Linearized oscillation theory for a nonlinear equation with a distributed delay

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Abstract

We obtain linearized oscillation theorems for the equation with distributed delays

\[ \dot{x}(t) + \sum_{k=1}^{m} r_{k}(t) \int_{-\infty}^{t} f_{k}(x(s)) \, ds \, R_{k}(t, s) = 0. \]

The results are applied to logistic, Lasota–Wazewska and Nicholson’s blowflies equations with a distributed delay. In addition, the “Mean Value Theorem” is proved which claims that a solution of (1) also satisfies the linear equation with a variable concentrated delay

\[ \dot{x}(t) + \left( \sum_{k=1}^{m} r_{k}(t) f'_{k}(\xi_{k}(t)) \right) x(g(t)) = 0. \]

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1. Introduction

It is usually believed that equations with a distributed delay provide a more realistic description for models of population dynamics and mathematical biology in general. For example, if maturation delay is involved in the equation, then the maturation time is, generally, not constant, but is distributed around its expectancy value.

Historically, equations with a distributed delay were studied even before relevant models with concentrated delays appeared. For example, Volterra considered the logistic equation with a distributed delay in 1926 \cite{Volterra1926}

\[ \dot{N}(t) = r N(t) \int_{0}^{\infty} k(\tau) \left[ 1 - \frac{N(t - \tau)}{K} \right] \, d\tau, \]

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before Hutchinson’s equation (the logistic equation with a concentrated delay)

\[ \dot{N}(t) = rN(t) \left[ 1 - \frac{N(t-\tau)}{K} \right] \]  

was introduced in 1948 [2].

To the best of our knowledge, the first systematic study of equations with a distributed delay can be found in the monograph of Myshkis [3], the results obtained by 1993 are summarized in the book of Kuang [4]. Presently equations with distributed delays are intensively studied. For various models of Mathematical Biology with distributed and concentrated delays see the monographs [4–7]. We also refer the reader to [8–35] and to the references therein for recent progress in the theory of delay differential equations with a distributed delay, especially asymptotics and stability, as well as justification of various applied models including a distributed delay. In most publications integrodifferential equations are studied, however sometimes applied models, like in the present paper, incorporate both integral terms and equations with concentrated delays (see, for example, [13,34]). Here we do not mention extensive literature on neural networks and control theory for equations with distributed delays, as well as partial differential equations including distributed delays.

However, in most of these publications authors either concentrate on a specific applied model [8,10,11,13,15,17,18, 23,28,33–35] with a distributed delay or present an equation with a distributed delay as an illustration to some other results. There are relatively few papers concerned with the systematic study of equations with a distributed delay, for example, [9,14,16,22,27,31,36,37]. Most of the obtained results are not relevant for time-dependent models and do not involve equations with a concentrated delay as a special case.

Let us notice that in the present paper we study a general form of delay and coefficients in the following sense.

1. The distributed delay allows us, for an appropriate choice of the distribution, to consider integrodifferential equations, equations with several variable concentrated delays and equations with both delayed and integral terms. All parameters, unlike (1), are time dependent.

2. Solutions are absolutely continuous, not necessarily continuously differentiable functions. This corresponds to the measurable locally essentially bounded (not necessarily continuous) kernels of integrals and coefficients.

Finally, let us refer the reader to the monograph [38] for an overview of some recent progress in oscillation theory and to [39–46] for previous linearization results for delay differential equations with concentrated delays.

The paper is organized as follows. After the preliminaries in Section 2 we prove some existence and uniqueness results for equations with a distributed delay in Section 3. Section 4 contains our main linearization theorems, which are applied in Section 5 to the various models of mathematical biology (logistic, Lasota–Wazewska and Nicholson blowflies equations). Finally, in Section 6 we establish “the mean value theorem”, which claims that a solution of a nonlinear equation with a distributed delay also satisfies a linear equation with a variable concentrated delay.

2. Preliminaries

We consider a nonlinear differential equation with a distributed delay

\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{-\infty}^{t} f_k(x(s)) \, d_k R_k(t, s) = 0, \]  

as well as this equation with a nondelay term

\[ \dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{-\infty}^{t} f_k(x(s)) \, d_k R_k(t, s) = 0, \]  

for \( t > t_0 \geq 0 \), assuming that for each \( t \) the memory is finite. Thus we can introduce the functions

\[ h_k(t) = \inf \{ s \leq t \mid R_k(t, s) \neq 0 \} \]

and rewrite (3), (4) in the form

\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k(x(s)) \, d_k R_k(t, s) = 0, \]
\[ \dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k(x(s)) \, ds \, R_k(t, s) = 0, \]  
\[ t > t_0. \]

Together with (6), (7) we assume, for each \( t_0 \geq 0 \), that the initial condition
\[ x(t) = \varphi(t), \quad t \leq t_0, \]  
is satisfied. We consider the Eqs. (6) and (7) under the following assumptions:

(a1) \( r_k(t) \geq 0, k = 1, \ldots, m, b(t) \geq 0 \) are the Lebesgue measurable functions bounded on the halfline: \( r_k(t) < r_k, b(t) < b, t \geq 0; \)

(a2) \( h_k : [0, \infty) \to \mathbb{R}, k = 1, \ldots, m, \) are the Lebesgue measurable functions, \( h_k(t) \leq t, \lim_{t \to \infty} h_k(t) = \infty; \)

(a3) \( R_k(t, \cdot) \) are left continuous nondecreasing functions for any \( t, R_k(\cdot, s) \) are locally integrable for any \( s, R_k(t, h_k(t)) = 0, R_k(t, t^+) = 1. \)

In (a3) the condition \( R_k(t, h_k(t)) = 0 \) means that the delay is finite, while \( R_k(t, t^+) = 1 \) corresponds to any delay equation, which is “normalized” with the coefficient \( r_k(t) \).

Now let us proceed to the initial function \( \varphi \). This function should satisfy such conditions that the integral on the left-hand side of (6) exists almost everywhere. In particular, if \( R(t, \cdot) \) is absolutely continuous for any \( t \) (which allows us to write (6) as an integrodifferential equation), then \( \varphi \) can be chosen as a Lebesgue measurable essentially bounded function. If \( R(t, \cdot) \) is a combination of step functions (which correspond to an equation with concentrated delays) then \( \varphi \) should be a Borel measurable bounded function. For any choice of \( R \) the integral exists if \( \varphi \) is bounded and continuous. Thus, we assume that

(a4) \( \varphi : (-\infty, 0] \to \mathbb{R} \) is a bounded continuous function;
and the following hypothesis for \( f_k \)

(a5) \( f_k : \mathbb{R} \to \mathbb{R}, k = 1, \ldots, m \) are continuous differentiable functions and \( f_k' \) are locally essentially bounded functions.

