Exponential Boundedness of Solutions for Impulsive Delay Differential Equations

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Abstract—For an impulsive delay differential equation

$$\dot{x}(t) + \sum A_k(t)x[h_k(t)] = 0, \quad t \geq 0, \quad x(\xi) = \phi(\xi), \quad \xi < 0,$$
$$x(\tau_j) = B_jx(\tau_j - 0), \quad \lim_{j \to \infty} \tau_j = \infty,$$

exponent estimates of solutions have been obtained.

Keywords—Impulsive delay differential equations, Exponential boundedness, Estimates.

1. INTRODUCTION

Functional differential equations with impulses provide an adequate mathematical description of numerous phenomena and processes studied by physics, chemistry, radio engineering, etc. Existence of an exponential estimate for solutions is an important characteristic of differential equations. Under certain constraints such estimates have been obtained for many actual classes of nonimpulsive functional differential equations, namely, for delay differential equations, integro-differential and neutral equations [1].

These estimates are applied in stability and oscillation analysis, control theory, and so on. For example, applying methods of operational calculus presupposes that solutions have exponential estimates. In the paper [2], an oscillation criterion for autonomous neutral delay equations is obtained using Laplace transform. The authors claim that their method also leads to an oscillation criterion for argument deviation of different signs (both delay and advance). However, Laplace transform cannot be applied to such equations since there is no result on exponential estimates of solutions in this case.

As demonstrated in the present paper, under natural constraints solutions of impulsive differential equations are bounded by certain exponential functions. It is to be emphasized that

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the coefficient and the power of the exponent are obtained explicitly from the parameters of the equation and do not depend on the initial point.

2. PRELIMINARIES

Let 0 = \tau_0 < \tau_1 < \ldots be fixed points, \lim_{j \to \infty} \tau_j = \infty, \mathbb{R}^n be the space of \(n\)-dimensional column vectors \(x = \text{col} \{x_1, \ldots, x_n\}\) with a norm \(\|x\|\), by the same symbol \(\| \cdot \|\) we will denote the corresponding matrix norm in the space \(\mathbb{R}^{n \times n}\) of \(n \times n\) matrices, and \(E_n\) is an \(n \times n\) identity matrix.

For every \(t_0 > 0\) we will consider the linear delay differential equation

\[
\dot{x}(t) + \sum_{i=1}^{m} A_i(t)x[\xi_1(t)] = 0, \quad t \geq t_0, \quad x(t) \in \mathbb{R}^n, \quad x(\xi) = \varphi(\xi), \quad \xi < t_0, \quad (1)
\]

\[
x(\tau_j) = B_jx(\tau_j - 0), \quad \tau_j > t_0. \quad (2)
\]

under the following assumptions:

(a1) \(0 = \tau_0 < \tau_1 < \tau_2 < \ldots\) are fixed points, \(\lim_{j \to \infty} \tau_j = \infty\);

(a2) columns of \(A_i : [0, \infty) \to \mathbb{R}^{n \times n}, \ i = 1, \ldots, m,\) are measurable and integrable on each segment \([a, b], \ b > a > 0,\) functions;

(a3) \(h_i : [0, \infty) \to \mathbb{R}\) are Lebesgue measurable functions, \(h_i(t) \leq t, \ i = 1, \ldots, m;\)

(a4) \(\varphi : (+\infty, 0) \to \mathbb{R}^n\) is a Borel measurable bounded function.

**DEFINITION.** A function \(\zeta : [t_0, \infty) \to \mathbb{R}^n\) absolutely continuous on each interval \([\tau_j, \tau_{j+1})\) is said to be a solution of the impulsive equation (1),(2), if (1) is satisfied for almost all \(t \in [t_0, \infty)\) and the equalities (2) hold.

The purpose of this paper is to prove the existence of an exponential estimate for solutions of (1),(2), with estimate parameters not depending on the initial point \(t_0\).

