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On Impulsive Beverton-Holt Difference Equations and their Applications

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The asymptotic properties of the impulsive Beverton-Holt difference equation

\[ x_{n+1} = \frac{\alpha_n x_n}{1 + B_n x_n}, \quad x_{n+k} = b_k x_{n+k} - d_k, \quad n, k = 1, 2, \ldots, \]

where \( p \) is a fixed positive integer, are considered. The results are applied to an impulsive logistic equation with non-constant coefficients

\[ \dot{x}(t) = x(t)(r(t) - a(t)x(t)), \quad x(t_k) = b_k x(t_k^-) - d_k, \quad \lim_{k \to \infty} \tau_k = \infty. \]

In particular, sufficient extinction and non-extinction conditions are obtained for both equations.

Keywords: Beverton-Holt difference equation; Impulsive harvesting; Asymptotic behavior; Logistic equations

Mathematics Subject Classifications: 39A11; 34K45; 92B99

\section*{INTRODUCTION}

The simplest discrete model describing population growth is the Beverton and Holt equation \cite{[1]}

\[ x_{n+1} = \frac{\alpha x_n}{1 + B x_n} \quad (1) \]

(which is also sometimes called the Pielou logistic difference equation). In some bottom-feeding fish populations, like North Atlantic plaice and haddock studied in Ref. \cite{[1]}, recruitment appears to be essentially unaffected by fishing, and this is true over a wide range of fishing effort. These species have very high fertility rates and very low survivorship to adulthood. The Beverton-Holt model is equivalent to the discrete Verhulst equation

\[ x_{n+1} = \frac{R x_n}{x_n + A}, \]

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which is a discrete version of the logistic differential equation [14,15]

\[ \frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \quad x(t_0) = x_0, \]  

(2)

where \( x(t) \) is the size of the resource population, \( r \) is an intrinsic growth rate, \( K \) is a carrying capacity, or saturation level; for more sophisticated models of the population growth (see Refs. [2–5]).

We can extend the Beverton-Holt model to the case when the recruitment is affected by some harvesting (e.g. fishing or hunting). Let us assume that at every \( p \)-th step the system is subject to a perturbation which incorporates the proportional decrease (only \( bx_{pk} \) is left at \( pk \)-th step, where \( 0 < b < 1 \) and \( x_{pk} \) is the value before the perturbation) and a deduction \( d \) which does not depend on the size of the population \( x_{pk} \). After the perturbation at step \( pk \) the size of the population \( x_{pk}^+ \) becomes

\[ x_{pk}^+ = bx_{pk} - d, \quad k = 1, 2, \ldots, \]  

(3)

where \( x_{pk} \) is the size of the population at step \( pk \) before the impulsive perturbation.

The impulsive conditions include both cases of the proportional and constant harvesting. A proportional part can be interpreted as fishery or hunting with a constant fishing (hunting) effort, while a constant part can correspond to hunting with a constant number of licences or predation with a constant number of predators that control a certain territory. Alternatively, if the model describes a farm with finite resources, the proportional reduction can correspond to some kind of taxation, while the constant part can describe a producer’s consumption. In both examples the problem of non-extinction of the population becomes a description of a reasonable hunting rate or of a judicious taxation.

As will be shown in the sequel, Eqs. (3) and (1) can be treated as a discrete analogue of the logistic equation (2) subject to short-time harvesting which is described by impulsive perturbations

\[ x(\tau_n) = bx(\tau_n^-) - d, \quad \lim_{n \to \infty} \tau_n = \infty. \]

We will study the Beverton-Holt model with non-constant coefficients and variable impulsive conditions of a linear type

\[ x_{n+1} = \frac{a_n x_n}{1 + B_n x_n}, \quad x_{pk}^+ = b_k x_{pk} - d_k, \quad 0 < b_k \leq 1, \quad d_k \geq 0. \]  

(4)

In the model with non-constant coefficients (4), we assume that the hunting rate and effort are not constant (however do not depend on the size of the population). Let us note that if \( d_k = 0 \) then all solutions of Eq. (4), with \( x_0 > 0 \), are positive, while \( d_k > 0 \) can lead to the extinction of the population.

The results on Eq. (4) are applied to the logistic equation with non-constant growth rates and carrying capacities

\[ \dot{x} = x(t)(r(t) - a(t)x(t)) \]  

(5)

and variable impulsive conditions

\[ x(\tau_n) = b_n x(\tau_n^-) - d_n, \quad \lim_{n \to \infty} \tau_n = \infty. \]  

(6)

Logistic equations with impulsive harvesting and general impulsive perturbations were recently considered in Ref. [6].
The model (5) and (6) is more general than most of previously considered impulsively perturbed logistic systems. First, the impulsive conditions (6) do not describe the proportional harvesting only. Thus the solution can become negative in some finite time which corresponds to the extinction of the population. Secondly, functions $a(t)$, $r(t)$ are not considered to be piecewise continuous but locally essentially bounded only. No additional condition (say, the periodicity of $a$, $r$) is imposed.

Discrete and continuous models of mathematical biology are closely connected (see, for example, the monograph [7] and the recent paper [8]): similar methods are sometimes applied to study linear difference and differential equations, continuous models are constructed on the base of discrete ones and vice versa. Impulsive differential equations incorporate elements of continuous and discrete systems. The main results of the present paper are concerned with the difference equation (4), the detailed analysis is due to the recent publications [9,10]. Based on these results, the asymptotics of the impulsive differential equation (5) and (6) is studied.

The paper is organized as follows. The Second Section includes relevant notations and auxiliary theorems on Riccati difference equations from Refs. [9,10]. Two modifications of these results (Lemmas 2 and 5) are applied to the analysis of impulsive difference equations. In Third Section an impulsive Beverton-Holt equation with constant and non-constant parameters is considered. Conditions for positiveness of solutions and asymptotics are described. In Fourth Section, the equivalence of the positiveness of a solution of (5) and (6) and the positiveness of the solution of a specially constructed impulsive Beverton-Holt equation is established. The results of Third Section are applied to the study of the logistic equation with impulsive harvesting, including positiveness of solutions and limit sets. Comparison results and a numerical example are presented. The Fifth Section discusses possible developments.

We note that delay logistic equations with a steady proportional harvesting were studied in Ref. [11], however, methods were completely different.

**PRELIMINARIES**

We will need the following auxiliary results concerning the first order Riccati difference equation

$$w_{n+1} = 1 - \frac{R}{w_n}, \quad n \geq 0, \quad R \neq 0.$$  \hspace{1cm} (7)

These results were obtained in Ref. [9] and can be also found in the monograph [10].

