On Nonoscillation of Systems of Delay Equations

By

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Abstract. The paper investigates nonnegativity of all entries of the fundamental matrix for the system of linear delay differential equations

\[ \dot{X}(t) + \sum_{k=1}^{m} A_k(t)X(h_k(t)) = 0 \]

in the case when the non-diagonal entries of matrices \( A_k \) are nonpositive. The results are applied to study nonoscillation of high order differential equations, as well as exponential stability for systems of delay equations.

Key Words and Phrases. Systems of delay equations, Nonoscillation, Nonnegative fundamental matrix, High order delay equations, Exponential stability.

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1. Introduction

This paper deals with nonoscillation for systems of delay differential equations. There are several different nonoscillation definitions for such systems.

In [17] a system was called nonoscillatory if there exists a solution for which at least one component is eventually positive. In [13] a nonoscillatory system by definition had a solution for which all components are eventually positive. Due to applications we will be interested in nonnegativity of the fundamental matrix for systems of linear delay differential equations. It means that all entries of this matrix are nonnegative functions, which will imply existence of a solution with positive components.

For ordinary differential equations nonnegativity of the fundamental matrix is equivalent to the following classical Wazewski’s result.

For solutions of the vector differential equation

\[ X'(t) + A(t)X(t) = 0 \]

and the vector differential inequality

\[ Y'(t) + A(t)Y(t) \leq 0, \]

where \( X(t_0) = Y(t_0) \), the inequality \( Y(t) \leq X(t) \) holds if and only if \( a_{ij} \leq 0, \)

\( i \neq j \).
We extend the sufficient part of this result to vector delay differential equations and discuss necessity of the condition $a_{ij} \leq 0$, $i \neq j$. We also consider the following related problems: comparison of nonoscillation properties and comparison of solutions, existence of positive solutions, connection between nonoscillation and asymptotic stability. For scalar delay differential equations these topics are well studied. Some of them have also been investigated for systems of delay equations: positivity of the fundamental matrix and comparison results in [11], connection between nonoscillation and stability in [3, 4]. In the present paper we obtain new results on all mentioned above problems. In particular, positivity of the fundamental matrix implies exponential stability of the vector delay differential equation, under some quite natural restrictions.

The paper is organized as follows. Section 2 contains relevant definitions and a solution representation formula. Section 3 presents the result on nonnegativity of the fundamental matrix of the vector delay equation. Section 4 includes some comparison results: nonoscillation properties of two delay equations and solutions of these equations are compared. In Section 5 previous results are applied to a higher order scalar delay differential equation. In Section 6 conditions on the initial function and the initial value which imply positivity for the solution of the initial value problem are established. In Section 6 we also prove that under some natural conditions an equation with a nonnegative fundamental matrix is exponentially stable. We also present an instability condition based on the comparison results. Finally, in Section 7 the results of the present paper are discussed and some open problems are stated.

2. Preliminaries

We consider for $t \geq 0$ systems of linear delay differential equations in two equivalent forms:

- the system of scalar equations

$$\dot{x}_i(t) + \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ij}^k(t)x_j(h_{ij}^k(t)) = 0, \quad i = 1, \ldots, n \tag{2.1}$$

and the vector equation

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t)X(h_k(t)) = 0, \tag{2.2}$$

where $A_k(t)$ are $n \times n$ matrices with entries $a_{ij}^k$, $i, j = 1, \ldots, n$, $k = 1, \ldots, m$.

Since system (2.1) can be rewritten in the form (2.2), we will formulate assumptions only for vector equation (2.2).
We consider vector delay differential equation (2.2) under the following conditions:

(a1) coefficients \(a_{ij}^k\) are Lebesgue measurable locally essentially bounded functions;
(a2) delays \(h_k : [0, \infty) \to \mathbb{R}\) are Lebesgue measurable functions, \(h_k(t) \leq t, \lim_{t \to \infty} h_k(t) = \infty, k = 1, \ldots, m\).