**Remark 1.** For existence and uniqueness results, in (a5) we can assume that the functions \( f_k \) are locally Lipschitz rather than differentiable: for each \([a, b]\) there is an \( M_k > 0 \) (generally, depending on \([a, b]\)), such that \(|f_k(x) - f_k(y)| < M_k|x - y|\) for any \( x, y \in [a, b] \).

**Definition.** An absolutely continuous function \( x : R \to R \) is called a solution of the problem (6) and (8) if it satisfies Eq. (6) for almost all \( t \in [t_0, \infty) \) and conditions (8) for \( t \leq t_0 \).

We will also consider the linear equation
\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) \, ds \, R_k(t, s) = 0 \]  
and the corresponding inequalities
\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) \, ds \, R_k(t, s) \leq 0, \]  
\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) \, ds \, R_k(t, s) \geq 0. \]

**Definition.** For each \( t \geq t_0 \) the solution \( X(t, s) \) of the problem
\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) \, ds \, R_k(t, s) = 0, \quad t \geq s, \quad x(t) = 0, \quad t < s, \quad x(s) = 1, \]  
is called the fundamental function of Eq. (9). It is assumed that \( X(t, s) = 0, 0 \leq t < s \).

Eq. (6) has a nonoscillatory solution if it has an eventually positive or an eventually negative solution. Otherwise, all solutions of (6) are oscillatory.
Denote by \( h(t) \) and \( H(t) \) the maximal and minimal delay functions

\[
h(t) = \min_{k=1,\ldots,m} h_k(t), \quad H(t) = \max_{k=1,\ldots,m} h_k(t).
\]

**Lemma 1** \((36)\). Suppose \((a1)\)–\((a4)\) hold. Then the following hypotheses are equivalent:

1. Eq. \((9)\) has an eventually positive \((\text{an eventually negative})\) solution.
2. Inequality \((10)\) has an eventually positive solution \((\text{inequality} \,(11)\, \text{has an eventually negative solution})\).
3. There exists \( t_1 \geq 0 \) such that the inequality

\[
\limsup_{t \to \infty} \frac{1}{t} \left( \int_{h(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[ \int_{h_k(\tau)}^{\tau} d_s R_k(t, s) \right] d\tau \right) < \frac{1}{e},
\]

then \((9)\) has a nonoscillatory solution.

If

\[
\liminf_{t \to \infty} \frac{1}{t} \left( \int_{H(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[ \int_{h_k(\tau)}^{\tau} d_s R_k(t, s) \right] d\tau \right) > \frac{1}{e},
\]

then all solutions of \((9)\) are oscillatory. Here \( h(t), H(t) \) are defined by \((13)\).

Considering the equation with a nondelay term

\[
\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) d_s R_k(t, s) = 0
\]

and substituting

\[
z = x \exp \left\{ \int_{h_0}^{t} b(\xi) d\xi \right\},
\]

we obtain the equation

\[
\dot{z}(t) + \exp \left\{ \int_{h_0}^{t} b(\xi) d\xi \right\} \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} z(s) \exp \left\{ -\int_{h_0}^{s} b(\xi) d\xi \right\} d_s R_k(t, s) = 0,
\]

which can be rewritten as

\[
\dot{z}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} z(s) \exp \left\{ \int_{h_0}^{s} b(\xi) d\xi \right\} d_s R_k(t, s) = 0.
\]

Applying \textbf{Lemma 1} to this equation, we immediately obtain the following result.

**Lemma 2.** Suppose \((a1)\)–\((a4)\) hold and \( b(t) \) is a measurable locally essentially bounded function.

If

\[
\limsup_{t \to \infty} \frac{1}{t} \left( \int_{h(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[ \int_{h_k(\tau)}^{\tau} \exp \left\{ \int_{s}^{\tau} b(\xi) d\xi \right\} d_s R_k(t, s) \right] d\tau \right) < \frac{1}{e}.
\]
then (19) has a nonoscillatory solution. If

\[ \liminf_{t \to \infty} \int_{H(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[ \int_{h_k(\tau)}^{\tau} \exp \left( \int_{s}^{\tau} b(\xi) \, d\xi \right) \, d_s R(\tau, s) \right] \, d\tau > \frac{1}{e}, \tag{22} \]

then all solutions of (19) are oscillatory. Here \( h(t), H(t) \) are defined by (13).

We will also apply the following auxiliary result.

**Lemma 3** ([47, IV.6.26]). The set \( M \subset L_\infty[a, b] \) is compact if and only if for any \( \varepsilon > 0 \) the segment \([a, b] \) can be presented as a union of a finite number of measurable subsets \( E_i \subset [a, b] \) such that for every \( E_i, u \in M \) and any \( t, s \in E_i \) we have

\[ |u(t) - u(s)| < \varepsilon. \]

### 3. Existence and uniqueness of solutions

By \( L_\infty[a, b] \) we denote the space of Lebesgue measurable essentially bounded functions \( f : [a, b] \to \mathbb{R} \), by \( C[a, b] \) the space of continuous functions, both spaces have the sup-norm, by \( L^2[a, b] \) denote the space of real Lebesgue measurable functions \( y(t) \), such that \( A = \int_{a}^{b} (y(t))^2 \, dt < \infty, \|y\|_{L^2[a, b]} = \sqrt{A} \).

We will use the following result from the book of Corduneanu [48, Theorem 4.5, p. 95]. We recall that operator \( N \) is causal (or Volterra) if for any two functions \( x \) and \( y \) and each \( t \) the fact that \( x(s) = y(s), s \leq t \), implies that \( (Nx)(s) = (Ny)(s), s \leq t \).

**Lemma 4** ([48]). Consider the equation

\[ \dot{y}(t) = (Ly)(t) + (Ny)(t), \quad t \in [a, b], \tag{23} \]

where \( L \) is a linear bounded causal operator, \( N \) is a nonlinear causal operator, \( N : C([a, b]) \to L^2([a, b]) \) satisfies

\[ \|Nx - Ny\|_{L^2([a, b])} \leq \lambda \|x - y\|_{C([a, b])} \tag{24} \]

for \( \lambda \) sufficiently small. Then there exists a unique absolutely continuous solution of (23) in \([a, b]\), with the initial function being equal to zero for \( t < a \).

**Theorem 1.** Suppose (a2)–(a5) hold, \( r_k \) are the Lebesgue measurable bounded functions satisfying \( |r_k(t)| \leq r_k, t \geq 0, k = 1, \ldots, m \).

Then there exists a unique local solution of both (6), (8) and (7), (8).

