Karmarkar's method

The problem (C):

$$\min(c, x)$$

subject to conditions:

$$Ax = 0,$$

$$(e, x) = n,$$

$$x \geq 0,$$

where A is a $(m \times n)$-matrix and $e = (1, \ldots, 1)$.

Let Ω be the set feasible solutions of (C), $\Omega^0 = \{x \in \Omega : x > 0\}$.

Assumptions:

(1) $Ae = 0$ (so e is a feasible solution);

(2) $(c, x) > 0$ for all $x \in \Omega^0$;

(3) $\min_{x \in \Omega}(c, x) = 0$.

Let $f(x) = (c, x) / \prod_{j=1}^{n} x_j$. Then $(c, x) \leq f(x)$, for all $x \in \Omega^0$.

Algorithm:

Initialization. $x^0 = e$.

k-th step. x^k is known.

1. Form $D = diag(x_j^k)$ and

$$B = \begin{pmatrix} AD \\ e \end{pmatrix}.$$

2. Project Dc to the space $S = \{x \in R^n : Bx = 0\}$:

$$c^* = (E - B^T(BB^T)^{-1}B)Dc.$$

If $c^* = 0$, terminate – solution is optimal.

3. Normalize direction:

$$d = \frac{c^*}{\|c^*\| \sqrt{n(n-1)}}.$$

4. Move in projected space:

$$y = e - \alpha d,$$

where α is a fixed step size (for example, $\alpha = 0.25$).

5. Project back into x-space:

$$x^{k+1} = \frac{nDy}{(e, Dy)}.$$

Theorem. For each k,

$$f(x^{k+1}) < \gamma f(x^k),$$

where $\gamma < 1$.