**Definition**  The set of initial conditions $w_0 \in \mathbb{R}$ through which the denominator $w_n$ will become zero for some $n \geq 0$ is called the forbidden set $F$ of Eq. (7).

**Definition**  A point $\tilde{w}$ is called an equilibrium point if $\tilde{w} = 1 - R/\tilde{w}$. An equilibrium point is called a global attractor if for every $w_0 \notin F$ we have $\lim_{n \to \infty} w_n = \tilde{w}$. An equilibrium point is called a repeller if there exists such $r > 0$ that for any $w_0 \notin F$ satisfying $|w_0 - \tilde{w}| < r$ there exists such $N$ that $|w_N - \tilde{w}| \geq r$.

**Lemma** 1 [10]  Let $R < 1/4$, $R \neq 0$. Then

$$w_- = \frac{1 - \sqrt{1 - 4R}}{2} \quad \text{and} \quad w_+ = \frac{1 + \sqrt{1 - 4R}}{2}$$  \hspace{1cm} (8)
are the only equilibrium points of Eq. (7). The forbidden set $F$ of Eq. (7) is the sequence of points

$$f_n = \left(\frac{w_{n-1} - w_{n-1}}{w_{n-1}^2 - w_n^2}\right)w_n, \quad n = 1, 2, \ldots$$

If $w_0 \notin F \cup \{w_\ast\}$, then $w_\ast$ is a global attractor, $w_\ast$ is a repeller, and the solution is given by

$$w_n = \frac{(w_0 - w_-)w_+^{n+1} - (w_+ - w_-)w_n^{n+1}}{(w_0 - w_-)w_+^{n} - (w_+ - w_-)w_n^{n}}, \quad n = 1, 2, \ldots$$

(9)

In addition to asymptotic behavior of solutions we will need the following result.

**Lemma 2** Let $R < 1/4$ and $w_0 > -1 > 0$ (we keep the notation $w_-, w_+$ of Lemma (1)). Then every next element of the sequence is closer to $w_+$ than the previous one $|w_{n+1} - w_+| < |w_n - w_+|$ and $w_+ > w_-$. Moreover, the sequence $\{w_n\}$ is monotone: if $w_n < w_+$, then $w_n \leq w_{n+1} < w_+$; if $w_n > w_+$, then $w_n \geq w_{n+1} > w_+$.

**Proof** First let us prove that $w_n > w_-$ and $\text{sgn}(w_n - w_+) = \text{sgn}(w_0 - w_+)$. To this end let us note that $w_+ w_- = R$. Assuming $w_n > w_-$ we have

$$w_{n+1} = 1 - \frac{R}{w_n} = 1 - \frac{w_+ w_-}{w_n} > 1 - \frac{w_+ w_-}{w_-} = 1 - w_+ = w_-,$$

so by induction $w_n > w_-$. If $w_n < w_+$, then

$$w_{n+1} = 1 - \frac{w_+ w_-}{w_n} < 1 - \frac{w_+ w_-}{w_+} = 1 - w_- = w_+.$$

Similarly, if $w_n > w_+$, then $w_{n+1} > w_+$.

Further, since $w_{n+1} = 1 - R/w_n$ and $R = w_- w_+$, then

$$|w_{n+1} - w_+| = \left|1 - \frac{R}{w_n} - \frac{\sqrt{1 - 4R}}{2}\right| = \left|1 - \frac{\sqrt{1 - 4R}}{2} - \frac{w_- w_+}{w_n}\right| = \left|w_- - \frac{w_- w_+}{w_n}\right|$$

$$= \left|\frac{w_- - w_+}{wn}\right| w_n - w_+ < |w_n - w_+|,$$

which completes the proof. \hfill \Box

**Lemma 3** [10] Let $R = 1/4$. Then the only equilibrium of Eq. (7) is $\tilde{w} = 1/2$. The forbidden set $F$ is the sequence of points $f_n = n - 1/2n$ for $n = 1, 2, \ldots$, which converges to the equilibrium from the left. The equilibrium point is a sink (attractor) for all $w_0 > \tilde{w}$ and is a repeller for all $w_0 < \tilde{w}$, where $w_0 \notin F$.

**Lemma 4** [10] Let $R > 1/4$ and let $\phi \in (0, \pi/2)$ be such that

$$\cos \phi = \frac{1}{2\sqrt{R}} \quad \text{and} \quad \sin \phi = \frac{\sqrt{4R - 1}}{2\sqrt{R}}.$$

Then the forbidden set $F$ is a sequence of points

$$f_n = \frac{1}{2} - \frac{\sqrt{4R - 1}}{2} \cot(n\phi), \quad n = 1, 2, \ldots$$
In addition, denote

\[ r = \sqrt{4R - 1 + (2w_0 - 1)^2}. \]

Let \( \theta \in (-\pi/2, \pi/2) \) be such that

\[ \cos \theta = \frac{\sqrt{4R - 1}}{r}, \quad \sin \theta = \frac{2w_0 - 1}{r}. \]

Then for any \( w_0 \notin F \)

\[ w_n = \sqrt{R} \frac{\cos(n\theta + \phi - \theta)}{\cos(n\theta - \theta)}. \]

There is no equilibrium points in this case.

Consider also the first order Riccati difference equation with a non-constant coefficient

\[ w_{n+1} = 1 - \frac{R_n}{w_n}, \quad n \geq 0. \tag{10} \]

Suppose \( \{c_n\} \) is a sequence of numbers. Consider the following problem: when does there exist a solution of Eq. (10) satisfying \( w_n > c_n \)? The following lemma presents sufficient conditions.