**Proof.** To reduce (6), (7) to an equation with a zero right-hand side, we can present the integral as a sum of two integrals for any \( t_0 \)

\[ \dot{y}(t) = - \sum_{k=1}^{m} r_k(t) \int_{t_0}^{t} f_k(y(s)) \, d_s R_k(t, s) - \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k(\varphi(s)) \, d_s R_k(t, s), \quad t > t_0, \tag{25} \]

where \( y(t) = 0, t < t_0, y(t) = x(t), t \geq t_0, \varphi(t) = 0, t \geq t_0 \). Then in (23) either \( Ly = 0 \) for (6) or \( (Ly)(t) = -b(t)y(t) \) for (7) and

\[ Ny = - \sum_{k=1}^{m} r_k(t) \int_{t_0}^{t} f_k(y(s)) \, d_s R_k(t, s) + u(t), \]

where \( u(t) = - \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f(\varphi(s)) \, d_s R_k(t, s) \). Let us fix some \( t^* > t_0 \). For any finite segment \([t_0, t^*] \) continuous functions \( x(t) \) and \( y(t) \) are bounded, say, \( x(t), y(t) \in [c, d], t \in [t_0, t^*] \). By (a5) derivatives of \( f_k \) are locally essentially bounded: \( |f_k'(x)| < M_k \) for almost all \( x \in [c, d] \). Then

\[ |f_k(x(t)) - f_k(y(t))| \leq M_k |x(t) - y(t)|, \quad t \in [t_0, t^*]. \]
Thus for any \( \lambda > 0 \) there exist \( t_1, t_0 < t_1 \leq t^* \), such that

\[
\|Nx - Ny\|_{L^2[0, t_1]} \leq \sum_{k=1}^{m} r_k \left\| \int_{t_0}^{t} |f_k(x(s)) - f_k(y(s))| \, ds \right\|_{L^2[0, t_1]}
\]

\[
\leq \sum_{k=1}^{m} r_k M_k \max_{s \in [t_0, t_1]} |x(s) - y(s)| \left( \int_{t_0}^{t} \|d_s R_k(t, s)\| \right)_{L^2[0, t_1]}
\]

\[
\leq \sum_{k=1}^{m} r_k M_k \|x - y\|_{C[0, t_1]} |t_1 - t_0| \leq \lambda \|x - y\|_{C[0, t_1]}.
\]

For example, we can choose any \( t_1 \leq \min \left\{ t_0 + \lambda \left( \sum_{k=1}^{m} r_k M_k \right)^{-1}, t^* \right\} \). Here \( \lambda \) can be chosen small enough. By Lemma 4 this implies the uniqueness and the existence of a local solution for either (6) or (7), which completes the proof. \( \square \)

Now let us consider some particular cases of (6) and (7), where bounded \( r_k(t) \) may have an arbitrary sign. The existence of the local solution was obtained in Theorem 1.

The equation

\[
\dot{x}(t) = -b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} d_s f_k(x(s)) \, ds, \quad t > 0,
\]

(26)

generalizes several models of mathematical biology (Lasota–Wazewska, Nicholson’s blowflies, Mackey–Glass equations).

**Corollary 1.** Suppose in addition to (a1)–(a5) the following conditions hold

\[
x f_k(x) > 0 \quad \text{for any } x \neq 0, \quad f_k(x) \text{ are bounded for } x > 0,
\]

(27)

\[
\varphi(t) \geq 0, \quad t \leq 0, \quad \varphi(0) > 0.
\]

(28)

Then there exists a unique global solution of (26) and (8) which is positive for \( t \geq 0 \).

**Proof.** By Theorem 1 there exists a unique local solution. This solution is either global or there exists such a \( t_2 \) that either

\[
\liminf_{t \to t_2^-} x(t) = -\infty
\]

(29)

or

\[
\limsup_{t \to t_2^-} x(t) = \infty.
\]

(30)

Let us demonstrate that under (27), (28) the solution of (26) is positive. In fact, by (27) as far as \( x(t) \) is positive, the derivative is not less than \(-b(t)x(t)\). Since by (28) \( x(0) > 0 \), then the solution of the ordinary differential equation \( \dot{x}(t) = -b(t)x(t) \) is positive for \( t > 0 \), as well as the solution of (26), which disproves (29).

Now let us prove that (30) is impossible. By (27) and (a1) functions \( f_k \) and \( r_k \) are bounded, let us assume that \( f_k(x) \leq M_k, M_k > 0, r_k(t) \leq r_k \). The solution of (26) does not exceed the solution of the equation

\[
\dot{y}(t) = \sum_{k=1}^{m} r_k M_k, \quad t > 0.
\]

(31)

Thus \( x(t) \leq x(0) + \sum_{k=1}^{m} r_k M_k t \), so there is no point \( t_2 \) where (30) can be valid, which completes the proof. \( \square \)

The equation

\[
\dot{x}(t) = x(t) \left[ \sum_{k=1}^{m} r_k(t) \left( 1 - \frac{1}{K} \int_{h_k(t)}^{t} x(s) \, ds \right) \right], \quad t > 0,
\]

(32)

generalizes the well-known logistic equation.
Corollary 2. Suppose $K > 0$, (a1)–(a4) and (28) hold. Then there exists a unique global solution of (32), (8) which is positive for $t \geq 0$.

**Proof.** First, let us demonstrate that under (28) the solution is positive. Let us fix $T > 0$. The solution does not exceed the solution of the ordinary differential equation $\dot{y}(t) = \sum_{k=1}^{m} r_k y(t)$ with the same initial value, i.e., $x(t) \leq A = x(0) \exp(\sum_{k=1}^{m} r_k T)$ for any $t \in [0, T]$. Denoting $B = \max\{A, \sup_{t<0} \varphi(t)\}$, we obtain that the solution is not less than the solution of the ordinary differential equation $\dot{z}(t) = -\frac{B}{K} \sum_{k=1}^{m} r_k z(t)$, or $x(t) \geq x(0) \exp\left\{ -\frac{B}{K} \sum_{k=1}^{m} r_k T \right\} > 0$ for any $t \in [0, T]$. $T > 0$ is arbitrary, so $x(t) > 0$ for any $t \geq 0$, which makes (29) impossible. Further, the inequality $x(t) \leq A = x(0) \exp(\sum_{k=1}^{m} r_k T)$ for any $t \in [0, T]$ contradicts (30), which completes the proof. \[\square\]

4. **Linearized oscillation**

Throughout this section we assume the existence of a global solution for $t \geq 0$.

**Theorem 2.** Suppose (a1)–(a4) hold and

$$
\int_{0}^{\infty} \sum_{k=1}^{m} r_k(t) \, dt = \infty, \quad xf_k(x) > 0, \quad x \neq 0.
$$

(33)

Then for any nonoscillatory solution $x(t)$ of (6)

$$
\lim_{t \to \infty} x(t) = 0.
$$

(34)

**Proof.** The first equality in (33) implies that at least one of the integrals of $r_k$ diverges. Let it be $r_j$. We also recall that all $r_k$ are nonnegative.

