Linearized oscillation theory for a nonlinear nonautonomous delay differential equation

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Abstract

Oscillation properties of two following equations are compared: a scalar nonlinear delay differential equation

\[ \dot{y}(t) + \sum_{k=1}^{m} r_k(t)f_k[y(h_k(t))] = 0 \]

with \( r_k(t) \geq 0 \), \( h_k(t) \leq t \), and a linear delay differential equation

\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0. \]

Coefficients \( r_k(t) \) and delays are not assumed to be continuous. As an application, explicit oscillation and nonoscillation conditions are established for nonlinear equations arising in population dynamics.

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1. Introduction

Nonlinear delay differential equations arise as models of population dynamics, economics, mechanics and technology where the evolution of a system depends not only on its present state but also on its history. Usually the study of nonlinear delay differential equations is more complicated than...
the investigation of linear equations. However in certain cases it is possible to deduce the properties of a nonlinear equation from an associated linear equation. The purpose of the linearized oscillation theory is to study the oscillation of an associated linear equation rather than the original nonlinear equation. Such a theory is very well developed for autonomous nonlinear delay differential equations and some generalizations (see monographs [4,6] and references therein). Nevertheless, for nonlinear nonautonomous delay differential equations most oscillation results were obtained by direct methods, without reducing these equations to linear ones [5,10–12,14].

Only few works are concerned with the linearized theory of such equations (see [6–9,13]). Paper [7] deals with a rather general nonlinear nonautonomous delay differential equation

\[ \dot{x}(t) + f(t, x(h_1(t)), \ldots, x(h_m(t))) = 0 \]  

and an associated linear equation

\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0. \]  

As shown in [7], under appropriate hypotheses oscillation (nonoscillation) of linear equation (2) implies oscillation (nonoscillation) of nonlinear equation (1). In addition, paper [7] presents necessary and sufficient conditions for the equivalence of oscillation properties of nonlinear and linear equations. However some results of [7] are incorrect.

In the present paper, we consider the following equation:

\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t)f_k[x(h_k(t))] = 0, \]  

which is a special case of (1). We revise linearized oscillation conditions and improve linearized nonoscillation conditions obtained in the paper [7] for Eq. (3). These results are applied to equations of mathematical biology.

2. Preliminaries

Consider the scalar delay differential equation (3) under the following assumptions:

(a1) \( r_k(t) \geq 0, \ k = 1, \ldots, m \) are Lebesgue measurable locally essentially bounded functions;

(a2) \( h_k : [0, \infty) \to R, \ k = 1, \ldots, m \) are Lebesgue measurable functions, \( h_k(t) \leq t, \lim_{t \to \infty} h_k(t) = \infty; \)

(a3) \( f_k : R \to R, \ k = 1, \ldots, m \) are continuous functions, \( x f_k(x) > 0, x \neq 0. \)

Together with (3) we consider for each \( t_0 \geq 0 \) an initial value problem

\[ \dot{x}(t) + \sum_{k=1}^{m} r_k(t)f_k[x(h_k(t))] = 0, \quad t \geq t_0, \]  

\[ x(t) = \phi(t), \quad t < t_0, \quad x(t_0) = x_0. \]  

We also assume that the following hypothesis holds

(a4) \( \phi : (-\infty, t_0) \to R \) is a Borel measurable bounded function.
**Definition.** An absolutely continuous in each interval \([t_0, b]\) function \(x : \mathbb{R} \rightarrow \mathbb{R}\) is called a solution of problem (4) and (5), if it satisfies Eq. (4) for almost all \(t \in [t_0, \infty)\) and equalities (5) for \(t \leq t_0\). Eq. (3) has a nonoscillatory solution if it has an eventually positive or an eventually negative solution. Otherwise all solutions of (3) are oscillatory.

We will present here a lemma which will be used in the proof of the main results.

**Lemma 1** (Győri and Ladas [6]). Let (a1) and (a2) hold for (2). Then the following hypotheses are equivalent:

1. differential inequality
   \[
   \dot{x}(t) + \sum_{k=1}^{m} r_k(t)x(h_k(t)) \leq 0, \quad t \geq 0,
   \]
   has an eventually positive solution;
2. there exists \(t_0 \geq 0\) such that the following inequality:
   \[
   u(t) \geq \sum_{k=1}^{m} r_k(t) \exp \left\{ \int_{h_k(t)}^{t} u(s) \, ds \right\}, \quad t \geq t_0, \quad u(t) = 0, \quad t < t_0,
   \]
   has a nonnegative locally integrable solution;
3. Eq. (2) has a nonoscillatory solution.