Together with (2.2) we consider for each \(t_0 \geq 0\) the initial value problem

\[
\dot{X}(t) + \sum_{k=1}^{m} A_k(t)X(h_k(t)) = F(t), \quad t \geq t_0,
\]

\[
X(t) = \Phi(t), \quad t < t_0, \quad X(t_0) = X_0,
\]

where \(F\) and \(\Phi\) satisfy the following hypothesis:

(a3) \(F : [t_0, \infty) \to \mathbb{R}^n, F(t) = [f_1(t), \ldots, f_n(t)]^T\), is a Lebesgue measurable locally essentially bounded function, \(\Phi : (-\infty, t_0) \to \mathbb{R}^n, \Phi(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T\) is a Borel measurable bounded function.

Here \(A^T\) is the transposed matrix.

**Definition.** A vector function \(X : \mathbb{R} \to \mathbb{R}^n\) is called a solution of problem (2.3), (2.4), if it is absolutely continuous on each interval \([t_0, b]\), satisfies equation (2.3) for almost all \(t \in [t_0, \infty)\) and equalities (2.4) for \(t \leq t_0\).

In addition to problem (2.3), (2.4), where \(X, F\) and \(\Phi\) are column vector functions, we will consider this problem, where \(F(t), \Phi(t)\) and solution \(X(t)\) are \(n \times n\) matrix functions.

**Definition.** For each \(s \geq 0\) the solution \(C(t,s)\) of the problem

\[
\dot{X}(t) + \sum_{k=1}^{m} A_k(t)X(h_k(t)) = 0, \quad X(t) = 0, \quad t < s, \quad X(s) = I,
\]

is called the fundamental matrix (or the Cauchy matrix) of equation (2.2), where \(C(t,s)\) is an \(n \times n\) matrix function, \(I\) is the identity matrix.

By 0 we will also denote the zero column vector and the zero matrix. We assume \(C(t,s) = 0, 0 \leq t < s\). As a corollary, Theorem 5.1.1 [2] gives the following result for equation (2.3).

**Lemma 2.1 ([2]).** Let (a1)–(a3) hold. Then there exists one and only one solution of problem (2.3), (2.4), and it can be presented in the form

\[
X(t) = C(t,t_0)X_0 + \int_{t_0}^{t} C(t,s)F(s)ds - \sum_{k=1}^{m} \int_{t_0}^{t} C(t,s)A_k(s)\Phi(h_k(s))ds,
\]

where \(\Phi(h_k(s)) = 0, \text{ if } h_k(s) > t_0\).
We will write $X \geq 0$, $A \geq 0$ if all components of vector $X$ or matrix $A$ are nonnegative.

Consider the scalar equations

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0 \tag{2.7}$$

and

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = 0, \tag{2.8}$$

as well as the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = f(t), \quad t \geq t_0 \quad x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0. \tag{2.9}$$

Here we assume that coefficients $a_k$, $b_k$ are Lebesgue measurable locally essentially bounded functions, delays $h_k$ and $g_k$ are Lebesgue measurable, $\limsup_{t \to \infty} h_k(t) = \infty$, $\limsup_{t \to \infty} g_k(t) = \infty$. The following results will be applied in the sequel.

**Lemma 2.2** ([17], [5, Theorem 2]). Assume that $a_k(t) \geq 0$ and equation (2.7) has a positive fundamental function $C(t,s) > 0$ for $t \geq s \geq t_0$. If

$$b_k(t) \leq a_k(t), \quad h_k(t) \leq g_k(t), \tag{2.10}$$

then equation (2.8) also has a positive fundamental function $D(t,s) > 0$ for $t \geq s \geq t_0$.

**Lemma 2.3** ([17], [5, Theorems 1 and 5]). If $a_k(t) \geq 0$, $f(t) \geq 0$ and there exists a nonnegative function $u$ which is integrable on each interval $[t_0, b]$ and satisfies $u(t) = 0$, $t < t_0$, and the inequality

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}, \tag{2.11}$$

then the fundamental function of (2.7) is positive for $t \geq s \geq t_0$.