Let us assume that $x(t) > 0$, $t \geq t_0$ (the case of negative $x(t)$ is treated similarly). Then by (a2) there is such a $t_1 \geq t_0$ that $x(h(s)) > 0$, $s > t_1$. So $\dot{x}(t) < 0$, or $x(t)$ is decreasing for $t > t_1$, $x(t)$ is bounded, $x(t) > 0$, $t \geq t_1$. Consequently, there exists a limit $d = \lim_{t \to \infty} x(t) \geq 0$. Let $d > 0$. By (33) we also have $f_k(d) = d_k > 0$. Then there exists $t_2 > 0$, such that $f_k(x(t)) \geq d_k/2$, $t > t_2$, and $t_3 \geq t_2$, such that $h(t) > t_3$, $t > t_3$. Integrating from $t_3$ to infinity, we obtain

$$
\int_{t_3}^{\infty} \dot{x}(\tau) \, d\tau = d - x(t_3) = -\int_{t_3}^{\infty} \sum_{k=1}^{m} r_k(\tau) \, d\tau \int_{h_k(\tau)}^{\tau} f_k(x(s)) \, ds \, R_k(\tau, s) \\
\leq -\int_{t_3}^{\infty} \sum_{k=1}^{m} \frac{d_k}{2} r_k(\tau) \, d\tau \leq -\frac{d_j}{2} \int_{t_3}^{\infty} r_j(\tau) \, d\tau = -\infty.
$$

Since $d - x(t_3)$ is finite, then we obtain a contradiction, which completes the proof. \[\square\]

**Remark 2.** The example of an ordinary differential equation $x' = x(x - 1)^2$ (all solutions of this equations with $x(0) \geq 1$ converge to the equilibrium $x = 1$) illustrates that the condition $xf_k(x) \geq 0$ (the nonstrict inequality for $x \neq 0$) is not enough for the convergence to the zero equilibrium. Let us also comment that $f_k(x) > 0$, $x > 0$, implied convergence to zero for positive solutions, while $f_k(x) < 0$, $x < 0$, for negative solutions.

**Theorem 3.** Suppose (a1)–(a5), (33) hold and

$$
\lim_{x \to 0} \frac{f(x)}{x} = 1.
$$

(35)

If for some $\varepsilon > 0$ all solutions of the equation

$$
\dot{x}(t) + (1 - \varepsilon) \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{\tau} x(s) \, ds \, R_k(t, s) = 0
$$

(36)

are oscillatory, then all solutions of (6) are oscillatory.
Proof. Let $x(t)$ be an eventually positive solution of (6). Then $\lim_{t \to \infty} x(t) = 0$ by Theorem 2. By (35) for any $\varepsilon > 0$ there exists $t_1$, such that $f_k(x(t)) \geq (1 - \varepsilon) x(t), t \geq t_1, k = 1, \ldots, m$. Thus

$$
\dot{x}(t) + (1 - \varepsilon) \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) \, ds \, R_k(t, s) \leq \dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k(x(t)) \, ds \, R_k(t, s) = 0.
$$

By Lemma 1 Eq. (36) has a nonoscillatory solution. In the case $x < 0$ for any $\varepsilon > 0$ there exists $t_1$, such that $f_k(x(t)) \leq (1 - \varepsilon) x(t), t \geq t_1, k = 1, \ldots, m$. Similar to the previous case, Eq. (36) has a nonoscillatory solution, which completes the proof. □

Now let us proceed to nonoscillation.

Theorem 4. Suppose (a1)–(a5) hold and for all $k = 1, \ldots, m$ either

$$
0 < f_k(x) \leq x, \quad x > 0,
$$

or

$$
0 > f_k(x) \geq x, \quad x < 0,
$$

and there exists a nonoscillatory solution of (9). Then there exists a nonoscillatory (positive or negative, respectively) solution of (6).

Proof. First suppose (37) holds and there exists a nonoscillatory solution of (9). Then by Lemma 1 there exists $w_0(t) \geq 0$ which is a solution of (14) for $t \geq t_1$:

$$
w_0(t) \geq \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} \exp \left\{ \int_{s}^{t} w_0(\tau) \, d\tau \right\} \, ds \, R_k(t, s).
$$

Let us fix $b > t_1$ and define the operator

$$(Tu)(t) = \exp \left\{ \int_{t_1}^{t} u(\tau) \, d\tau \right\} \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k \left( \exp \left\{ - \int_{t_1}^{t} u(\tau) \, d\tau \right\} \right) \, ds \, R_k(t, s), \quad t_1 \leq t \leq b
$$

(we assume that $u(s) = 0, s < t_1$). For any $u$ from the interval $0 \leq u \leq w_0$ we have by (37)

$$
0 \leq (Tu)(t) \leq \exp \left\{ \int_{t_1}^{t} u(\tau) \, d\tau \right\} \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} \exp \left\{ - \int_{t_1}^{t} u(\tau) \, d\tau \right\} \, ds \, R_k(t, s)
$$

$$
= \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, ds \, R_k(t, s) \leq w_0(t),
$$

so $0 \leq Tu \leq w_0$. Thus $T$ maps a closed segment in $L_{\infty}[t_1, b]$ onto itself.

Now let us prove that for any $b > t_1$ the operator $T$ is compact in $L_{\infty}[t_1, b]$. Let us fix $k$ and omit this index (the sum of $m$ compact operators is compact). Denote

$$(T_1 u)(t) = \int_{t_1}^{t} u(\tau) \, d\tau, \quad (T_2 u)(t) = r(t) e^{\alpha(t)} \int_{h(t)}^{t} e^{-\alpha(s)} \, ds \, dR(t, s).
$$

For any $u$ in the unit ball $B_1$ the function $y(s) = \int_{t_1}^{s} u(\tau) \, d\tau$ is continuous; moreover, all such functions are bounded ($|y(s)| \leq b - t_1$) and equicontinuous:

$$
|y(t) - y(s)| = \left| \int_{s}^{t} u(\tau) \, d\tau \right| \leq \text{ess sup}_r |u(\tau)||t - s| \leq |t - s|.
$$

Thus the image of the unit ball is compact by Lemma 3.

Then operator $T_1$ is a compact operator in the space $L_{\infty}[t_1, b]$. Moreover, it is compact as an operator $T_1 : L_{\infty}[t_1, b] \to C[t_1, b]$. Evidently the operator $T_2 : C[t_1, b] \to L_{\infty}[t_1, b]$ is continuous. Then the composition $T = T_2 T_1$ is a compact operator in the space $L_{\infty}[t_1, b]$. 
Thus by the Schauder fixed-point theorem there exists a nonnegative solution of the equation \( u = Tu \). Then the function
\[
x(t) = \begin{cases} 
\exp \left\{ - \int_{t_1}^t u(\tau) \, d\tau \right\}, & t \geq t_1, \\
1, & t < t_1,
\end{cases}
\]
is an eventually positive solution of (6).