**3. Oscillation conditions**

In Sections 3 and 4, we assume that (a1)–(a4) hold for Eq. (3). We will use the following lemma.

**Lemma 2** (Kocić et al. [7]). Suppose there exists index \(k\) such that
\[
\int_{0}^{\infty} r_k(t) \, dt = \infty
\]  
and \(x(t)\) is a nonoscillatory solution of (3). Then \(\lim_{t \to \infty} x(t) = 0\).

**Theorem 1.** Suppose (6) holds and
\[
\lim_{x \to 0} \frac{f_k(x)}{x} = 1, \quad k = 1, \ldots, m.
\]  
If for some \(\varepsilon > 0\) all solutions of the linear equation
\[
\dot{x}(t) + (1 - \varepsilon) \sum_{k=1}^{m} r_k(t)x(h_k(t)) = 0
\]  
are oscillatory, then all solutions of Eq. (3) are also oscillatory.
Proof. First suppose \( x(t) > 0, \ t \geq t_1 \), is an eventually positive solution of (3) and
\[
h_k(t) \geq t_1, \quad \text{for} \ t \geq t_2. \tag{9}\]
Lemma 2 and (7) imply that there exists \( t_3 \geq t_2 \) such that
\[
f(x(h_k(t))) > (1 - \varepsilon)x(h_k(t)), \quad t \geq t_3.
\]
Hence
\[
\dot{x}(t) + (1 - \varepsilon) \sum_{k=1}^{m} r_k(t) x(h_k(t)) \leq 0, \quad t \geq t_3.
\]
Lemma 1 yields that Eq. (8) has a nonoscillatory solution which leads to a contradiction.

Suppose now \( x(t) < 0 \) for \( t \geq t_1 \) and (9) holds for \( t \geq t_2 \). Denote \( y(t) = -x(t) \), \( g_k(y) = -f_k(-y) \).
Hence functions \( g_k \) satisfy all the assumptions for \( f_k \), \( y(t) \) is an eventually positive solution of the equation
\[
\dot{y}(t) + \sum_{k=1}^{m} r_k(t) g_k(y(h_k(t))) = 0.
\]
As was shown above, we have
\[
\dot{y}(t) + (1 - \varepsilon) \sum_{k=1}^{m} r_k(t) y(h_k(t)) \leq 0, \quad t \geq t_2,
\]
for some \( t_2 \geq t_1 \). Then Eq. (8) has a nonoscillatory solution. This contradiction proves the theorem. \( \square \)

Remark. Theorem 1 in [7] contains a stronger oscillation result than our Theorem 1. However the proof of Theorem 1 in [7] is based on Lemma 3 of [7] which is not correct. Really, by this lemma two equations
\[
\dot{x}(t) + \frac{1}{e} x(t - 1) = 0, \tag{10}
\]
\[
\dot{y}(t) + \left( \frac{1}{e} + \frac{1}{t} \right) y(t - 1) = 0, \tag{11}
\]
have the same oscillation properties while in practice (10) has a nonoscillatory solution while all solutions of (11) are oscillatory [3].

Theorem 2. Suppose for all \( k = 1, \ldots, m \),
\[
either \ f_k(x) \leq x, \ x > 0, \quad or \quad f_k(x) \geq x, \ x < 0 \tag{12}
\]
and there exists a nonoscillatory solution of the linear delay differential equation (2).

Then there exists a nonoscillatory solution of Eq. (3) as well.

Proof. Suppose \( f_k(x) \leq x, \ x > 0 \), \( k = 1, \ldots, m \). By Lemma 1 there exist \( t_0 \geq 0 \) and \( w_0(t) \geq 0 \), \( t \geq t_0 \), \( w_0(t) = 0 \), \( t < t_0 \), such that
\[
w_0(t) \geq \sum_{k=1}^{m} r_k(t) \exp \left\{ \int_{h_k(t)}^{t} w_0(s) \, ds \right\}, \quad t \geq t_0.
\]
Let us fix \( b \geq t_0 \) and define the operator \( T : L_\infty[t_0, b] \to L_\infty[t_0, b] \) by the following equality

\[
(Tu)(t) = \sum_{k=1}^{m} r_k(t) f_k \left( \exp \left\{ - \int_{t_0}^{h_k(t)} u(s) \, ds \right\} \right) \exp \left\{ \int_{t_0}^{t} u(s) \, ds \right\},
\]

where \( L_\infty[t_0, b] \) is the space of all essentially bounded on \( [t_0, b] \) functions with the usual norm.

