## Problems

1. Check if the following function has a saddle point

$$F(x,y) = (x-y)^2, \ \ 0 \le x, y \le 1;$$

Solution. We should check that

$$\min_{0\le x\le 1} \max_{0\le y\le 1} F(x,y) = \max_{0\le y\le 1} \min_{0\le x\le 1} F(x,y).$$

Let $x \in [0,1]$ be fixed. Find $\max_{0\le y\le 1} F(x,y)$. It is clear that $\max_{0\le y\le 1}(x-y)^2$ is attained at $y=0$ or $y=1$. Therefore

$$g(x) = \max_{0\le y\le 1}(x-y)^2 = \max(x^2, (x-1)^2).$$

Then $\min_{0\le x\le 1} g(x)$ is attained at $x = 0.5$ and equal to $0.25$. Thus

$$\min_{0\le x\le 1} \max_{0\le y\le 1} F(x,y) = 0.25.$$

Now calculate $\max_{0\le y\le 1} \min_{0\le x\le 1} F(x,y)$. Let $y \in [0,1]$ be fixed. Then $\min_{0\le x\le 1}(x-y)^2 = 0$. Hence

$$\max_{0\le y\le 1} \min_{0\le x\le 1} F(x,y) = 0,$$

and

$$\min_{0\le x\le 1} \max_{0\le y\le 1} F(x,y) \ne \max_{0\le y\le 1} \min_{0\le x\le 1} F(x,y),$$

that is, $F(x,y)$ does not have a saddle point.

2. Find a solution of the following problems:

(b)

$$\min(8x_1^2 + 2x_2^2)$$

subject to

$$x_1^2 + x_2^2 \le 9, \ 1 \le x_1 \le 2, \ x_2 \ge 1.$$

Solution. It is clear that $\min(8x_1^2+2x_2^2)$ is attained at the point $x = (x_1, x_2)$ with minimal possible values of $x_1$ and $x_2$, that is, at $x^* = (1,1)$. This is a feasible point, and we should verify that it satisfies the conditions of the Kuhn-Tucker theorem.

In our case

$$f(x) = 8x_1^2 + 2x_2^2,$$

$$g_1(x) = x_1^2 + x_2^2 - 9,$$

$$g_2(x) = 1 - x_1,$$

$$g_3(x) = x_1 - 2,$$

$$g_4(x) = 1 - x_2,$$

$$\nabla f(x) = (16x_1, 4x_2),$$

$$\nabla g_1(x) = (2x_1, 2x_2),$$

$$\nabla g_2(x) = (-1, 0),$$

$$\nabla g_3(x) = (1, 0),$$

$$\nabla g_4(x) = (0, -1).$$

Verify that there exist non-negative numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\nabla f(x^*) + \sum_{i=1}^{4} \lambda_i \nabla g(x^*) = 0, \tag{1}$$

and $\lambda_i = 0$ for non-active constraints, that is, for $i$ with $g_i(x^*) < 0$. Since $g_1$ and $g_3$ are non-active at $x^*$, $\lambda_1 = \lambda_3 = 0$, and we have from (1):

$$16 + \lambda_2(-1) = 0,$$

$$4 + \lambda_4(-1) = 0.$$

Hence $\lambda_1 = 0,\ \lambda_2 = 16,\ \lambda_3 = 0,\ \lambda_4 = 4$ are non-negative and satisfy (1).

Since $f$ and $g_i$ are convex functions and the regularity condition is satisfied, point $x^* = (1, 1)$ is the solution of the problem.

5. Find $a$ and $b$ such that the function

$$f(x) = \begin{cases} b(x - \alpha), & x \le \alpha, \\ a(x - \alpha), & x > \alpha \end{cases}$$

is a convex function.

Solution. A function of one variable is convex iff its derivative increases. The derivative of $f$ is $b$, for $x \le \alpha$, and $a$ for $x > \alpha$. Hence $f$ is a convex function iff $a \ge b$.

6. Find an area where the following functions are convex

$$f(x) = \sum_{i=1}^{n} x_j \ln x_j;$$

Solution. The Hessian is a diagonal matrix with $h_{ii} = 1/x_i$. Hence the Hessian is a positive definite matrix, and $f$ is a convex function, for $x > 0$.

$$f(x, y) = x^2 - 3xy + y^2.$$

Solution. Calculate the Hessian

$$H = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

Since $\det H = -5$, $f$ is not a convex function.

$$f(x) = x_1^4 + x_2^4 - x_1 x_2$$

Solution. Calculate the Hessian

$$H = \begin{pmatrix} 12x_1^2 & -1 \\ -1 & 12x_2^2 \end{pmatrix}$$

$H$ is a positive semi-definite matrix if its diagonal elements are non-negative as well as its determinant. The diagonal elements are non-negative for all $x$, and $\det H = 144x_1^2x_2^2 - 1$. Therefore $f$ is convex on a convex set which is contained in $\{x : x_1^2x_2^2 \geq 1/144\}$.

7. Construct a dual problem for the following problem:

$$min \sum_{i=1}^{n} e^{x_i}$$

subject to

$$\sum_{i=1}^{n} x_i \geq 1.$$

Solution. Construct the Lagrange function:

$$L(x, \lambda) = \sum_{i=1}^{n} e^{x_i} + \lambda(1 - \sum_{i=1}^{n} x_i).$$

The dual problem (D) is:

find

$$\max_{\lambda \geq 0} \min_{x} L(x, \lambda).$$

Since $L(x, \lambda)$ is a convex and differentiable function with respect to $x$, problem (D) is equivalent to

$$\max_{\lambda \geq 0} \text{Ł}(x, \lambda).$$

subject to conditions:

$$\nabla_x L(x, \lambda) = 0,$$

that is

$$e^{x_i} - \lambda = 0, \text{ for all } i.$$

8. We have a bar 140 cm, and we can cut it into blanks:
20 cm, price \$3,
40 cm, price \$8,
60 cm, price \$12,
100 cm, price \$16.
Find a cutting of maximal worth.
Solution. The problem is:
find

$$\max(3x_1 + 8x_2 + 12x_3 + 16x_4)$$

subject to

$$20x_1 + 40x_2 + 60x_3 + 100x_4 \leq 140,$$

$x_j$ are non-negative integers denoting how many blanks of length $l_j$ should be cut from the bar.

The problem can be solved by dynamic programming:

$$F_{k+1}(l) = \max_x (c_{k+1}x + F_k(l - l_{k+1}x)),$$

F(0)=0.