If (38) holds, i.e., \( f(x) \geq x, x < 0 \), then we consider the segment \(-w_0(t) \leq u(t) \leq 0\) and the operator
\[
(Tu)(t) = \exp \left\{ - \int_{t_1}^t u(\tau) \, d\tau \right\} \sum_{k=1}^m r_k(t) \int_{h_k(t)}^t f_k \left( - \exp \left\{ \int_{t_1}^s u(\tau) \, d\tau \right\} \right) \, d_s R_k(t, s)
\]
which satisfies \(-w_0(t) \leq (Tu)(t) \leq 0\), as far as \(-w_0(t) \leq u(t) \leq 0\). Similarly, we demonstrate that
\[
x(t) = \begin{cases} 
- \exp \left\{ - \int_{t_1}^t u(\tau) \, d\tau \right\}, & t \geq t_1, \\
-1, & t < t_1,
\end{cases}
\]
is an eventually negative solution of (6). □

5. Applications

5.1. Logistic equation

Consider the logistic equation with a distributed delay
\[
\dot{N}(t) = N(t) \sum_{k=1}^m r_k(t) \left( 1 - \frac{1}{K_0} \int_{h_k(t)}^t N(s) \, d_s R_k(t, s) \right),
\]
(39)
where (a1)–(a4), (28) are satisfied, \( K_0 > 0 \). The existence of a global solution is due to Corollary 2 and this solution is positive.

After the substitution \( N(t) = Ke^{x(t)} \) (39) becomes
\[
\dot{x}(t) = - \sum_{k=1}^m r_k(t) \int_{h_k(t)}^t f(x(s)) \, d_s R_k(t, s),
\]
(40)
where \( f(x) = e^x - 1 \) satisfies (35) and (38).

The results of Section 4 and Lemma 2 imply the following result.

**Theorem 5.** Suppose (a1)–(a4), (28) hold. If
\[
\limsup_{t \to \infty} \int_{h(t)}^t \sum_{k=1}^m r_k(\tau) \left[ \int_{h_k(\tau)}^\tau \, d_s R_k(t, s) \right] \, d\tau < \frac{1}{e},
\]
(41)
then (39) has a nonoscillatory solution about \( K \). If
\[
\liminf_{t \to \infty} \int_{H(t)}^t \sum_{k=1}^m r_k(\tau) \left[ \int_{h_k(\tau)}^\tau \, d_s R_k(t, s) \right] \, d\tau > \frac{1}{e},
\]
(42)
then all solutions of (39) are oscillatory about \( K \). Here \( h(t), H(t) \) are defined by (13).

The result of Theorem 5 was obtained in [13, Theorem 5] using a different method. It implies the known oscillation and nonoscillation results for logistic equations with concentrated (constant or nonconstant) delays, see [40].
5.2. Lasota–Wazewska equation

Consider the generalized Lasota–Wazewska equation [49] for the survival of red blood cells with a distributed delay

\[
\dot{N}(t) = -\mu N(t) + p \int_{h(t)}^{t} e^{-\gamma N(s)} \, d_s R(t, s),
\]

where \(\mu > 0\), \(p > 0\), \(\gamma > 0\). Here the function \(e^{-\gamma x}\) is bounded for positive \(x\), so Corollary 1 can be applied to establish the existence of a global solution.

The equilibrium is

\[
N^* = \frac{p}{\mu} e^{-\gamma N^*}.
\]

After the change of variables \(N(t) = N^* e^{\lambda(t)}\), (43) becomes

\[
\dot{x}(t) + \mu x(t) + \mu \gamma N^* \int_{h(t)}^{t} f(x) \, d_s R(t, s) = 0,
\]

where

\[
f(x) = 1 - e^{-x}
\]

satisfies (35) and (37).

Thus the results of Section 4 imply the following theorems.

**Theorem 6.** Suppose (a2)–(a3) and (28) hold, where \(m = 1\) and \(R_1(t, s) = R(t, s), h_1(t) = h(t)\). If there exists \(\varepsilon > 0\), such that all solutions of the linear equation

\[
\dot{x}(t) + (1 - \varepsilon)\mu x(t) + (1 - \varepsilon)\mu \gamma N^* \int_{h(t)}^{t} x(s) \, d_s R(t, s) = 0
\]

are oscillatory, then all solutions of (43) oscillate about \(N^*\).

**Theorem 7.** Suppose (a2), (a3), and (28) hold, where \(m = 1\) and \(R_1(t, s) = R(t, s), h_1(t) = h(t)\), and there exists a nonoscillatory solution of the linear equation

\[
\dot{x}(t) + \mu x(t) + \mu \gamma N^* \int_{h(t)}^{t} x(s) \, d_s R(t, s) = 0.
\]

Then there exists a solution of (43) which is nonoscillatory about \(N^*\).

For the particular case of (43) with a variable concentrated delay

\[
\dot{N}(t) = -\mu N(t) + p e^{-\gamma N(h(t))}
\]

we obtain the following corollary which was earlier deduced in [39].

**Corollary 3.** Let \(\lim_{t \to \infty} \sup (t - h(t)) < \infty\). If \(\lim_{t \to \infty} \inf [\mu \gamma N^* \int_{h(t)}^{t} \exp(\mu(s - h(s))) \, ds] > \frac{1}{\varepsilon}\), then all solutions of (49) are oscillatory about \(N^*\).

If \(\lim_{t \to \infty} \sup [\mu \gamma N^* \int_{h(t)}^{t} \exp(\mu(s - h(s))) \, ds] < \frac{1}{\varepsilon}\), then there exists a nonoscillatory about \(N^*\) solution of (49).

For the integrodifferential equation (43)

\[
\dot{N}(t) = -\mu N(t) + p \int_{h(t)}^{t} k(t, s) e^{-\gamma N(s)} \, ds,
\]

Theorem 6 implies the following result.
Corollary 4. Let \( k(t, s) \) be a Lebesgue measurable locally essentially bounded function, \( k(t, s) \geq 0, \mu > 0, p > 0, \gamma > 0, \int_{h(t)}^{t} k(t, s) \, ds = 1 \) for any \( t > 0 \). If
\[
\liminf_{t \to \infty} \left[ \mu \gamma N^* \int_{h(t)}^{t} \int_{h(\tau)}^{T} k(\tau, s) \exp\{\mu(s - h(s))\} \, ds \right] > \frac{1}{e}
\]
then all solutions of \( (50) \) are oscillatory about \( N^* \). If
\[
\limsup_{t \to \infty} \left[ \mu \gamma N^* \int_{h(t)}^{t} \int_{h(\tau)}^{T} k(\tau, s) \exp\{\mu(s - h(s))\} \, ds \right] < \frac{1}{e}
\]
then there exists a nonoscillatory about \( N^* \) solution of \( (50) \).

To illustrate the applicability of Theorem 6 to different models, consider the mixed equation
\[
\dot{N}(t) = -\mu N(t) + p \left[ \int_{h(t)}^{t} k(t, s)e^{-\gamma N(s)} \, ds + \alpha(t)N(g(t)) \right],
\]
(51)
where \( k(t, s) \geq 0, \mu > 0, p > 0, \gamma > 0, \alpha(t) \geq 0 \).