For any function \( u \) from the interval \( 0 \leq u \leq w_0 \) we have

\[
0 \leq (Tu)(t) \leq \sum_{k=1}^{m} r_k(t) \exp \left\{ - \int_{t_0}^{h_k(t)} u(s) \, ds \right\} \exp \left\{ \int_{t_0}^{t} u(s) \, ds \right\}
\]

\[
\leq \sum_{k=1}^{m} r_k(t) \exp \left\{ \int_{h_k(t)}^{t} w_0(s) \, ds \right\} \leq w_0(t).
\]

Hence \( 0 \leq Tu \leq w_0 \). Lemma 3 [2] implies that operator \( T \) is a compact operator in the space \( L_\infty[t_0, b] \). Then by Schauder fixed point theorem there exists a nonnegative solution of equation \( u = Tu \).

Denote

\[
x(t) = \begin{cases} 
\exp \left\{ - \int_{t_0}^{t} u(s) \, ds \right\}, & t \geq t_0, \\
0, & t < t_0.
\end{cases}
\]

Then \( x(t) \) is an eventually positive solution of Eq. (3).

If \( f_k(x) \geq x \), \( x < 0 \), \( k = 1, \ldots, m \), then (3) has an eventually negative solution, which completes the proof of the theorem. \( \Box \)

**Remark.** For Eq. (3) Theorem 2 improves Theorem 2 of [7] since in [7] functions \( f_k \) are assumed to be increasing.

### 4. Applications

As a corollary of Theorems 1 and 2 the following well-known linearized oscillation result [6] can be obtained.

**Suppose** \( xf_k(x) > 0 \), \( x \neq 0 \) and conditions (7), (12) hold, \( r_k \geq 0, \tau_k > 0 \). **Nonlinear autonomous equation**

\[
\dot{y}(t) + \sum_{k=1}^{m} r_k f_k(y(t - \tau_k)) = 0,
\]

has a nonoscillatory solution if and only if an algebraic equation

\[
z = \sum_{k=1}^{m} r_k e^{\tau_k z}
\]

has a positive solution \( z > 0 \).
Consider now the delay logistic equation
\[ \dot{N}(t) = N(t) \sum_{k=1}^{m} r_k(t) \left( 1 - \frac{N(h_k(t))}{K} \right), \]  \hspace{1cm} (13)
where \( r_k, h_k \) satisfy conditions (a1) and (a2), \( K > 0 \), and the initial function \( \psi \) satisfies (a4). There exists a unique solution of (13) with the initial condition
\[ N(t) = \psi(t) \geq 0, \quad t < t_0, \quad N(t_0) = y_0 > 0. \]  \hspace{1cm} (14)

Similar to the autonomous case [6], the solution of (13) and (14) are positive.

A positive solution \( N \) of (13) is said to be oscillatory about \( K \) if there exists a sequence \( t_n; t_n \to \infty \), such that \( N(t_n) = K \); \( n = 1, 2, \ldots \), \( N \) is said to be nonoscillatory about \( K \) if there exists \( t_0 \geq 0 \) such that \( |N(t) - K| > 0 \) for \( t \geq t_0 \). A solution \( N \) is said to be eventually positive (eventually negative) about \( K \) if \( N - K \) is eventually positive (eventually negative).

We recall that the oscillation (nonoscillation) of \( N \) about \( K \) is equivalent to oscillation (nonoscillation) of \( x \).

By applying Theorems 1 and 2 we obtain the following results for Eq. (13).

**Theorem 3.** Suppose (6) holds and for some \( \varepsilon > 0 \) all solutions of linear equation (8) are oscillatory. Then all solutions of Eq. (13) are oscillatory about \( K \).

Explicit oscillation results for linear delay differential equations are well-known. Thus by Theorem 3 these conditions imply explicit conditions for Eq. (13). For example, we have the following result.

**Corollary.** Suppose there exist indices \( i_l \in \{1, \ldots, m\} \), with \( l = 1, \ldots, n \), such that
\[ \lim_{t \to \infty} \inf [t - h_{i_l}(t)] > 0, \quad \lim_{t \to \infty} \inf \sum_{l=1}^{n} r_{i_l}(t) > 0 \]
and
\[ \lim_{t \to \infty} \inf \sum_{k=1}^{m} r_k(t)(t - h_k(t)) > \frac{1}{e}. \]  \hspace{1cm} (15)

Then all solutions of Eq. (13) are oscillatory about \( K \).