For each \(s \geq 0\), consider an auxiliary matrix initial value problem:

\[
\dot{y}(t) + \sum_{i=1}^{m} A_i(t)y[\xi_1(t)] = 0, \quad t \geq s, \quad y(t) \in \mathbb{R}^{n \times n}, \quad y(\xi) = 0, \quad \xi < s, \quad y(s) = E_n; \quad (3)
\]

\[
y(\tau_j) = B_jy(\tau_j - 0), \quad \tau_j > s. \quad (4)
\]

**DEFINITION.** The solution \(X(t, s)\) of the problem (3),(4) is said to be a fundamental matrix of the equation (1),(2). We assume \(X(t, s) = 0, t < s\).

**LEMMA 1.** [3] Suppose (a1)-(a4) are satisfied. Then there exists one and only one solution of the initial value problem (1),(2), \(x(0) = \alpha_0\) which can be presented as

\[
x(t) = X(t, t_0)\alpha_0 - \sum_{i=1}^{m} \int_{t_0}^{t} X(t, s)A_i(s)\varphi[h_i(s)] \, ds, \quad (5)
\]

wherein \(\varphi(\xi) = 0, \text{if } \xi \geq 0\).

3. MAIN RESULT

**LEMMA 2.** For the fundamental matrix of (1),(2), the following estimate holds:

\[
\|X(t, s)\| \leq \prod_{s < \tau_j \leq t} \left[ \|B_j\| + \int_{\tau_j}^{t+1} \sum_{i=1}^{m} \|A_i(\xi)\| \, d\xi \right] \exp \left\{ \int_{s}^{t} \sum_{i=1}^{m} \|A_k(\xi)\| \, d\xi \right\}, \quad (6)
\]

wherein \(\prod_{s < \tau_j \leq t} = 1\), if there are no impulse points \(\tau_j\) in \((s, t]\).
Proof. Let us fix $s$ and choose such index $k$ that $\tau_{k-1} \leq s < \tau_k$. First consider the case $t \in (s, \tau_k)$. Then $y(t) = X(t, s)$ is a solution of the nonimpulsive problem (3), hence

$$y(t) = \mathbb{E}_n - \int_s^t \sum_{i=1}^m A_i(\zeta)y[\eta_i(\zeta)] \, d\zeta.$$ 

Thus,

$$\|y(t)\| \leq 1 + \int_s^t \sum_{i=1}^m \|A_i(\zeta)\| \sup_{s < \zeta \leq \zeta} \|y(\zeta)\| \, d\zeta.$$  (7)

After denoting $z(t) = \sup_{s < \zeta \leq t} \|y(\zeta)\|$, (7) yields

$$z(t) \leq 1 + \int_s^t \sum_{i=1}^m \|A_i(\zeta)\| z(\zeta) \, d\zeta.$$ 

By applying the Gronwall-Bellman inequality one obtains

$$z(t) \leq \exp \left\{ \int_s^t \sum_{i=1}^m \|A_i(\zeta)\| \, d\zeta \right\},$$

therefore,

$$\|X(t, s)\| \leq \exp \left\{ \int_s^t \sum_{i=1}^m \|A_i(\zeta)\| \, d\zeta \right\}, \quad t \in [s, \tau_k).$$  (8)

The inequality (8) proves the lemma in case the interval $(s, t)$ does not contain points $\tau_j$. Now let $t \in [\tau_k, \tau_{k+1})$. Then $X(t, s)$ is a solution of the problem

$$\dot{y}(t) + \sum_{i=1}^m A_i(t)y[\eta_i(t)] = 0, \quad t \in [\tau_k, \tau_{k+1}), \quad y(t) \in \mathbb{R}^{n \times n},$$  (9)

$$y(\xi) = \varphi(\xi), \quad \xi < \tau_k, \quad y(\tau_k) = B_ky(\tau_k - 0),$$  (10)

wherein

$$\varphi(\xi) = \begin{cases} X(\xi, s), & s \leq \xi < \tau_k, \\ 0, & \xi < s. \end{cases}$$