Corollary 5. Let \( k(t, s) \) be a Lebesgue measurable locally essentially bounded function, \( (a2) \) holds for \( h(t) \) and \( g(t) \), \( k(t, s) \geq 0, \mu > 0, p > 0, \gamma > 0, \int_{h(t)}^{t} k(t, s) \, ds + \alpha(t) = 1 \) for any \( t > 0 \). If
\[
\liminf_{t \to \infty} \mu \gamma N^* \int_{\max[h(t), g(t)]}^{t} \int_{h(\tau)}^{T} k(\tau, s) \exp\{\mu(s - h(s))\} \, ds + \alpha(\tau) \exp\{\mu(\tau - g(\tau))\} \right] > \frac{1}{e}
\]
then all solutions of \( (51) \) are oscillatory about \( N^* \). If
\[
\limsup_{t \to \infty} \mu \gamma N^* \int_{\min[h(t), g(t)]}^{t} \int_{h(\tau)}^{T} k(\tau, s) \exp\{\mu(s - h(s))\} \, ds + \alpha(\tau) \exp\{\mu(\tau - g(\tau))\} \right] < \frac{1}{e}
\]
then there exists a nonoscillatory about \( N^* \) solution of \( (51) \).

5.3. Nicholson’s Blowflies equation

Now let us apply the above results to Nicholson’s blowflies equation with a distributed delay
\[
\dot{N}(t) - p \int_{h(t)}^{t} N(s)e^{-aN(s)} \, ds \, R(t, s) + \delta N(t) = 0, \quad t > 0,
\]
(52)
p > \delta > 0, a > 0, which has a positive equilibrium
\[
N^* = \frac{1}{a} \ln \frac{p}{\delta}.
\]
(53)
Here the function \( e^{-ax} \) is bounded for positive \( x \), so Corollary 1 can be applied to establish the existence of a global solution, see also [50].

We can apply the linearization argument after the transformation
\[
N = N^* + \frac{1}{a} x,
\]
(54)
where \( N^* \) is defined in (53). Then (52) becomes
\[
\dot{x}(t) + \delta x(t) - \delta \int_{h(t)}^{t} x(s)e^{-x(s)} \, ds \, R_k(t, s) + \delta \int_{h(t)}^{t} \ln \left( \frac{P}{\delta} \right) \left[ 1 - e^{-x(s)} \right] \, ds \, R_k(t, s) = 0,
\]
which can be rewritten as
\[
\dot{x}(t) + \delta x(t) + \delta \int_{h(t)}^{t} \left[ \ln \left( \frac{P}{\delta} \right) \left[ 1 - e^{-x(s)} \right] - x(s)e^{-x(s)} \right] \, ds \, R(t, s) = 0.
\]
(55)
Consider the function
\[ f(x) = \frac{1}{\ln \left( \frac{P}{\delta} \right) - 1} \left[ \ln \left( \frac{P}{\delta} \right) \left[ 1 - e^{-x} \right] - xe^{-x} \right]. \]  
(56)

Then (55) has the form
\[ \dot{x}(t) + \delta x(t) + \delta \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \int_{h(t)}^{t} f(x(s)) \, ds \, R(t, s) = 0. \]  
(57)

**Lemma 5.** Let \( f(x) \) be defined in (56), \( p > \delta > 0 \). Then:

1. (35) holds.
2. If \( p > \delta e, x \neq 0, x > 1 - \ln(p/\delta) \), then \( xf(x) > 0 \).
3. If \( p > \delta e^2 \), then (37) is satisfied.

**Proof.** Since \( \lim_{x \to 0} \frac{1-e^{-x}}{x} = \lim_{x \to 0} e^{-x} = 1 \), then (35) holds for \( p > \delta \).

The function \( f(x) \) in (56) vanishes at zero. For \( p > \delta e \) its derivative is positive
\[ f'(x) = e^{-x} + \frac{xe^{-x}}{\ln \left( \frac{P}{\delta} \right) - 1} > 0 \quad \text{for} \quad x > 1 - \ln \left( \frac{P}{\delta} \right), \]
and is negative otherwise, so \( f(x) > 0 \) for \( x > 0 \), \( f(x) < 0 \), \( 1 - \ln(p/\delta) < x < 0 \).

Further, consider \( g(x) = f(x) - x \),
\[ g'(x) = e^{-x} - 1 + \frac{xe^{-x}}{\ln \left( \frac{P}{\delta} \right) - 1}, \quad g''(x) = \frac{2 - \ln \left( \frac{P}{\delta} \right) - x}{\ln \left( \frac{P}{\delta} \right) - 1} e^{-x}. \]

Thus \( g(0) = 0, g'(0) = 0 \) and \( \ln \left( \frac{P}{\delta} \right) > 2 \) implies \( g''(x) < 0 \) for \( x > 0 \). Consequently, for \( p > \delta e^2 \) the first derivative is negative for \( x > 0 \) and \( g(x) < g(0) = 0 \), or \( f(x) < x, x > 0 \). Since also \( f(x) > 0 \) for \( x > 0 \), then (37) holds, which completes the proof. \( \square \)

The only obstacle in applying Theorems 2–4 is that \( xf(x) > 0 \) for \( x < 0 \) is not satisfied. Thus the result that any solution less than \( N^* \) tends to \( N^* \) is not a corollary of Theorem 2. Lemma 6 demonstrates that, without loss of generality, we can assume that for any negative solution of (55) there exists \( t_1 \geq 0 \), such that for \( t > t_1 \) the solution \( x(t) \) satisfies \( x(t) > 1 - \ln(p/\delta) \). Then, Theorems 2 and 3 can be applied. For illustrative purposes, we will prove the equivalent result for (52). According to the transformation (54), the condition \( x(t) > 1 - \ln(p/\delta) \) is equivalent to the inequality
\[ N(t) = N^* + \frac{1}{a} x > \frac{1}{a} \left[ \ln \left( \frac{P}{\delta} \right) + 1 - \ln \left( \frac{P}{\delta} \right) \right] = \frac{1}{a}. \]
We recall that \( N^* > 1/a \) for \( p > \delta e \).

**Lemma 6.** Suppose \( p > \delta e \) and a solution \( N(t) \) of (52) is below the equilibrium \( N(t) < N^* \) for any \( t > t_1 \geq 0 \). Then there exists \( t^* \), such that
\[ N(t) > \frac{1}{a}, \quad t > t^*. \]  
(58)

**Proof.** Denote
\[ g(x) = \frac{P}{\delta} x e^{-ax}. \]  
(59)

According to (a2), there exists \( t_2 \geq t_1 \), such that \( h(t) > t_1 \) for \( t > t_2 \). Since the solution \( N(t) \) is positive and continuous, then there exists
\[ N_1 = \min_{t \in [t_1, t_2]} N(t) < N^*. \]  
(60)
we have \( g(N_1) \) already exceeds \( 1/a \). \( N_2 \) in part (3) of the proof exceeds \( g(N_1) \), \( N_3 = N^* \).

