colon \pi_1(M,z) \to \mathbf{R}$ is an essential quasimorphism. Thus in order to show that the fragmentation norm is stably unbounded it is enough to show that an essential quasimorphism

$$\overline{\Psi} \colon \mathrm{Ham}(M,\omega) \to \mathbf{R}$$

vanishes on Hamiltonian diffeomorphisms supported in balls of volume at most one, see e.g. [3, Lemma 5.2].

Notice that the proof of (1) in Lemma 3.3 does not work in this case because it is not known whether a Hamiltonian diffeomorphism of M supported in a ball $B \in \mathcal{B}'$ can be isotoped to the identity through a Hamiltonian isotopy supported in B (such an issue does not occur in dimension two [22, Corollary (2.6)]). We overcome this problem by adjusting the definition of the fragmentation norm as follows. We define it by

$$\|f\|_{\mathrm{FRAG}} = \min\{n \in \mathbf{N} \mid f = g_1 \cdots g_n\},$$

where $\mathrm{supp}(g_i) \subset B_{g_i} \in \mathcal{B}'$ and $g_i \in \mathrm{Ham}(B_{g_i})$. Note that in this case the fragmentation norm is well defined, since $\mathrm{Ham}(M,\omega)$ is a simple group when M is closed [1]. After this modification the proof of (1) in Lemma 3.3 goes through as before and so does the proof of the version of Theorem 1.4 for the fragmentation norm defined as above.

4.4. **An example for a question of Burago, Ivanov and Polterovich.**
One of the open problems of [7] asks for an example of a group which is perfect, has stably vanishing commutator length and admits a stably positive conjugation invariant norm. The following example, has been suggested by an anonymous referee.

Consider the group $H = \mathrm{Ham}(T^*\Sigma_g \times \mathbf{R}^{2n})$, where Σ_g is a closed surface of genus $g > 1$. The commutator subgroup of H is perfect and has stably vanishing commutator length (every compact subset of $T^*\Sigma_g \times \mathbf{R}^{2n}$ is displaceable by a compactly supported Hamiltonian isotopy which lies in the commutator subgroup). Observe that the fragmentation norm on H restricted to the commutator subgroup $G = [H, H]$ is still stably unbounded because the relative quasimorphism $\overline{\Psi}$ does not vanish on the commutator subgroup. So this restriction yields a required example for the Burago, Ivanov, Polterovich problem.

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