If there exists a nonnegative function $u$ which is integrable on each interval $[t_0, b]$ and satisfies inequality (2.11), where $u(t) = 0$ for $t < t_0$, $f(t) \geq 0$ and also

$$0 < \varphi(t) \leq x(t_0) = x_0, \quad t \leq t_0, \tag{2.12}$$

then the solution of problem (2.9) is positive for $t \geq t_0$. 

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In particular, condition (2.11) holds if

\[
\int_{t_0}^{t} \sum_{j=1}^{m} a_j(s) ds \leq \frac{1}{e}, \quad t \geq t_0.
\]

Finally, let us present an upper estimate of the fundamental function.

**Lemma 2.4** ([6, Lemma 6]). Suppose that for equation (2.7)

\[
a_k(t) \geq 0, \quad X(t,s) > 0, \quad t \geq s \geq t_0, \quad t - h_k(t) \leq H, \quad t \geq t_0.
\]

Then for the fundamental function \(X(t,s)\) of equation (2.7) the estimate

\[
0 < X(t,s) \leq 1, \quad s \leq t \leq s + H,
\]

is valid and

\[
\int_{t_0}^{t} X(t,s) \sum_{k=1}^{m} a_k(s) ds \leq 1.
\]

3. **Positivity of the fundamental matrix**

**Theorem 3.1.** Suppose \(a_{ij}^k(t) \leq 0, \ i \neq j, \ k = 1, \ldots, m, \ t \geq t_0\) and the fundamental functions \(X_i(t,s)\) of the scalar equations

\[
\dot{x}(t) + \sum_{k=1}^{m} [a_{ii}^k(t)]^+ x(h_k(t)) = 0, \quad i = 1, \ldots, n, \quad a^+ = \max\{a, 0\},
\]

are positive for \(t \geq s \geq t_0\). Then for the fundamental matrix \(C(t,s)\) of vector equation (2.2) we have \(C(t,s) \geq 0, \ t \geq s \geq t_0\).

**Proof.** Consider first the case \(a_{ii}^k \geq 0\). After introducing diagonal matrices \(B_k(t) = \text{diag}\{a_{ii}^1, \ldots, a_{ii}^m\}\) and defining \(D_k(t) = A_k(t) - B_k(t), \ k = 1, \ldots, m\), it is evident that for the entries \(d_{ij}^k\) of \(D_k(t)\) we have \(d_{ij}^k \leq 0, \ i \neq j, \ d_{ii}^k = 0\).

Denote by \(Y(t,s)\) the fundamental matrix of the vector equation

\[
\dot{Y}(t) + \sum_{k=1}^{m} B_k(t) Y(h_k(t)) = 0.
\]

Since \(B_k(t)\) are diagonal matrices then \(Y(t,s) = \text{diag}\{X_1(t,s), \ldots, X_n(t,s)\}\), where \(X_i(t,s)\) are the fundamental functions of scalar equations (3.1), \(i = 1, \ldots, n\), and \(X_i\) are positive for \(t \geq s \geq t_0\). Then \(Y(t,s) \geq 0, \ t \geq s \geq t_0\).

Consider the following problem

\[
\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X(h_k(t)) = F(t), \quad t \geq t_0, \ X(t) = 0, \ t \leq t_0,
\]
which can be rewritten in the form
\[
\dot{X}(t) + \sum_{k=1}^{m} B_k(t)X(h_k(t)) + \sum_{k=1}^{m} D_k(t)X(h_k(t)) = F(t), \quad t \geq t_0, \ X(t) = 0, \ t \leq t_0.
\]