**Lemma 5** Suppose \( R = \sup_n R_n < 1/4 \) and \( \{c_n\} \) is a sequence. Denote

\[
\begin{align*}
    m_n &= \frac{1 - \sqrt{1 - 4R_n}}{2}, \quad M_n = \frac{1 + \sqrt{1 - 4R_n}}{2}, \quad c = \sup_n c_n, \\
    m &= \sup_n m_n, \quad M = \inf_n M_n = \frac{1}{2}(1 + \sqrt{1 - 4R}), \quad \mathcal{M} = \sup_n M_n.
\end{align*}
\]

Suppose \( c < M. \) Then any solution with the initial condition

\[ w_0 > \max\{c, m\} \]

satisfies \( w_n > c_n. \) Besides,

\[ M \leq \liminf_{n \to \infty} w_n \leq \limsup_{n \to \infty} w_n \leq \mathcal{M} \]  \hspace{1cm} (12)

**Proof** Let \( w_0 > \max\{c, m\}. \) As the first step, we will prove \( w_n > c_n. \)

Suppose first \( w_0 < M. \) Then \( w_1 = 1 - R_0/w_0 \) satisfies

\[
    M - w_1 = M - 1 + \frac{R_0}{w_0} < M_0 - 1 + \frac{R_0}{w_0} = \frac{1}{2} + \frac{\sqrt{1 - 4R}}{2} - 1 + \frac{m_0 M_0}{w_0} = -\frac{1}{2} + \frac{\sqrt{1 - 4R}}{2} + \frac{m_0 M_0}{w_0} = m_0 \left( \frac{M_0}{w_0} - 1 \right) > 0,
\]

so \( w_1 < M. \) Similarly, if \( w_n < M, \) then \( w_{n+1} < M. \) In addition (see the proof of Lemma 2), \( |w_{n+1} - M_n| < |w_n - M_n|, \) so if \( w_n < M \) then

\[ M - w_{n+1} = (M - M_n) + (M_n - w_{n+1}) < (M - M_n) + (M_n - w_n) = M - w_n. \]

Consequently the sequence \( \{w_n\} \) is increasing as far as \( w_n < M. \) Hence if \( w_0 < M, \)
then \( w_n \geq w_0 > c \geq c_n. \)
Now suppose \( w_0 \geq M \). If all \( w_n \geq M \) then obviously \( w_n > c_n \); if there exists such \( n \) that \( w_n < M \), then the argument is similar to the previous case \( w_0 < M \).

We note that since \( R_n < 1/4 \) then all \( m_n \) are positive, thus \( w_n \) are also positive.

At the second step, we will prove \( M \leq \liminf_{n \to \infty} w_n \leq \limsup_{n \to \infty} w_n \leq \mathcal{M} \) for any initial condition \( w_0 > M \). Note that \( m_n R_n = R_n, m_n + R_n = 1 \). If \( w_n < M \), then

\[
w_{n+1} = 1 - \frac{m_n M_n}{w_n} < 1 - \frac{m_n M_n}{M_n} = 1 - m_n = R_n.
\]

Similarly, if \( w_n > M_n \), then \( w_{n+1} > M_n > M \).

In addition, since \( w_{n+1} = 1 - R_n/w_n \) and \( R = m_n M_n \) then

\[
|w_{n+1} - M_n| = \left| 1 - \frac{R_n}{w_n} - \frac{\sqrt{1 - 4R_n}}{2} \right| = \left| 1 - \frac{\sqrt{1 - 4R_n}}{2} - \frac{m_n M_n}{w_n} \right| = \left| m_n - \frac{m_n M_n}{w_n} \right|
\]

so \( w_n < M_n \) implies \( w_{n+1} > w_n \). If \( w_n > M_n \), then \( w_n > w_{n+1} > M_n \) and still \( w_n < \mathcal{M} \), \( w_{n+1} < \mathcal{M} \).

Thus if \( w_n \) belongs to the interval \([M, \mathcal{M}]\), then all \( w_k \in [M, \mathcal{M}] \), \( k \geq n \), belong to this interval: if \( M \leq w_k < M_{k+1} \), then \( M \leq w_{k+1} < M_{k+1} \leq \mathcal{M} \), if \( \mathcal{M} \geq w_k \geq M_{k+1} \), then \( \mathcal{M} \geq w_{k+1} \geq M_{k+1} \geq M \). Consequently, as far as \( w_n \) enters the interval \([M, \mathcal{M}]\), so are all \( w_k \), \( k \geq n \), which implies Eq. (12).

Let us consider two additional cases: all \( w_n < M \) and all \( w_n > \mathcal{M} \).

If \( w_n < M \) for all \( n \), then necessarily \( w_n < M_n \). Consider the sequence \( v_n \), with \( v_0 = w_0 \) and

\[v_{n+1} = 1 - \frac{R}{v_n}, \quad \text{where} \quad R = \sup_n R_n.\]

Then \( v_1 = 1 - (R/v_0) < 1 - (R_n/w_0) = 1 - (R_n/w_n) = w_1 \).

Further, if \( v_n < w_n \) then

\[v_{n+1} = 1 - \frac{R}{v_n} < 1 - \frac{R_n}{v_n} < 1 - \frac{R_n}{w_n} = w_{n+1}.\]

Consequently, \( v_n < w_n \). Since \( \lim_{n \to \infty} v_n = M \) (\( M \) is an attractor of the Riccati equation with a constant coefficient) then \( \liminf_{n \to \infty} w_n \geq M \). If all \( w_n < M \) then obviously \( \limsup_{n \to \infty} w_n \leq M \leq \mathcal{M} \).

Similarly the case when \( w_n > \mathcal{M} \) can be considered, which completes the proof. \( \square \)

### AN IMPULSIVE BEVERTON-HOLT DIFFERENCE EQUATION

First let us consider the impulsive Beverton-Holt difference equation with constant parameters (1) and (3). Since Eq. (1) leads to a similar mapping for \( n = pk \)

\[x_{pk+1} = \frac{\alpha x_{pk}}{1 + B x_{pk}},\]
then without loss of generality we can assume an impulsive perturbation at each step \( p = 1 \)
\[
    x_{n+1} = b \left( \frac{\alpha x_n}{1 + B x_n} \right) - d = \frac{b \alpha x_n - d - dB x_n}{1 + B x_n} = \frac{(b \alpha - dB) x_n - d}{1 + B x_n}.
\]

Let us denote \( \beta = b \alpha - dB \) and study the asymptotic properties and non-extinction conditions for the equation
\[
    x_{n+1} = \frac{\beta x_n - d}{1 + B x_n},
\]
which is equivalent to the impulsive Beverton-Holt difference equation (1) and (3) with \( p = 1 \).

Denote
\[
    R = \frac{\beta + dB}{(\beta + 1)^2}, \quad w_+ = \frac{1 + \sqrt{1 - 4R}}{2}, \quad w_- = \frac{1 - \sqrt{1 - 4R}}{2}.
\]

**Theorem 1** Suppose \( R < 1/4 \), where \( R \) is defined in Eq. (14).

1. If \( (\beta + 1)(1 + \sqrt{1 - 4R}) < 2 \) then every solution of Eq. (13) becomes negative in a finite time
2. If \( \beta > 1 \), then any solution of Eq. (13), with
\[
    x_0 > \max \left\{ 0, \frac{\beta + 1}{B} w_- - \frac{1}{B} \right\},
\]
is positive for \( n \geq 0 \) and tends to \( p = (\beta + 1/B) w_+ - 1/B \) as \( n \to \infty \), where \( w_-, w_+ \) are defined in Eq. (14).

