Optimization with constraints

The problem P1:

$$\min f(x),$$

subject to

$$x \in C = \{g_i(x) \leq 0 \;\; i \in 1 : m\}.$$

Penalty method

Introduce a penalty function:

$$h(x) \begin{cases} = 0 & \text{for } x \in C, \\ > 0 & \text{for } x \bar{\in} C. \end{cases}$$

An example of a penalty function

$$h(x) = \sum_{i=1}^{m} (\max(g_i(x), 0))^2.$$

Consider a sequence of unconstrained problems $\{P_k\}$:

$$\min_{x \in R^n} (f(x) + A_k h(x))$$

where $\lim_{k \to \infty} A_k = \infty$.

Let x^k be a solution of the problem P_k. If f and g_i are continuous functions, the set $\{x^k\}$ is compact, the solution of the problem P1 exists and subsequence $\{x^{k_i}\}$ converges, then $x^* = \lim x^{k_i}$ is a solution of the problem P1.

Barrier method

b is a barrier function if

$$b(x) \geq c \;\text{ for some } c \text{ if } x \in C,$$

$$b(x) \to \infty, \text{ if } g_i(x) \to 0 \text{ for some } i.$$

Examples of barrier functions:

$$b(x) = \sum_{i=1}^{m} (-1/g_i(x)),$$

$$b(x) = \sum_{i=1}^{m} \log(-g_i(x)).$$

Consider a sequence of unconstrained problems $\{B_k\}$:

$$\min_{x \in R^n} (f(x) + r_k b(x))$$

where $r_k > 0$, $\lim_{k \to \infty} r_k = 0$.

Let x^k be a solution of the problem B_k. If f and g_i are continuous functions, set C is compact and $Int(C)$ is not empty, the solution of the problem P1 exists and subsequence $\{x^{k_i}\}$ converges, then $x^* = \lim x^{k_i}$ is a solution of the problem P1.

Method of feasible directions

Let $A(x, \delta) = \{i \in 1 : m \ : g_i(x) > -\delta\}$.

Let $x^0 \in C$ be an initial point, $\delta_0 > 0$.

Algorithm.

Let us have $x^k \in C$ and $\delta_k > 0$.

Find a feasible direction p solving the problem L1:

$$\min \eta$$

subject to

$$(\nabla f(x^k), p) \le \eta,$$

$$(\nabla g_i(x^k), p) \le \eta, \ i \in A(x^k, \delta_k),$$

$$\|p\| \le 1.$$

If $\|p\| = \max_j |p_j|$, L1 is a linear programming problem.

Let p^k and η_k be a solution of L1.

(1) If $\eta_k \ge -\delta_k$, set $x^{k+1} = x^k$, $\delta_{k+1} = \delta_k/2$.

(2) If $\eta_k < -\delta_k$, search for α_k:

Start with $\alpha = 1$. Check the condition:

$$f(x^k + \alpha p^k) \le f(x^k) + \alpha \eta_k/2,$$

$$g_i(x^k + \alpha p^k) \le 0, \ i \in 1 : m.$$

If the conditions are satisfied set $\alpha_k = \alpha$, otherwise set $\alpha := \alpha/2$ until the conditions will be satisfied.

Set $x^{k+1} = x^k + \alpha_k p^k$, $\delta_{k+1} = \delta_k$.

If f and g_i are convex functions, the regularity condition is satisfied and set C is compact, then $f(x^k) \to f(x^*)$, where x^* is a solution of the problem P1.

How to find a feasible solution?

Consider the following problem (P2):

$$\min \eta,$$

subject to

$$g_i(x) - \eta \leq 0, \ i \in 1 : m.$$

An initial point (x^0, η_0) for P2 is: x^0 is an arbitrary point in R^n and $\eta_0 = \max_{i \in 1:m} g_i(x_0)$.

Solve the problem P2 by the method of feasible directions until $\eta_k \leq 0$, then x^k is a feasible solution of P1.

If $\min \eta > 0$ in P2, then P1 have no feasible solution.