Hence by (2.6) for the solution of (3.2) we have
\[
(3.3) \quad X(t) = \int_{t_0}^{t} Y(t, s)F(s)ds - \int_{t_0}^{t} Y(t, s) \sum_{k=1}^{m} D_k(s)X(h_k(s))ds.
\]

If we introduce operator $H$ as
\[
(HG)(t) = -\int_{t_0}^{t} Y(t, s) \sum_{k=1}^{m} D_k(s)X(h_k(s))ds, \quad \text{where } X(h_k(s)) = 0, \ h_k(s) \leq t_0,
\]
then vector equation (3.3) has the form
\[
(3.4) \quad X - HX = G,
\]
where $G(t) = \int_{t_0}^{t} Y(t, s)F(s)ds \geq 0$ if $F(t) \geq 0$. Let $L_{\infty}[t_0, b]$ be the space of vector-functions which are essentially bounded on $[t_0, b]$ functions; we assume the essential supremum norm in this space, $b > t_0$ is arbitrary. Then $H : L_{[t_0, b]} \rightarrow L_{\infty}[t_0, b]$ is a sum of compact integral Volterra operators, hence its spectral radius equals 0: $r(H) = 0 < 1$. If $X \geq 0$ then $HX \geq 0$, i.e., $H$ is positive. Thus for a solution of equation (3.4) we have $X = (I - H)^{-1}G \geq 0$ for $G \geq 0$ since $(I - H)^{-1} = I + H + H^2 + \cdots$. Hence for any $F \geq 0$ the solution of equation (3.2) satisfies $X(t) \geq 0$. The solution representation $X(t) = \int_{t_0}^{t} C(t, s)F(s)ds$ implies $C(t, s) \geq 0$, $t \geq s \geq t_0$.

In the general case we can write $B_k(t) = B_k^+(t) - B_k^-(t)$, where
\[
B_k^+(t) = diag\{(a_{11}^k)^+, \ldots, (a_{nn}^k)^+\}, \quad B_k^-(t) = B_k^+(t) - B_k(t)
\]
and prove the statement similarly. 

Remark 3.1. Theorem 3.1 was first proven in [11] using a different method.

Corollary 3.1. Suppose $a_{ij}^k(t) \leq 0, \ i \neq j, \ t \geq t_0$ and
\[
(3.5) \quad \sup_{t \geq t_0} \int_{\max\{t_0, \min\ h_k(t)\}}^{t} \sum_{j=1}^{m} [a_{ij}^k(s)]^+ ds \leq \frac{1}{e}, \quad i = 1, \ldots, n.
\]

Then the fundamental matrix of vector equation (2.2) satisfies $C(t, s) \geq 0, \ t \geq s \geq t_0$. 

Unlike systems of ordinary differential equations, the condition $a_{ij}^k(t) \leq 0$, $i \neq j$, $t \geq t_0$, generally, is not necessary for positivity of the fundamental matrix of a system of linear delay equations, as the following example demonstrates.

**Example 3.1.** Consider the system

\[
\begin{align*}
\dot{x}(t) + x(t) &= 0 \\
\dot{y}(t) + 3x(t) + y(t) &= 0 \\
\dot{z}(t) + e^{-3}x(t - 3) - 3y(t) + z(t) &= 0
\end{align*}
\]

Denote by $X(t, s)$, $t \geq s \geq 0$ the fundamental matrix of this system. Simple calculations lead to the following structure of $C(t, s)$.

The first row of $C(t, s)$ is $(e^{-3(t-s)}, 0, 0)$. The second row is $(3(t-s)e^{-(t-s)}, e^{-3(t-s)}, 0)$. The third row is $(4.5(t-s)^2e^{-(t-s)}, 3(t-s)e^{-(t-s)}, e^{-(t-s)})$ for $t \in [s, s+3]$ and $([4.5(t-s)^2/(t-s) + 3]e^{-(t-s)}, 3(t-s)e^{-(t-s)}, e^{-(t-s)})$ for $t \in [s+3, \infty)$. So $C(t, s) \geq 0$ while one of the nondiagonal coefficients is positive.