**Proof** After the transformation
\[
    x_n = \frac{\beta + 1}{B} w_n - \frac{1}{B}, \quad \text{with } R = \frac{\beta + dB}{(\beta + 1)^2}.
\]

Let us note that the positive solutions \( x_n \) of Eq. (13) after transformation (16) correspond to such \( w_n \) that \( (\beta + 1) w_n - 1 > 0 \), or \( w_n > (1/\beta + 1) \).

Thus by Lemma (1) \( w_+ \) is a global attractor of the difference equation (17) while \( w_- \) is a repeller, where \( w_+ \), \( w_- \) are defined in Eq. (14). If \( w_+ < 1/\beta + 1 \), i.e. \( (\beta + 1) \times (1 + \sqrt{1 - 4R}) < 2 \), then the global attractor corresponds to negative values of \( x \) and thus every solution becomes negative in some finite time, which corresponds to the extinction of the population. This completes the proof of (1).

Now assume \( (\beta + 1)(1 + \sqrt{1 - 4R}) > 2 \), or \( \sqrt{1 - 4R} > 2/\beta + 1 - 1 \). This is obviously satisfied if \( \beta > 1 \) (the right hand side is negative while the left hand side is positive). Thus the conditions of (2) imply that the global attractor \( w_+ \) satisfies \( w_+ > 1/\beta + 1 \).

Besides, if \( w_0 > w_- \) and \( w_0 > 1/\beta + 1 \) then all the values of \( w_n \) are above \( \max \{ w_-, 1/\beta + 1 \} \) (since \( f(w) = 1 - R/w \) is concave, monotone and its fixed points are \( w_- \), \( w_+ \)). After substituting bounds for \( w_n \) in Eq. (16) we obtain that any solution
Lemma (4) of Eq. (17) becomes negative then so is the corresponding solution of Eq. (13). According to Similar to Theorems (1) and (2), the Eq. (13) can be reduced to Eq. (17); if a solution Proof

Remark It is to be noted that if \( \beta < 1 \) then \( \sqrt{1 - 4R} > (2/\beta + 1) - 1 \) is equivalent to

\[
1 - 4R > \frac{4}{(\beta + 1)^2} - \frac{4}{\beta + 1} + 1.
\]

After substituting \( R = \beta + dB/(\beta + 1)^2 \) and multiplying by \((\beta + 1)^2\) we have \( 4 + 4(\beta + dB) < 4(\beta + 1) \) which implies the impossible inequality \( dB < 0 \). Thus the condition \( \beta \geq 1 \) is a necessary one for the existence of a positive solution.

Theorem 2 Let \( R = 1/4 \), where \( R \) is defined in Eq. (14).

(1) If \( \beta < 1 \) then every solution of Eq. (13) becomes negative in a finite time.

(2) If \( \beta > 1 \) and \( x_0 \geq \beta - 1/2B \), then any solution of Eq. (13) is positive for \( n \geq 0 \) and tends to \( p = \beta - 1/2B \) as \( n \to \infty \).

Proof is similar to the proof of Theorem (1), where Lemma 3 is applied. Here there is the only equilibrium state \( \bar{x} = (\beta + 1/2B) - 1/B = (\beta - 1)/2B \). If \( \beta > 1 \) then any solution with \( x_0 \geq \bar{x} \) is positive for \( t \geq t_0 \) and tends to the equilibrium.

Theorem 3 Let \( R > 1/4 \), where \( R \) is defined in Eq. (14). Then for any initial value a solution of Eq. (13) becomes negative in a finite time.

Proof Similar to Theorems (1) and (2), the Eq. (13) can be reduced to Eq. (17); if a solution of Eq. (17) becomes negative then so is the corresponding solution of Eq. (13). According to Lemma (4)

\[
w_n = \sqrt{R \frac{\cos(n\phi + \phi - \theta)}{\cos(n\phi - \theta)}},
\]

where \( 0 < \phi < \pi/2 \). Thus there exists such \( n \) that \( \cos(n\phi + \phi - \theta)/\cos(n\phi - \theta) < 0 \) (beginning with some \( k > 1 \) it is enough to find the smallest \( n \) such that \( \cos(n\phi + \phi - \theta) < 0 \), then \( \cos(n\phi - \theta) > 0 \). Thus there exists a negative \( w_n \) and so a negative \( x_n \), which completes the proof.

Now let us proceed to the impulsive Beverton-Holt equation with variable parameters (4) which can be reduced to

\[
x_{n+1} = \frac{\beta_n x_n - d_{n+1}}{1 + B_n x_n}, \quad \beta_n = b_{n+1} \alpha_n - d_{n+1} B_n,
\]

similar to Eq. (13):

\[
x_{n+1} = b_{n+1} \left( \frac{\alpha_n x_n}{1 + B_n x_n} \right) - d_{n+1} = \frac{b_{n+1} \alpha_n x_n - d_{n+1} - d_{n+1} B_n x_n}{1 + B_n x_n}
\]

\[
= \frac{(b_{n+1} \alpha_n - d_{n+1} B_n) x_n - d_{n+1}}{1 + B_n x_n} = \frac{\beta_n x_n - d_{n+1}}{1 + B_n x_n}.
\]
Theorem 4 Denote
\[ R_n = \frac{b_n B_n + 1}{B_n + 1} \frac{\beta_n + 1}{B_n + 1}, \quad n \geq 1, \]
and
\[ \rho = \inf_n R_n, \quad R = \sup_n R_n, \quad m = \frac{1 - \sqrt{1 - 4R}}{2}, \quad M = \frac{1 + \sqrt{1 - 4R}}{2}, \quad \mathcal{M} = \frac{1 + \sqrt{1 - 4\rho}}{2}. \]

If \( R < 1/4, m < M \) and
\[ \mathcal{B} = \sup_{n \geq 1} \left( \frac{B_n \beta_{n-1} + B_{n-1}}{(\beta_n + 1)^2 B_{n-1}} \right) < M, \]
then any solution of Eq. (18) with the initial condition \( x_0 \), such that
\[ \frac{B_1 x_1 + 1}{B_1 \beta_0 + 1} > \max\{ \mathcal{B}, m \}, \quad \text{where} \quad x_1 = \frac{B_0 x_0 - d_1}{1 + B_0 x_0}, \]
is positive.

