Mathematical programming problem

To find

$$\min f(x)$$

subject to

$$x \in C \ (x \in R^n)$$

Usually $C = \{x \in R^n : g_i(x) \leq 0, \ i \in 1 : m\}$.

If f is a convex function and C is a convex set, the problem is a *convex programming problem.*

A point $x \in C$ is called a *feasible point* or a *feasible solution.*

A point $x^* \in C$ such that $f(x^*) \leq f(x)$ for each $x \in C$ is called an *optimal point* or an *optimal solution.*

Necessary conditions

Let x be a feasible point. A vector d is a *feasible direction* for x, if $x + \tau d \in C$ for $0 < \tau < \delta$, $\delta > 0$.

Let $D(x)$ be the set of all feasible directions for x.

Properties

1. $D(x)$ is a cone.
2. If C is a convex set then $D(x)$ is a convex cone.
3. Let x^* be an optimal solution. Then $(\nabla f(x^*), d) \geq 0$ for all $d \in D(x^*)$.
4. Let x^* be an optimal solution. Then $(\nabla f(x^*), d) \geq 0$ for all $d \in \bar{D}(x^*)$.

Let $C = \{x \in R^n : g_i(x) \leq 0, \ i \in 1 : m\}$.

Constraint i is active for point x if $g_i(x) = 0$. Let $A(x)$ be the set of all active constraints.

Let $\tilde{D}(x) = \{d \in R^n : (\nabla g_i(x), d) \leq 0 \text{ for all } i \in A(x)\}$.

5. $\bar{D}(x) \subset \tilde{D}(x)$.

If $\bar{D}(x^*) = \tilde{D}(x^*)$, we say that the regularity condition is satisfied.

6. If $g_i(x)$ are linear functions, then $\bar{D}(x^*) = \tilde{D}(x^*)$.

7. If $g_i(x)$ are convex functions, and there exists an $x \in D$ such that $g_i(x) < 0,\ i \in 1:m$, then $\bar{D}(x^*) = \tilde{D}(x^*)$.

8. (Farkas' lemma). A vector b will satisfy $(b, y) \geq 0$ for all y satisfying $Ay \geq 0$ iff there exists a $w \geq 0$ such that $wA = b$.

9. (Kuhn - Tucker theorem). Let x^* be an optimal solution of the mathematical programming problem

$$\min f(x)$$

subject to

$$g_i(x) \leq 0,\ i \in 1:m.$$

and regularity condition is satisfied. Then there exist numbers $\lambda_i \geq 0,\ i \in 1:m$ such that

$$\lambda_i g_i(x^*) = 0, i \in 1:m$$

and

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 0.$$