However there are several types of delay systems for which the condition that nondiagonal entries of matrices $A_k$ are nonpositive

\[ a_{ij}^k(t) \leq 0, \quad i \neq j, \ t \geq t_0 \]

is necessary. In particular, we consider here two classes of such systems: first, when nondiagonal terms are nondelayed and second, when there are two equations and nondiagonal terms involve a constant delay. In the first case the system can be rewritten in the form

\[
\dot{X}(t) + A_0(t)X(t) + \sum_{k=1}^m A_k(t)X(h_k(t)) = 0,
\]

where $a_{ij}^k(t) = 0$, $i \neq j$, $k = 1, \ldots, m$.

**Theorem 3.2.** Suppose that the fundamental functions of the scalar equations

\[
\dot{x}(t) + [a_{ii}^0(t)]^+ x(t) + \sum_{k=1}^m [a_{ij}^k(t)]^+ x(h_k(t)) = 0, \quad i = 1, \ldots, n
\]

are positive for $t \geq s \geq t_0$. If there exists a nonnegative coefficient $a_{ij}^0(t) \geq 0$, $i \neq j$ such that $a_{ij}^0(t) \geq a_0 > 0$ in some interval, then the fundamental matrix $C(t, s)$ of vector equation (3.7) is not nonnegative.
Proof. Denote by $X_i(t,s)$ the fundamental function of the equation

$$\dot{x}(t) + a_i^0(t)x(t) + \sum_{k=1}^{m} a_i^k(t)x(h_k(t)) = 0, \quad i = 1, \ldots, n.$$ 

Since $a_i^0(t) \leq [a_i^0(t)]^+, \quad k = 0, 1, \ldots, m, \quad i = 1, \ldots, n$ and the fundamental function of equation (3.8) is positive then by Lemma 2.2 we have $X_i(t,s) > 0, \quad t \geq s \geq t_0, \quad i = 1, \ldots, n$.

Let us assume the contrary: $C(t,s) \geq 0$ for $t \geq s \geq t_0$ and there exist a pair of indices $(i,j)$ and a number $t_1 > t_0$ such that $a_i^0(t) \geq a_0 > 0$ for $t \in [t_0, t_1]$; without loss of generality we can assume $j = 1$. Coefficients $a_i^0$ are locally bounded, so for some $\alpha > 0$ we have $|a_i^0(t)| \leq \alpha, \quad r \neq 1, \quad t_0 \leq t \leq t_1$.

Consider now the solution $X(t)$ of equation (3.7) with initial conditions $X(t) = 0, \quad t < t_0, \quad X(t_0) = B := [1, 0, \ldots, 0]^T$, which is the first column of the fundamental matrix $C(t,t_0)$. For the solution of this problem $X = [x_1, \ldots, x_n]^T$ we have $X(t) = C(t,t_0)B \geq 0$.

Let us choose $\delta > 0$ satisfying $\delta < a_0/(a_0 + \alpha(n-2))$, then there exists $t_2 \in (t_0, t_1)$ such that for $t \in [t_0, t_2]$ we have $x_1(t) > 1 - \delta, \quad x_j(t) < \delta, \quad j \neq 1$, which implies

$$\sum_{j=1,\ldots,n, j \neq i} a_i^0(t)x_j(t) \geq a_0(1-\delta) - \alpha\delta(n-2) > 0.$$ 

The $i$-th equation in system (3.7) has the form

$$\dot{x}_i(t) + a_i^0(t)x(t) + \sum_{k=1}^{m} a_i^k(t)x(h_k(t)) = -\sum_{j=1,\ldots,n, j \neq i} a_i^0(t)x_j(t).$$ 

Hence

$$x_i(t) = -\int_{t_0}^{t} X_i(t,s) \sum_{j=1,\ldots,n, j \neq i} a_i^0(s)x_j(s)ds < 0, \quad t_0 \leq t \leq t_2,$$

which contradicts nonnegativity of the $i$-th component $x_i$ of the first column of the fundamental matrix for $t_0 \leq t \leq t_2$. \hfill \Box