Denote

$$\tilde{y}(t) = \begin{cases} y(t), & t \in [\tau_k, \tau_{k+1}), \\ 0, & t \not\in [\tau_k, \tau_{k+1}), \end{cases} \quad \tilde{\varphi}(t) = \begin{cases} 0, & t \in [\tau_k, \tau_{k+1}), \\ \varphi(t), & t \not\in [\tau_k, \tau_{k+1}). \end{cases}$$

The solution of (9),(10) satisfies the inequality

$$\|y(t)\| \leq \|y(\tau_k)\| + \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| \|g[\eta_i(\zeta)]\| \, d\zeta + \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| \|\varphi[\eta_i(\zeta)]\| \, d\zeta.$$  (11)

Denote

$$z(t) = \sup_{\tau_k \leq \zeta \leq t} \|y(\zeta)\|.$$ 

Then (8), (10) and (11) yield

$$z(t) \leq \|B_k\| \exp \left\{ \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| \, d\zeta \right\} + \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| z(\zeta) \, d\zeta$$

$$+ \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| \exp \left\{ \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| \, d\zeta \right\} \, d\zeta$$

$$\leq \left( \|B_k\| + \int_{\tau_k}^{\tau_{k+1}} \sum_{i=1}^m \|A_i(\zeta)\| \, d\zeta \right) \exp \left\{ \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| \, d\zeta \right\}$$

$$+ \int_{\tau_k}^t \sum_{i=1}^m \|A_i(\zeta)\| z(\zeta) \, d\zeta.$$
By applying again the Gronwall-Bellman inequality one obtains

\[ z(t) \leq \left( \| B_k \| + \int_{\tau_k}^{\tau_{k+1}} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right) \times \exp \left\{ \int_{s}^{t} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right\} \exp \left\{ \int_{\tau_k}^{t} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right\} = \left( \| D_k \| + \int_{\tau_k}^{\tau_{k+1}} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right) \exp \left\{ \int_{s}^{t} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right\} . \]

Consequently,

\[ \| X(t, s) \| \leq \left( \| B_k \| + \int_{\tau_k}^{\tau_{k+1}} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right) \exp \left\{ \int_{s}^{t} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \right\} , \quad \tau_k \leq t < \tau_{k+1}. \tag{12} \]

Thus, the general case of the formula (6) one can prove by induction in the number of points \( \tau_j \in (s, t] \), which completes the proof.

**Theorem 1.** Suppose for (1),(2) the hypotheses (a1)–(a4) are satisfied and there exist positive numbers \( B, \rho, \sigma \) and \( M \) such that \( \| B_j \| \leq B, \) \( \rho \leq \tau_{j+1} - \tau_j \leq \sigma, \)

\[ \sup_{t>0} \int_{t}^{t+1} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi = M < \infty. \tag{13} \]

Then the solution of (1),(2) has the following estimate:

\[ \| x(t) \| \leq K e^{\lambda (t-s_0)} \left( \| x(t_0) \| + \sup_{\xi < t_0} \| \varphi(\xi) \| \right), \]

wherein

\[ \lambda = \begin{cases} M, & B + (\sigma + 1)M \leq 1, \\ M + \frac{\ln[B + (\sigma + 1)M]}{\rho}, & B + (\sigma + 1)M > 1, \\ \max \left\{ e^M, \frac{Me^{M+2\lambda}}{e^\lambda - 1} \right\}, & B + (\sigma + 1)M \leq 1, \\ \max \left\{ e^{M+2\lambda}M, \frac{e^{M+2\lambda}[B + (\sigma + 1)M]}{e^\lambda - 1} \right\}, & B + (\sigma + 1)M > 1. \end{cases} \]

\[ K = \begin{cases} e^M, & B + (\sigma + 1)M \leq 1, \\ e^{M+2\lambda}[B + (\sigma + 1)M], & B + (\sigma + 1)M > 1. \end{cases} \]