(1) Let us demonstrate that if \( N_1 = C > 1/a \), then \( N(t) > 1/a \) for any \( t > t_1 \). Assume the contrary. Denote \( \tilde{t} = \inf \{ t > t_2 \mid N(t) < \frac{1}{a} \} \). By definition \( N(\tilde{t}) = 1/a \) and \( N^* > N(t) > 1/a \) for \( t \in [t_1, \tilde{t}] \). Thus \( g(N(t)) > N^* \) for \( t \in [t_1, \tilde{t}] \) (see Fig. 1). Consequently, from (52) we have

\[
\dot{N}(t) \geq \delta \left( \inf_{s \in [t_1, \tilde{t}]} g(N(s)) - N(t) \right) > \delta (N^* - N^*) = 0, \quad t \in [t_2, \tilde{t}]
\]

almost everywhere, the nonnegative derivative in the segment \( [t_2, \tilde{t}] \) contradicts the assumption \( N(\tilde{t}) = 1/a < N(t_2) \).

(2) Next, let us assume that \( m < N^* \) and prove that once \( N(t) > m, t \in [t_1, t_2] \), then, first, \( N(t) > m \) for any \( t \geq t_1 \) and, second, if \( c \leq g(m) < N^* \) and there is \( t_3 \), such that \( N(t_3) = c \), then \( N(t) \geq c, t \geq t_3 \).

As in (1), first assume that there are points where \( N(t) \) does not exceed \( m \) and denote \( \tilde{t} = \inf \{ t > t_2 \mid N(t) < m \} \). By definition \( N(\tilde{t}) = m \) and \( N^* > N(t) > m \) for \( t \in [t_1, \tilde{t}] \). Since \( N(t) \) is continuous and \( g(m) > m \), then there exists \( \varepsilon > 0 \) such that \( N(t) < g(m) \) for \( t \in [\tilde{t} - \varepsilon, \tilde{t}] \). Besides, \( N(t) < N^* \) for any \( t \). Let us notice that \( \min_{x \in [m, N^*]} g(x) = \min \{ g(m), N^* \} \) and \( g(N(t)) > \min \{ g(m), N^* \}, t \in [t_1, \tilde{t}] \). Hence for \( t \in [\tilde{t} - \varepsilon, \tilde{t}] \) we have \( N(t) < g(m) \) and

\[
\dot{N}(t) \geq \delta \left( \inf_{s \in [t_1, \text{barr}]} g(N(s)) - N(t) \right) > \delta (\min \{ g(m), N^* \} - \min \{ g(m), N^* \}) = 0
\]

almost everywhere, which contradicts the assumption \( N(\tilde{t}) = m < N(\tilde{t} - \varepsilon) \).

Next, assume that \( g(m) < N^* \), \( N(t_4) = c \leq g(m) \) and there is \( t_4 > t_3 \) where \( N(t_4) < c \). According to the previous part, \( N(t) > m \) and \( g(N(t)) > g(m) \) for any \( t > t_1 \). Then, like in (61), \( \dot{N}(t) \geq 0 \) in \( [t_4 - \varepsilon_1, t_4] \), which contradicts \( N(t_4) < N(t_4 - \varepsilon_1) \).

(3) Finally, assuming \( N_1 < 1/a \), we build a sequence of \( N_k \), which eventually exceeds \( 1/a \) and a sequence of increasing points \( s_k \), such that \( t \geq s_k \) implies \( N(t) \geq N_k, t > s_k \). Let \( N_1 \leq 1/a \). Consider \( N_2 = \min \{ 0.5(N^* + 1/a), g(N_1) \}, N_2 \geq N_1 \) (see Fig. 1). According to part (2) of the proof, there may be 2 possibilities: for some \( s_2 = t_3 > t_2 \) we have \( N(t_3) = N_2 \) and also \( N(t) \geq N_2 \) for \( t \geq s_2 \), or \( N \) is increasing (see (61)) and is less than \( N_2 \) for any \( t > t_1 \). The latter is impossible. In fact, assuming \( N < N_2 \) implies that \( \dot{N} > \delta (g(N_1) - N_1) > 0 \). Thus \( N(t) \to \infty \) as \( t \to \infty \), which contradicts \( N(t) < N^* \). Hence \( N(t_3) = N_2 \) for some \( t_3 \). Similarly, we define \( N_k = \min \{ 0.5(N^* + 1/a), g(N_{k-1}) \} \). By induction, we prove that for some \( s_k > s_{k-1} \) we have \( N(t) \geq N_k, t > s_k \). The sequence \( \{ N_k \} \) is nondecreasing, i.e. each element is less than \( N^* \) and eventually exceeds \( 1/a \). Let \( N_k > 1/a \). Then \( s_k = t^* \), where the existence of \( t^* \) is claimed in the statement of the lemma, which completes the proof. \( \square \)
Remark 3. Continuing the proof of Lemma 6, we could obtain that any solution of (52), which is less than the equilibrium, converges to $N^*$. 

Thus applying Theorems 3 and 4 and Lemma 1 we get the following results. Let us also note that according to Theorem 2, any nonoscillatory solution tends to zero.

Theorem 8. Suppose $h(t)$, $R(t, s)$ and initial conditions satisfy (a2)–(a4), $p > \delta e$. If

$$
\delta \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \liminf_{t \to \infty} \int_{h(t)}^{t} \int_{h(\tau)}^{T} e^{\delta(t-s)} \, ds \, R(\tau, s) > \frac{1}{e},
$$

(62)

then all solutions of (52) are oscillatory about $N^*$. If in addition $p > \delta e^2$ and

$$
\delta \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \limsup_{t \to \infty} \int_{h(t)}^{t} \int_{h(\tau)}^{T} e^{\delta(t-s)} \, ds \, R(\tau, s) < \frac{1}{e},
$$

(63)

then (52) has a nonoscillatory about $N^*$ solution. For nonoscillatory solutions $\lim_{t \to \infty} N(t) = N^*$.

To deduce some corollaries, let us consider particular cases of (52): the equations with several concentrated delays

$$
\dot{N}(t) - p \sum_{k=1}^{m} N(h_k(t)) e^{-aN(h_k(t))} + \delta N(t) = 0,
$$

(64)

the autonomous equation with a constant delay

$$
\dot{N}(t) - pN(t - \tau) e^{-aN(t-\tau)} + \delta N(t) = 0,
$$

(65)

and the integrodifferential equation

$$
\dot{N}(t) - p \int_{h(t)}^{T} K(t, s) N(s) e^{-aN(s)} ds + \delta N(t) = 0.
$$

(66)

Corollary 6. If $p > \delta e$ and

$$
\liminf_{t \to \infty} \frac{\delta}{m} \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \sum_{k=1}^{m} \int_{h_k(t)}^{t} e^{\delta(h_k(t)-\tau)} \, d\tau > \frac{1}{e},
$$

(67)

then all solutions of (64) are oscillatory about $N^*$.