Proof In Eq. (18) let us make a substitution
\[ x_n = \frac{B_n + 1}{B_n} z_n - \frac{1}{B_n}. \]
Then
\[ z_n = \frac{B_n x_n + 1}{\beta_n + 1}, \quad z_{n+1} = \frac{B_n x_{n+1} + 1}{\beta_{n+1} + 1}, \]
so
\[ x_{n+1} = \frac{\beta_n x_n + \frac{\beta_n}{B_n} - \frac{d_n}{B_n}}{1 + B_n x_n} = \frac{\beta_n}{B_n} \frac{\beta_n + d_{n+1}}{1 + B_n x_n} = \frac{\beta_n}{B_n} \frac{\beta_n + d_{n+1}}{1 + B_n x_n} = \frac{\beta_n}{B_n} \frac{\beta_n + d_{n+1}}{1 + B_n x_n} = \frac{\beta_n}{B_n} \frac{\beta_n + d_{n+1}}{1 + B_n x_n} = \frac{\beta_n}{B_n} \frac{\beta_n + d_{n+1}}{1 + B_n x_n} \]
Consequently,
\[ z_{n+1} = \frac{B_n x_{n+1} + 1}{\beta_n + 1} = \frac{B_{n+1} \beta_n + B_{n+1}}{B_n (\beta_{n+1} + 1)}, \quad z_{n+1} = \frac{B_{n+1} \beta_n + B_{n+1}}{B_n (\beta_{n+1} + 1)}, \]
The Eq. (18) has a positive solution if and only if the latter equation has a solution satisfying
\[ z_n > \frac{1}{\beta_n + 1}. \]
Let us make an additional substitution to obtain a Riccati equation
\[ w_n = \frac{z_n B_{n-1} (\beta_n + 1)}{B_n (\beta_{n+1} + 1)}, \quad w_{n+1} = \frac{z_{n+1} B_n (\beta_{n+1} + 1)}{B_n (\beta_{n+1} + 1)}, \quad n \geq 1. \]
Thus
\[ w_{n+1} = 1 - \frac{b_{n+1} a_n b_{n+1}}{(\beta_{n+1} + 1)(\beta_{n+1} + B_n + B_n)} zn_n \]
\[ = 1 - \frac{b_{n+1} a_n b_{n+1} (\beta_{n+1} + 1)}{(\beta_{n+1} + 1)(\beta_{n+1} + B_n)(B_n B_{n-1} + B_{n-1}) w_n} \]

Denoting \( R_n \) as in Eq. (19) we obtain the Riccati equation with a non-constant coefficient which was considered in Lemma 5.

The original solution \( x_n \) is positive if and only if \( b_0 x_0 > d_1 \) and \( w_n > B_n B_{n-1} + B_{n-1}, \ n \geq 1 \).

The rest of the proof is an application of Lemma 5.

Finally, let us proceed to comparison results. To this end consider a comparison equation
\[ y_{n+1} = \gamma y_n - \delta_{n+1} \]
\[ = \gamma y_n - \delta_{n+1} \]

Thus we immediately get the following results.

**Theorem 5**

1. Let \( b_n \geq \gamma_n, B_n \leq \kappa_n, d_k \leq \delta_k, x_0 \geq y_0 \). Then
   a. If a solution of Eq. (23) is positive, then a solution of Eq. (18), with \( x_0 = y_0 \), is also positive;
   b. Generally, if \( \{x_n\} \) is a solution of Eq. (18) and \( \{y_n\} \) is a solution of Eq. (23), with \( x_0 = y_0 \), then
      \[ x_n \geq y_n. \]

2. Let \( b_n \leq \gamma_n, B_n \geq \kappa_n, d_k \geq \delta_k, x_0 \leq y_0 \). Then
   a. If Eq. (23) has no nonnegative solutions satisfying \( y_0 < N \), so does Eq. (18);
   b. If the solution of Eq. (23) is negative for \( n > n_0 \), then the solution of Eq. (23), with \( x_0 = y_0 \), is also negative for \( n > n_0 \);
   c. Generally, if \( \{x_n\} \) is a solution of Eq. (18) and \( \{y_n\} \) is a solution of Eq. (23), with \( x_0 = y_0 \), then
      \[ x_n \leq y_n. \]

Theorems (1)–(3), (5) imply the following result.

**Theorem 6**

Let \( d = \inf_i d_i \geq 0, \ D = \sup_i d_i, \ a = \inf_i B_i, \ A = \sup_i B_i, \ b = \inf_i \beta_i, \ B = \sup_i \beta_i, \)

1. If \( b > 1 \) and
   \[ R = \frac{b + AD}{(\beta + 1)^2} \leq \frac{1}{4}, \]
then Eq. (18) has a positive solution; moreover, all solutions satisfying the initial condition
\[ x_0 > \max \left\{ 0, \frac{(\beta + 1)(1 - \sqrt{1 - 4R})}{2a} - \frac{1}{a} \right\} \]
are positive for \( n \geq 1 \).

(2) If
\[ \frac{\beta + ad}{(\beta + 1)^2} > \frac{1}{4}, \]
then Eq. (18) has no positive solutions.

APPLICATIONS TO LOGISTIC IMPULSIVE DIFFERENTIAL EQUATIONS

Now consider a logistic equation with non-constant coefficients
\[ \dot{x}(t) = x(t)(r(t) - a(t)x(t)) \quad (24) \]
and linear impulsive perturbations
\[ x(\tau_k) = b_k x(\tau_k^-) - d_k \quad (25) \]
under the following conditions:

(a1) \( r(t), a(t) \) are locally essentially bounded functions, \( r(t) \geq 0, a(t) \geq 0 \);

(a2) \( \lim_{\tau \to -\infty} \tau_n = \infty, 0 < b_k \leq 1, d_k \geq 0 \), the inequalities mean that impulsive perturbations are of harvesting type.

This includes the cases of proportional and periodic constant harvesting.

