A deterministic dynamic programming process

Let us consider a system having a status set V. Let N_i be a set of possible controls for status $i \in V$. A process starts from some initial status $i_0 \in V$. Then one uses some control $u_0 \in N_{i_0}$ and the system passes to status $i_1 = g(i_0, u_0) \in V$, then again some control $u_1 \in N_{i_1}$ is chosen and the system passes to the next status $i_2 = g(i_1, u_1) \in V$, and so on.

The situation can be represented by a graph. Consider a graph G whose vertex set is V; there is an arc from vertex $i \in V$ to vertex $j \in V$ if there exists a control $u \in N_i$ such that $g(i, u) = j$. A process is a path in graph G. Let P be the set of all paths in graph G and P_i the set of all paths starting from vertex i.

A typical problem:

Suppose a function f is given whose domain is set P. Find a path $p^* \in P_i$ which minimizes (or maximizes) value $f(p)$ for $p \in P_i$.

The function f is supposed to satisfy the following condition:

Let path p be a sequence of vertices: $\{i_0, i_1, i_2, \ldots\}$, and let p_1 be a sub-path of p: $p_1 = \{i_1, i_2, \ldots\}$. Then

$$f(p) = c(u_0) + \alpha f(p_1),$$

where $\alpha \geq 0$ and u_0 is a control such that $i_1 = g(i_0, u_0)$.

Let $v(i) = \min_{p \in P_i} f(p)$. Then v satisfies the Bellman equation:

$$v(i) = \min_{u \in N_i} (c(u) + \alpha v(g(i, u)))$$

$$v(i) = f_i \text{ if } N_i = \emptyset.$$

(1) If graph G has no contours, the Bellman equation has a unique solution, and controls which realize the minimum in the Bellman equation generate a path with minimal value of $f(p)$ for each initial status.

(2) If $\alpha < 1$, the Bellman equation has a unique solution, and controls which realize the minimum in the Bellman equation generate a path with minimal value of $f(p)$ for each initial status.

Maxima and minima properties

$$|\max_{x\in A} f(x)| \leq \max_{x\in A} |f(x)| \quad (1)$$

$$\min_{x\in A} f(x) = -\max_{x\in A}(-f(x)) \quad (2)$$

$$\min_{x\in A}(f(x)+g(x)) \geq \min_{x\in A} f(x) + \min_{x\in A} g(x) \quad (3)$$

$$|\min_{x\in A} f(x) - \min_{x\in A} g(x)| \leq \max_{x\in A} |f(x)-g(x)| \quad (4)$$