**Proof.** Equality (13) implies for any \( a > 0, b > 0 \)

\[ \int_{a}^{b} \sum_{i=1}^{m} \| A_i(\xi) \| d\xi \leq (b - a + 1)M. \tag{14} \]

Applying (6) and (14) yields the following estimate for \( X(t, s) \):

\[ \| X(t, s) \| \leq [B + (\sigma + 1)M]^{i(a, b)} e^{M(t-s+1)}, \tag{15} \]

wherein \( i(a, b) \) is the number of points \( \tau_j \) in the interval \( (a, b] \).

The inequality \( \tau_{j+1} - \tau_j \geq \rho \) implies \( i(s, t) \leq (t-s)/\rho + 1 \). Thus (15) yields

\[ \| X(t, s) \| \leq \begin{cases} e^M e^{M(t-s)}, & B + (\sigma + 1)M \leq 1, \\ e^{M+M+2\lambda}[B + (\sigma + 1)M], & B + (\sigma + 1)M > 1. \end{cases} \tag{16} \]
The estimate (16) can be rewritten in the form

\[ \|X(t, s)\| \leq K_0 e^{\lambda(t-s)} \]  

(17)

where the constant \( \lambda \) is as in the statement of the theorem and

\[ K_0 = \begin{cases} 
    e^M, & B + (\sigma + 1)M \leq 1, \\
    e^M[B + (\sigma + 1)M], & B + (\sigma + 1)M > 1.
\end{cases} \]

Applying Lemma 1 to the solution of (1),(2) yields

\[ \|x(t)\| \leq \|X(t, t_0)\|\|x(t_0)\| + \int_{t_0}^{t} \|X(t, s)\| \sum_{i=1}^{m} \|A_i(s)\| \|\varphi[h_i(s)]\| \, ds, \]

where \( \varphi(\xi) = 0 \), if \( \xi \geq t_0 \).

The estimate (17) implies

\[ \|x(t)\| \leq K_0 e^{\lambda(t-t_0)}\|x(t_0)\| + K_0 \int_{t_0}^{t} e^{\lambda(t-s)} \sum_{i=1}^{m} \|A_i(s)\| \, ds \sup_{\xi < t_0} \|\varphi(\xi)\|. \]  

(18)

Let us estimate the integral included in the second term

\[ \int_{t_0}^{t} e^{\lambda(t-s)} \sum_{i=1}^{m} \|A_i(s)\| \, ds \leq \int_{k=t_0}^{[t]} e^{\lambda(t-s)} \sum_{i=1}^{m} \|A_i(s)\| \, ds \]

\[ \leq e^{\lambda t} \sum_{k=[t_0]}^{[t]} e^{-\lambda k} M e^{\frac{2\lambda}{e^\lambda - 1}} \frac{e^{-\lambda[t_0]}}{1 - e^{-\lambda}} \leq \frac{M e^{\frac{2\lambda}{e^\lambda - 1}}}{e^\lambda - 1} e^{\lambda(t-t_0)}, \]  

(19)

wherein by \([a]\) the greatest integer not exceeding \( a \) is denoted.

By substituting (19) into (18), one obtains

\[ \|x(t)\| \leq K_0 e^{\lambda(t-t_0)}\|x(t_0)\| + \frac{K_0 Me^{\frac{2\lambda}{e^\lambda - 1}}}{e^\lambda - 1} \sup_{\xi < t_0} \|\varphi(\xi)\| \]

\[ \leq Ke^{\lambda(t-t_0)} \left( \|x(t_0)\| + \sup_{\xi < t_0} \|\varphi(\xi)\| \right), \]

wherein

\[ K = \max \left\{ K_0, \frac{K_0 Me^{\frac{2\lambda}{e^\lambda - 1}}}{e^\lambda - 1} \right\}. \]

The latter inequality coincides with the statement of the theorem.

**REFERENCES**