Equation (24) has an explicit solution in every interval without an impulse \( [\tau_k, \tau_{k+1}) \) :
\[ x(t) = \frac{x(\tau_k)\exp\left\{ \int_{\tau_k}^{t} r(s)ds \right\}}{1 + x(\tau_k)\int_{\tau_k}^{t} a(s)\exp\left\{ \int_{\tau_k}^{s} r(\xi)d\xi \right\}ds}. \quad (26) \]

In order to study the impulsive equation (24) and (25) first consider a logistic equation with constant coefficients and a constant distance between impulses:
\[ \dot{x}(t) = rx\left(1 - \frac{x}{K}\right), \quad t \geq t_0, \quad (27) \]
\[ x(\tau) = bx(\tau^-) - d, \quad \tau_k = t_0 + k\tau, \quad (28) \]
under the following conditions:

(c1) \( r, K > 0, \tau > 0 \);

(c2) \( 0 < b \leq 1, d \geq 0, bK > d \).

We will also consider the initial value problem, with
\[ x(t_0) = x_0. \quad (29) \]
In addition, let us introduce the following notation
\[ \beta = e^{rt} \left( b - \frac{d}{K} \right), \quad B = \frac{e^{rt}}{K}. \]

**Definition** We will say that \( S \subset \mathbb{R} \) is a limit set of the problem equations (27)–(29) if for any \( y \in S \) and \( t_1 > t_0 \) there exists such \( t_2 > t_1 \) that \( x(t_2) = y \).

**Theorem 7** Suppose (c1)–(c2) are satisfied and \( R < 1/4 \), where \( \beta, B, R \) are defined in Eqs. (30) and (14).

1. If \( (\beta + 1)(1 + \sqrt{1 - 4R}) < 2 \) then every solution of Eqs. (27)–(29) becomes negative in a finite time.
2. If \( \beta > 1 \), then any solution of Eqs. (27)–(29), with
\[ x_0 > \max \left\{ 0, \frac{B + 1}{B} w_+ - \frac{1}{B} \right\}, \]
is positive for \( t > t_0 \) and the limit set of Eqs. (27)–(29) is an interval \( (p, q) \), where
\[ p = \frac{\beta + 1}{B} w_+ - \frac{1}{B}, \quad q = \frac{pe^{rt}}{1 + \frac{p}{K} e^{rt}}, \]
where \( w_-, w_+ \) are defined in Eq. (14).

**Proof** The solution of Eqs. (27)–(29) has the following form
\[ x(t) = \frac{x_0 e^{r(t-t_0)}}{1 + \frac{w_-}{K} e^{rt}}, \quad t_0 \leq t < t_0 + \tau = \tau_1, \quad x(\tau_1) = \frac{x_0 e^{r\tau}}{1 + \frac{w_-}{K} e^{rt}}, \]
and
\[ x(\tau_1) = \frac{b x_0 e^{rt}}{1 + \frac{w_-}{K} e^{rt}} - d = \frac{b x_0 e^{rt} - d - \frac{w_-}{K} e^{rt}}{1 + \frac{w_-}{K} e^{rt}} = \frac{(b - \frac{d}{K}) e^{rt} x_0 - d}{1 + \frac{w_-}{K} e^{rt}}. \]

Similarly, if we denote \( x_n = x(\tau_n) \), then
\[ x_{n+1} = \frac{(b - \frac{d}{K}) e^{rt} x_n - d}{1 + \frac{w_-}{K} e^{rt} x_n}. \]

Let us consider solutions (27)–(29) with \( x_0 < K \) which is not really a restriction: if \( b < 1, d > 0 \) then any solution eventually becomes less than \( K \). Suppose \( x_0 > K \), \( x(t) > K \) and \( y(t) \) is a solution of the problem (27) and (29) without impulses. Then \( x(t) \leq y(t) \) and \( \lim_{t \to \infty} y(t) = K \). Hence \( \lim_{t \to \infty} x(t) = K \). The limit case in Eq. (28) implies \( K = bK - d \) which contradicts \( b < 1, d > 0 \).

Thus any solution with \( x_0 < K \) is positive for any \( t > t_0 \) if and only if the difference equation (31), with any \( x_0 \) satisfying \( 0 < x_0 < K \), has a positive solution. Thus Theorem 1 can be applied. Since \( p = (\beta + 1/B) w_+ - 1/B \) is a global attractor of the difference equation (13) then \( \lim_{t \to \infty} x(\tau_t) = p \). The limit set is a set of all values of \( x(t) \) such that
\[ x(t) = \varphi(t) = \frac{pe^{rt}}{1 + \frac{p}{K} e^{rt}}, \quad 0 < t < \tau. \]

Since \( \varphi(t) \) is continuous, then the limit set is \( (p, \varphi(\tau)) \), which completes the proof of (2). \( \square \)
Thus we can apply Theorem 1.
By applying Theorems 2 and 3 we deduce the following result.

**Theorem 8** Suppose (c1)–(c2) are satisfied.

1. Let $R = 1/4$, where $R$ is defined in Eq. (14).
   - If $\beta < 1$ then every solution of Eqs. (27)–(29) becomes negative in a finite time.
   - If $\beta > 1$ and $x_0 \geq (\beta - 1)/2B$, then any solution of Eqs. (27)–(29) is positive for $t > t_0$ and the limit set of Eqs. (27)–(29) is $(p, q)$, where
     
     $$ p = \frac{\beta - 1}{2B}, \quad q = \frac{pe^{\pi t}}{1 + \frac{p}{q}e^{\pi t}}. $$

2. Let $R > 1/4$. Then for any initial value a solution of Eqs. (27)–(29) becomes negative in a finite time.

Now let us proceed to the impulsive logistic equation with variable coefficients (24) and (25).

Denote

$$ r_n = \int_{\tau_n}^{\tau_{n+1}} r(s)\, ds, \quad B_n = \int_{\tau_n}^{\tau_{n+1}} a(s)\exp\left\{\int_{\tau_n}^{s} r(\xi)\, d\xi\right\}\, ds, \quad \beta_n = b_{n+1}r_n - d_{n+1}B_n \quad (32) $$

The following result establishes the relation between a solution of the logistic impulsive equation and an impulsive Beverton-Holt difference equation.

**Theorem 9** Suppose (a1)–(a2) are satisfied and $a(t_0) > 0$. Then the solution of Eqs. (24),(25) and (29), with an initial condition $0 < x_0 < \lambda(t_0)/a(t_0)$, is positive if and only if the solution of the difference equation

$$ x_{n+1} = \frac{(b_{n+1}e^{\pi t_n} - d_{n+1}B_n)x_n - d_{n+1}}{1 + B_nx_n} = \frac{\beta_n x_n - d_{n+1}}{1 + B_n x_n} \quad (33) $$

with the same initial condition is positive.