**Proof.** Inequality (15) yields that for some \( \varepsilon > 0 \)
\[ \lim_{t \to \infty} \inf \sum_{k=1}^{m} (1 - \varepsilon)r_k(t)(t - h_k(t)) > \frac{1}{e}. \]

By Corollary 3.4.1 in [6] all solutions of Eq. (8) are oscillatory. Thus all solutions of Eq. (13) are oscillatory about \( K \). □
Theorem 4. Suppose there exists a nonoscillatory solution of the linear delay differential equation (2). Then there exists a nonoscillatory about \( K \) solution of Eq. (13).

Corollary. If there exists \( \mu > 0, \ t_0 \geq 0, \) such that
\[
\sum_{k=1}^{m} r_k(t)e^{\alpha[t-h_k(t)]} \leq \mu, \quad t \geq t_0,
\]
then there exists a nonoscillatory about \( K \) solution of Eq. (13).

Proof. Based on Theorems 4 and 3.3.2 [6]. \( \square \)

Remark. Theorems 3 and 4 were obtained in [1] using a different method.

Consider now the generalized Lasota–Wazewska equation for the survival of red blood cells (for details see [6])
\[
\dot{N}(t) = -\mu N(t) + pe^{-\gamma N(h(t))}, \quad t \geq 0,
\] (16)
where \( \mu, p, \gamma > 0 \) and for \( h(t) \) the condition (a2) holds.

We consider only those solutions of (16) which correspond to initial conditions (14). Then (16), (14) has a unique solution which is positive for all \( t \geq t_0. \)

The equilibrium \( N^* \) of Eq. (16) is positive and satisfies the equation
\[
N^* = \frac{p}{\mu} e^{-\gamma N^*}.
\]
The change of variables
\[
N(t) = N^* + \frac{1}{\gamma} x(t),
\]
turns Eq. (16) into the following one:
\[
\dot{x}(t) + \mu x(t) + \mu\gamma N^*[1 - e^{-x(h(t))}] = 0.
\] (17)

Eq. (17) has form (3), where
\[
n = 2, \quad r_1(t) = \mu, \quad r_2(t) = \mu\gamma N^*, \quad h_1(t) = t, \quad h_2(t) = h(t), \quad f_1(x) = x, \quad f_2(x) = 1 - e^{-x}.
\]
All solutions of (16) are oscillatory about \( N^* \) if and only if all solutions of (17) are oscillatory about zero.

Conditions (a3), (7) and (12) are satisfied for functions \( f_1 \) and \( f_2. \) As corollaries of Theorems 1 and 2 we obtain the following results.

Theorem 5. Suppose there exists \( \varepsilon > 0 \) such that all solutions of the linear equation
\[
\dot{x}(t) + (1 - \varepsilon)\mu x(t) + (1 - \varepsilon)\mu\gamma N^* x(h(t)) = 0
\] (18)
are oscillatory. Then all solutions of (16) are oscillatory about \( N^*. \)
Corollary. Suppose
\[
\lim_{t \to \infty} \sup (t - h(t)) < \infty, \quad \lim_{t \to \infty} \inf \mu(t) N^* \int_{h(t)}^{t} \exp \{\mu(s - h(s))\} \, ds > \frac{1}{e}.
\]  
Then all solutions of (16) are oscillatory about \( N^* \).

Proof. After the substitution \( x(t) = ye^{-(1-\varepsilon)\mu t} \) Eq. (18) takes the following form
\[
\dot{y}(t) + (1 - \varepsilon)\mu N^* \exp((1 - \varepsilon)\mu(t - h(t)) y(h(t)) = 0.
\]  
Inequalities (19) yield that for some \( \varepsilon > 0 \)
\[
\lim_{t \to \infty} \inf (1 - \varepsilon)\mu N^* \int_{h(t)}^{t} \exp \{\mu(1 - \varepsilon)(s - h(s))\} \, ds > \frac{1}{e}.
\]  
Theorem 3.4.1 [6] implies all solutions of (20) and therefore (18) are oscillatory. \( \square \)

Theorem 6. Suppose there exists a nonoscillatory solution of linear equation
\[
x(t) + \mu x(t) + \mu N^* x(h(t)) = 0.
\]  
Then there exists a nonoscillatory about \( N^* \) solution of (16).

Corollary. Suppose
\[
\lim_{t \to \infty} \sup \mu N^* \int_{h(t)}^{t} \exp \{\mu(s - h(s))\} \, ds < \frac{1}{e}.
\]  
Then there exists a nonoscillatory about \( N^* \) solution of (16).

Proofs of Theorem 6 and its corollary are similar to the proof of Theorem 5 and its corollary; they employ Theorem 3.3.1 in [6].

References