The reliability problem

There is an electronic system consisting of n components. Let p_j be a probability of failure in the j-th component. Then $1 - p_j$ is the probability that the j-th component will operate successfully. Assume that this probability is independent of what is done for the other components. Thus the reliability of the whole system is $\prod_{j=1}^{n}(1 - p_j)$. To increase the reliability of the total system, one might attempt to spend money to decrease the numbers p_j, for example, instead of using one component j, we use two or more in parallel. Let $p_j(x_j)$ be a probability that the j-th component will fail, if we spend x_j to improve it. Suppose that the total sum of money which can be spent cannot be greater than W. Within this budget, it is desired to build the most reliable system possible.

The mathematical problem is:

To find

$$\max \prod_{j=1}^{n}(1 - p_j(x_j))$$

subject to conditions

$$\sum_{j=1}^{n} x_j \leq W,\ x_j \geq 0,\ j \in 1 : n.$$

Algorithm.

Let

$$F_k(z) = \max \prod_{j=1}^{k}(1 - p_j(x_j))$$

subject to conditions

$$\sum_{j=1}^{k} x_j \leq z,\ x_j \geq 0,\ j \in 1 : k.$$

Then $F_0(z) = 1$ for all z.

$$F_{k+1}(z) = \max_{x_{k+1} \geq 0}((1 - p_{k+1}(x_{k+1}))F_k(z - x_{k+1}))$$

(the Bellman formula).

The shortest path problem

Consider a network with two special nodes: i_0 called the source and s called the sink. Each branch joining two nodes i and j has a direction from i to j and has a length l_{ij}. The problem is to find the shortest path from i_0 to s in the network.

Algorithm.

Let Γi is a set of such nodes j that there is a branch from i to j. Let $f(i)$ is the length of the shortest path from i to s. The function f is defined for all nodes of the network, and $f(s) = 0$. Function f satisfies the following equation (the Bellman equation):

$$f(i) = \min_{j \in \Gamma i}(l_{ij} + f(j)).$$

Iterations for the solution of the Bellman equation:

$$f_0(i) = \begin{cases} 0, & \text{if } i = s, \\ \infty, & \text{if } i \neq s. \end{cases}$$

$$f_{k+1}(i) = \min(f_k(i), \min_{j \in \Gamma i}(l_{ij} + f_k(j)).$$

A deterministic inventory problem

Consider a particular item stocked by an inventory system for n periods. The demands for the item are known: r_j in period j. The cost of y_j units ordered in period j is $\varphi_j(y_j)$. The cost of the stock of s_j units in period j is $\gamma_j(s_j)$. The problem is to find the numbers y_j - the amount of the order in all periods to provide the given demands and to minimize the total expenses for the order and for the stock. The initial status of the stock is known: s_0.

Let s_j be a stock in period j. Then

$$s_j = s_{j-1} + y_j - r_j. \quad (1)$$

The total expenses are

$$F(y) = \sum_{j=1}^{n}((\varphi_j(y_j) + \gamma(s_j)).$$

The problem is to find y_j and s_j satisfying (1), $s_j \geq 0$ so that the total expenses $F(y)$ will be minimal.

Consider the following graph G. Nodes of the graph are pairs (j, s_j), where $j \in 0 : n$ is a period number and $s_j \geq 0$ is the amount of the stock at end of period j. We consider $(0, s_0)$ as a source node, and we add a node $n+1$ as a sink node. There is an arc from a node $(j-1, s_{j-1})$ to a node (j, s_j) if there exists y_j such that (1) is satisfied, and there are arcs from nodes (n, s) to the sink node $(n+1)$. Then the problem is equivalent to the shortest path problem on graph G.