Proof By the solution representation (26)

$$ x(\tau_{n+1}) = \frac{x(\tau_n)\exp\left\{\int_{\tau_n}^{\tau_{n+1}} r(s)\, ds\right\}}{1 + x(\tau_n)\int_{\tau_n}^{\tau_{n+1}} a(s)\exp\left\{\int_{\tau_n}^{s} r(\xi)\, d\xi\right\}\, ds} = \frac{x(\tau_n)e^{\pi t_n}}{1 + B_n x(\tau_n)}. $$

After the impulse we have

$$ x(\tau_{n+1}) = b_{n+1}x(\tau_{n+1}^\ast) - d_{n+1} = b_{n+1} \frac{x(\tau_n)e^{\pi t_n}}{1 + B_n x(\tau_n)} - d_{n+1} $$

$$ = \frac{(b_{n+1}e^{\pi t_n} - d_{n+1}B_n)x(\tau_n) - d_{n+1}}{1 + B_n x(\tau_n)}, $$

thus using notation (32) we obtain Eq. (33). Besides, equality (24) and inequalities $0 < x_0 < \lambda(t_0)/a(t_0)$ imply that the solution grows between impulse points. Thus the solution of Eqs. (24), (25) and (29), with an initial condition $0 < x_0 < \lambda(t_0)/a(t_0)$, is positive if and only if the solution of the difference equation (33) with the same initial condition is positive, which completes the proof of the theorem. \qed
Corollary 9.1 Suppose \((a_1)-(a_2)\) are satisfied and in Eqs. (24) and (25) \(b_n = b, d_n = d, \)

\[
\int_{t_n}^{t_{n+1}} a(s) \exp \left( \int_{t_n}^{s} r(\xi) d\xi \right) ds = B, \quad \int_{t_n}^{t_{n+1}} r(s) ds = r,
\]

where \(b, d, a, r\) are positive constants. Then the solution of Eqs. (24), (25) and (29), with an initial condition \(x_0 < \frac{(r(t_0))}{a(t_0)}\), is positive if and only if the solution of the difference equation

\[
x_{n+1} = \frac{(be^r - dB)x_n - d}{1 + ax_n}, \tag{34}
\]

with the same \(x_0\), is positive.

Corollary 9.2 Suppose \((a_1)-(a_2)\) are satisfied. If there exists such \(n\) that in Eq. (32) \(\beta_0 < 0\) then all solutions of Eqs. (24) and (25) become negative after \(t > \tau_{n+1}\).

By applying Theorem 4 we obtain

**Theorem 10** Suppose \((a_1)-(a_2)\) hold and

\[
R_n = \frac{b_{n+1}r_n B_{n+1} B_{n-1}(\beta_n + 1)}{(\beta_n + 1)(B_{n+1} \beta_n + B_n) (B_n \beta_{n-1} + B_{n-1})}, \quad n \geq 1,
\]

\[
\rho = \inf_n R_n, \quad R = \sup_n R_n, \quad m = \frac{1 - \sqrt{1 - 4R}}{2}, \quad M = \frac{1 + \sqrt{1 - 4R}}{2},
\]

\[
\mathcal{M} = \frac{1 + \sqrt{1 - 4\rho}}{2}.
\]

If \(R < 1/4, m < M\) and

\[
\mathcal{B} = \sup_{n \geq 1} \left\{ \frac{B_n \beta_{n-1} + B_{n-1}}{(\beta_n + 1)^2 B_{n-1}} \right\} < M,
\]

then any solution of Eqs. (24) and (25) with the initial condition \(x_0\), such that

\[
\frac{(B_1 x_1 + 1)B_0}{B_1 \beta_0 + B_1} > \max\{\mathcal{B}, m\}, \quad \text{where} \quad x_1 = \frac{B_0 x_0 - d_1}{1 + B_0 x_0}
\]

is positive.

Now let us proceed to comparison results. To this end consider an additional logistic equation

\[
\dot{y}(t) = y(t)(r_1(t) - a_1(t)y(t)) \tag{35}
\]

with linear impulsive perturbations

\[
y(t_k) = \bar{b}_k y(t^-_k) - \bar{d}_k \tag{36}
\]

and initial condition

\[
y(t_0) = y_0. \tag{37}
\]
Theorem 11  Suppose (a1)–(a2) are satisfied for the Eqs. (24), (25), (35) and (36).

1) Let \( r(t) \geq r_1(t), a(t) \leq a_1(t), b_k \geq \overline{b}_k, d_k \leq \overline{d}_k, x_0 \geq y_0 \). Then
(a) If all solutions of Eqs. (35)–(37) are positive, then all solutions of Eqs. (24)–(26) are positive;
(b) If for some \( N \) all solutions of Eqs. (35) and (36) satisfying \( N < y(t_0) < r_1(t_0)/a_1(t_0) \) are positive then all solutions of Eqs. (24) and (25) satisfying \( N < x(t_0) < r(t_0)/a(t_0) \) are positive;
(c) Generally, if \( x \) is a solution of Eqs. (24), (25), (29) and \( y \) is a solution of Eqs. (35)–(37), then
\[
x(t) \geq y(t).
\]

2) Let \( r(t) \leq r_1(t), a(t) \geq a_1(t), b_k \leq \overline{b}_k, d_k \geq \overline{d}_k, x_0 \leq y_0 \). Then
(a) If Eqs. (35) and (36) has no non-negative solutions satisfying \( y(t_0) < N \), so does Eqs. (24) and (25);
(b) If the solution of Eqs. (35) and (36), with the initial condition (37) is negative for \( t > t_1 \) then the solution of Eqs. (24), (25) and (29) is also negative for \( t > t_1 \);
(c) Generally, if \( x \) is a solution of Eqs. (24), (25) and (29) and \( y \) is a solution of Eqs. (35), (36) and (37), then
\[
x(t) \leq y(t).
\]

Proof  The statements 1(c) and 2(c) of the theorem are an immediate result of the solution representation between impulses (26), which can be rewritten in the form
\[
x(t) = \frac{1}{a(x)} + \int_{t_0}^{t} a(s) \exp \left\{ - \int_s^t r(\xi) d\xi \right\} ds
\]
and the inequality \( bx - d \geq b_1 x - d_1 \), if \( b \geq b_1, d \leq d_1 \). The other statements are a consequence of 1(c) and 2(c).

