Convex functions

Let C be a convex set in R^n. A function f on C is a *convex function* if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for each $x \in C$, $y \in C$ and $0 < \lambda < 1$.

If $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$ for each $x \in C$, $y \in C$ and $0 < \lambda < 1$, then f is called a *strictly convex function* on C.

A function g is called a *concave function* if $-g$ is a convex function.

Let f be a function on S in R^n. The set

$$\{(x, \mu) \in R^{n+1} : \; x \in S, \; \mu \in R^1, \; \mu \geq f(x)\}$$

is called the *epigraph* of f and is denoted by epi(f).

Properties

1. A function f is a convex function iff epi(f) is a convex set.

2. For any convex function f and any $\alpha \in R^1$, the level sets $\{x : \; f(x) < \alpha\}$ and $\{x : \; f(x) \leq \alpha\}$ are convex.

3. Let $f_\alpha, \; \alpha \in A$ be a set of convex functions. Then $f(x) = \sup_{\alpha \in A} f_\alpha(x)$ is a convex function.

4. Let $f_1, \dots f_m$ are convex functions on C and $\lambda_1 \geq 0, \dots \lambda_m \geq 0$. Then $f = \sum_{i=1}^m \lambda_i f_i$ is a convex function on C.

5. Any local minimum of a convex function on a convex set is the global minimum.

6. Any convex function on a convex set C is continuous on $Int(C)$.

7. Let f be a differentiable function on an open convex set C. Then f is convex iff

$$f(x) - f(x_0) \geq (\nabla f(x_0), x - x_0)$$

for any $x_0 \in C, x \in C$, and f is strictly convex iff

$$f(x) - f(x_0) > (\nabla f(x_0), x - x_0),$$

for any $x_0 \in C, x \in C, \; x_0 \neq x$.

8. Let f be a twice continuosly differentiable function on an open convex set C. Then f is convex on C iff its Hessian matrix is positive semi-definite for every $x \in C$, and f is strictly convex on C iff its Hessian matrix is positive definite for every $x \in C$.

(The Hessian matrix is $H(x) = (h_{ij}(x))$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(x)$).