The Lagrangian function

The problem P1:

$$\min f(x)$$

subject to

$$g_i(x) \leq 0 \ \ i \in 1 : m \ (x \in R^n)$$

A function

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \ x \in R^n, \ \lambda \in R^m$$

is called the *Lagrangian function.*

A function $L(x, \lambda)$ is said to have a *saddle point* at the point (x^*, λ^*) in the area $x \in X, \ \lambda \in \Lambda$ if

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*)$$

for all $x \in X, \ \lambda \in \Lambda$.

Theorem 1. Let f and g_i be convex functions defined on a convex set $C \subset R^n$ and the regularity condition is satisfied. Then a point x^* is a solution of problem (P1) iff there exists a vector $\lambda^* \in R^m$ such that (x^*, λ^*) is a saddle point of the Lagrangian function $L(x, \lambda)$, i. e.

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*)$$

for all $x \in C, \ \lambda \geq 0$.

Dual problem

1. The problem (P1) is equivalent to the problem:

Find

$$\min_{x \in R^n} \max_{\lambda \geq 0} L(x, \lambda) = \min_{x \in R^n} K(x),$$

where $K(x) = \max_{\lambda \geq 0} L(x, \lambda)$.

The problem: find

$$\max_{\lambda \geq 0} \min_{x \in R^n} L(x, \lambda)$$

is called a *dual problem* for (P1).

2.

$$\max_{\lambda \geq 0} \min_{x \in R^n} L(x, \lambda) \leq \min_{x \in R^n} \max_{\lambda \geq 0} L(x, \lambda).$$

3. Let f and g_i be convex functions defined on a convex set $C \subset R^n$ and the regularity condition is satisfied. A point x^* is a solution of problem (P1) iff iff there exists a vector $\lambda^* \in R^m$ such that

$$\max_{\lambda \geq 0} \min_{x \in R^n} L(x, \lambda) = \min_{x \in R^n} \max_{\lambda \geq 0} L(x, \lambda) = L(x^*, \lambda^*).$$

i.e. (x^*, λ^*) is a solution of the dual problem.

4. If f and g_i are differentiable functions, the dual problem is:

$$\max L(x, \lambda)$$

subject to

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$