Theorem 12  Suppose (a1)–(a2) are satisfied, \( a_i \) and \( r_i \) are defined in Eq. (32) and
\[
b = \inf_{i} b_i > 0, \quad B = \sup_{i} b_i, \quad d = \inf_{i} d_i \geq 0, \quad D = \sup_{i} d_i,
\]
\[
a = \inf_{i} A_i, \quad A = \sup_{i} A_i, \quad r = \inf_{i} r_i, \quad \rho = \sup_{i} r_i,
\]
\[
\beta = be^{\rho} - da, \quad \overline{\beta} = Be^{\rho} - DA.
\]

1) If \( \beta > 1 \) and
\[
R = \frac{be^{\rho}}{(\beta + 1)^2} \leq \frac{1}{4},
\]
then Eqs. (24) and (25) has a positive solution; moreover, all solutions satisfying the initial condition
\[
x(t_0) > \max \left\{ 0, \frac{(\beta + 1)(1 - \sqrt{1 - 4R})}{2a} - \frac{1}{a} \right\}
\]
are positive for \( t > t_0 \).
and (28) has no positive solutions.

Proof This result is a consequence of the comparison Theorem 11 and Corollary 9.1. In the case (1) the comparison equation is

\[ \dot{y}(t) = y(t) \left( \frac{r}{r_{t+1} - r_t} \chi_{[r_t, r_{t+1})}(t) - \frac{A}{\int_{r_t}^{r_{t+1}} \exp \left( \frac{\alpha - \eta}{\eta_{t+1} - \eta_t} \right) dx} \chi_{[r_t, r_{t+1})}(t) y(t) \right), \]

where \( \chi_{[z, t)} \) is a characteristic function of the interval \([z, t)\), with linear impulsive perturbations

\[ y(\tau_k) = By(\tau_k) - d. \]

If we denote \( b_1 = b, d_1 = D \),

\[ r_1(t) = \frac{r}{r_{t+1} - r_t} \chi_{[r_t, r_{t+1})}(t), \quad a_1(t) = \frac{A}{\int_{r_t}^{r_{t+1}} \exp \left( \frac{\alpha - \eta}{\eta_{t+1} - \eta_t} \right) dx} \chi_{[r_t, r_{t+1})}(t), \]

then \( r(t) \geq r_1(t), a(t) \leq a_1(t), b_k \geq b, d_k < D \). Consequently, if Eqs. (38) and (39) has a positive solution then Eqs. (24) and (25) has a positive solution. Finally, we apply Corollary 4.1 to the Eqs. (38) and (39).

In case (2) the comparison equation is

\[ \dot{y}(t) = y(t) \left( \frac{\rho}{r_{t+1} - r_t} \chi_{[r_t, r_{t+1})}(t) - \frac{a}{\int_{r_t}^{r_{t+1}} \exp \left( \frac{\alpha - \eta}{\eta_{t+1} - \eta_t} \right) dx} \chi_{[r_t, r_{t+1})}(t) y(t) \right), \]

where \( \chi_{[z, t)} \) is a characteristic function of the interval \([z, t)\), with linear impulsive perturbations

\[ y(\tau_k) = By(\tau_k) - d. \]

The rest of the proof is similar to case (1). \( \square \)

The following example illustrates Theorems 7 and 8.

Example Consider the logistic equation (27) with \( K = 1 \) and impulsive conditions (28) with \( b = 0.9, d = 0.2, \tau = 1 \). If \( r = 1 \) then \( R \approx 0.67 \), \( R > 1/4 \) which means that all solutions become negative in a finite time. If \( r = 2 \) then \( R = 0.17 < 1/4 \). Figure 1 illustrates the vanishing solution in the first case and a positive solution in the second case. Both solutions correspond to the same initial value \( x(0) = 0.2 \).

CONCLUSIONS AND OPEN PROBLEMS

We have demonstrated that the impulsive Beverton-Holt equation can be applied to the study of the asymptotics for a logistic equation with impulsive harvesting. The analysis of the difference equation led to explicit extinction and non-extinction results, comparison
theorems and provided a complete description of the asymptotic behavior in the case of constant coefficients.

In the framework of discrete population models with impulsive harvesting the following models can be studied.

1. If we assume that there is a time delay of time period 1 in the response of the growth rate per individual to a density change, we obtain the delay difference model [2], p. 79:

\[ x_{n+1} = \frac{a_n x_n}{1 + B_n x_{n-1}}. \]

If in addition impulsive conditions are imposed, we get a second order difference equation

\[ x_{n+1} = \frac{a_n x_n - B_n x_{n-1} - d_n}{1 + B_n x_{n-1}} \]

which is a particular case of the rational second order difference equation; such an equation was studied in Ref. [10]. We can also consider the recruitment type coefficient \( b_n \), this models the situation when both natural population growth and recruitment compensate an independent consumption \( d_n \). The situation with \( d_n < 0 \) and \( 0 < b_n < 1 \) models the situation when a constant recruitment and natural population growth are balanced by proportional harvesting (in the latter case all solutions are positive). High order difference equations can present a discrete analogue for delay impulsive logistic equations which were recently studied in [12,13].

2. In the harvesting model we assumed that the harvesting rate depends on the present state of the population. Let us consider a model with a harvesting (hunting) strategy where we do not know the exact size of the population \( x_n \). However it is required to determine the hunting quota. There is always a delay in processing and distributing field information, so the quota must be set long before the hunting season begins. This means that the harvesting rate depends on the previous value \( x_n \) at \( n + 1 \)-th step (we use the population size information
from the previous step $x_n$). Let the population $x_n$ be harvested at $(n + 1)$-th step at the rate $(1 - b_n) + d_n$. This leads to the first order difference equation of the form

$$x_{n+1} = \frac{\alpha_n x_n}{1 + B_n x_n} - (1 - b_n)x_n - d_n,$$

where $\alpha_n > 0$, $0 < b_n \leq 1$, $d_n > 0$.

(3) Other discrete models can be considered subject to impulsive perturbations, for example, the second Verhulst difference equation

$$x_{n+1} = \frac{r x^2_n}{x^2_n + A}$$

under linear impulsive perturbations takes the form

$$x_{n+1} = \frac{\beta x^2_n - Ad}{x^2_n + A}, \quad \beta = br - d.$$

(4) The analysis presented in this paper and proposed in items 1–3 can be applied to various differential equations

$$x'(t) = f(t, x)$$

with impulsive conditions (6). By applying the comparison technique one can deduce sufficient non-extinction conditions even in cases when the explicit solution is not available.

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References