If in addition $p > \delta e^2$ and

$$
\limsup_{t \to \infty} \frac{\delta}{m} \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \sum_{k=1}^{m} \int_{h_k(t)}^{t} e^{\delta(h_k(t)-\tau)} \, d\tau < \frac{1}{e},
$$

(68)

then (64) has a nonoscillatory about $N^*$ solution. For nonoscillatory solutions $\lim_{t \to \infty} N(t) = N^*$.

Corollary 7 ([40,42]). If $p > \delta e$ and

$$
\delta \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \tau e^{\delta \tau} > \frac{1}{e},
$$

(69)

then all solutions of (65) are oscillatory about $N^*$. If in addition $p > \delta e^2$ and

$$
\delta \left[ \ln \left( \frac{P}{\delta} \right) - 1 \right] \tau e^{\delta \tau} < \frac{1}{e},
$$

(70)

then (65) has a nonoscillatory about $N^*$ solution. For nonoscillatory solutions $\lim_{t \to \infty} N(t) = N^*$. 
Corollary 8. Let \( p > \delta e, K(t,s) \) be a Lebesgue measurable locally essentially bounded function, \( K(t,s) \geq 0, \mu > 0, p > 0, \gamma > 0, \int_{h(t)}^{t} k(t,s) \, ds = 1 \) for any \( t > 0 \). If
\[
\liminf_{t \to \infty} \delta \left[ \ln \left( \frac{p}{\delta} \right) - 1 \right] \int_{h(t)}^{t} \, d\tau \int_{h(\tau)}^{\tau} e^{\delta(\tau-s)} K(\tau,s) \, ds > \frac{1}{e},
\]
then all solutions of (66) are oscillatory about \( N^* \). If in addition \( p > \delta e^2 \) and
\[
\limsup_{t \to \infty} \delta \left[ \ln \left( \frac{p}{\delta} \right) - 1 \right] \int_{h(t)}^{t} \, d\tau \int_{h(\tau)}^{\tau} e^{\delta(\tau-s)} K(\tau,s) \, ds < \frac{1}{e},
\]
then (66) has a nonoscillatory about \( N^* \) solution. For nonoscillatory solutions \( \lim_{t \to \infty} N(t) = N^* \).

Let us note that oscillation properties for Nicholson’s Blowflies equation with \( \delta < p < \delta e \) are essentially different (see [51] for the constant concentrated delay and [50] for the distributed delay).

6. Distributed delay and variable concentrated delay

Let us demonstrate that from a certain point of view an equation with a distributed delay (6) can be reduced to a linear equation with a constant concentrated delay.

Denote
\[
\begin{align*}
& h_k(t) = \operatorname{sup} \{ s \in \mathbb{R} | R_k(t,s) = 0 \}, \quad h(t) = \min_k h_k(t), \\
& G_k(t) = \operatorname{inf} \{ s \leq t | R_k(t,s) = 1 \}, \quad G(t) = \max_k G_k(t).
\end{align*}
\]

Theorem 9. Suppose (a1)–(a5) hold, \( f_k(0) = 0, k = 1, \ldots, m \). Then there exist functions \( \xi_k(t) \) and \( g(t) \), where \( h(t) \leq g(t) \leq G(t) \) (\( h(t), G(t) \) are defined in (73), (74)), such that the solutions of (6), (8) also satisfy a linear equation with a single concentrated delay
\[
\dot{x}(t) + \left( \sum_{k=1}^{m} r_k(t) f'_k(\xi_k(t)) \right) x(g(t)) = 0.
\]

Proof. Since \( f_k(x(\cdot)) \) is a continuous function for any \( k \), then by the mean value theorem for any \( k \) there exist \( g_k(t) \), \( h_k(t) \leq g_k(t) \leq H_k(t) \), such that
\[
\int_{h_k(t)}^{t} f_k(x(s)) \, ds R_k(t,s) = f_k(x(g_k(t))) \int_{h_k(t)}^{t} ds R_k(t,s) = f_k(x(g_k(t))),
\]
i.e., \( x(t) \) is a solution of the equation
\[
\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k(x(g_k(t))) = 0,
\]
as well as
\[
\dot{x}(t) + \sum_{k=1}^{m} r_k(t) [ f_k(x(g_k(t))) - f_k(0) ] = 0.
\]

By the mean value theorem the expression in the brackets equals \( f'_k(\xi_k(t)) x(g_k(t)) \), where \( \xi_k(t) \) is between zero and \( x(g_k(t)) \), so \( x(t) \) is a solution of the equation
\[
\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f'_k(\xi_k(t)) x(g_k(t)) = 0.
\]
By Lemma 5 in [52], there is a function \( g(t), h(t) \leq \min_k g_k(t) \leq g(t) \leq \max_k g_k(t) \leq G(t) \), such that \( x(t) \) is also a solution of the equation with one delay

\[
\dot{x}(t) + \left( \sum_{k=1}^{m} r_k(t) f_k'(\xi_k(t)) \right) x(g(t)) = 0,
\]

(79)

which completes the proof. □

For the linear equation (9) we immediately obtain

**Corollary 9.** Suppose \((a1)\)–\((a4)\) hold. Then there exists \( g(t) \), where \( h(t) \leq g(t) \leq H(t) \) \((h(t), G(t)\) are defined in (73), (74)), such that the solution of (9), (8) also satisfies a linear equation with a single concentrated delay

\[
\dot{x}(t) + \left( \sum_{k=1}^{m} r_k(t) \right) x(g(t)) = 0.
\]

(80)

The following result is also an immediate corollary of Theorem 9.

**Corollary 10.** Suppose \((a1)\)–\((a5)\) hold and the linear equation

\[
\dot{x}(t) + \left( \sum_{k=1}^{m} A_k r_k(t) \right) x(g(t)) = 0
\]

(81)

has one of the following properties for any \( g(t) \) satisfying \( h(t) \leq g(t) \leq G(t) \) and any \( A_k \) satisfying

\[
\inf_{t \in \mathbb{R}} f_k'(t) \leq A_k \leq \sup_{t \in \mathbb{R}} f_k'(t)
\]

(82)

– all solutions of (81) are oscillatory;
– there exists a nonoscillatory solution of (81);
– the zero solution of (81) is stable (globally asymptotically stable);
– all solutions of (81) with nonnegative initial conditions and a positive initial value are positive (permanent, i.e., satisfy \( 0 < a < x(t) < b < \infty \) for any \( t \)).

Then (6) has the same property.

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