The second case when the condition $a_i^k(t) \leq 0, \quad i \neq j, \quad t \geq t_0$ becomes necessary is the delay system of two equations with constant delays of non-diagonal terms. For simplicity consider the following system:

$$\begin{align*}
\dot{x}_1(t) &= -a_{11}(t)x_1(h_{11}(t)) - a_{12}(t)x_2(t - \tau_{12}), \\
\dot{x}_2(t) &= -a_{21}(t)x_1(t - \tau_{21}) - a_{22}(t)x_2(h_{22}(t)),
\end{align*}$$

where $\tau_{ij} \geq 0$. 

\subsection{Remark}

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**Theorem 3.3.** Suppose that the fundamental functions of the scalar equations

\[(3.10) \quad \dot{x}(t) + [a_{ii}(t)]^+x(h_i(t)) = 0, \quad i = 1, 2\]

are positive for \(t \geq s \geq t_0\). If there exists a nonnegative coefficient \(a_{ij}(t) \geq 0\), \(i \neq j\), such that \(a_{ij}(t) \geq a_0 > 0\) in some interval \([t_0, t_1]\), \(t_1 - t_0 > \max\{\tau_i\}\), then the fundamental matrix \(C(t, s)\) of system (3.9) is not a nonnegative function.

**Proof.** Denote by \(X_i(t, s)\) the fundamental function of the equation

\[(3.11) \quad \dot{x}(t) + a_{ii}(t)x(h_i(t)) = 0, \quad i = 1, 2.\]

As in the proof of the previous theorem we can show that \(X_i(t, s) > 0\), \(t \geq s \geq t_0\).

Let us assume the contrary to the statement of the theorem: \(C(t, s) \geq 0\) for any \(t \geq s \geq t_0\) and for some \(t_1 > t_0\) while \(a_{ij}(t) \geq a_0 > 0\) for \(t \in [t_0, t_1]\) by the assumption of the theorem; without loss of generality we can assume \((i, j) = (2, 1)\).

Consider now the solution \(X(t)\) of equation (3.9) with the initial conditions \(X(t) = 0, \ t < t_0, \ X(t_0) = B := [1, 0]^T\), which is the first column of the fundamental matrix \(C(t, t_0)\). For the solution of this problem \(X = [x_1, x_2]^T\) we have \(X(t) = C(t, t_0)B \geq 0\). There exist an interval \([t_0, t_2]\) and a number \(\delta > 0\) such that

\[x_1(t) \geq 1 - \delta > 0, \quad t_0 \leq t \leq t_2 \leq t_0 + \tau_{21}.\]

The second equation in system (3.9) has the form

\[\dot{x}_2(t) + a_{22}(t)x_2(h_{22}(t)) = -a_{21}(t)x_1(t - \tau_{21}),\]

where \(x_1(t) = 0, \ t < t_0, \ x_2(t_0) = 0\). Then

\[x_2(t_1) = -\int_{t_0}^{t_1} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds\]
\[= -\int_{t_0}^{t_0 + \tau_{21}} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds - \int_{t_0 + \tau_{21}}^{t_1} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds\]
\[= -\int_{t_0}^{t_1} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds =: I_1 + I_2 + I_3.\]

It is evident that \(I_1 = 0, \ I_3 \leq 0\). In the second integral \(t_0 \leq s - \tau_{21} \leq t_2\). Hence in this integral \(x_1(s - \tau_{21}) \geq 1 - \delta > 0\). Then \(I_2 < 0\) and so \(x_2(t_1) < 0\), which contradicts our assumption. \(\square\)
4. Comparison results

Now we can compare two solutions of system (2.3) of differential equations.

**Corollary 4.1.** Suppose conditions of Theorem 3.1 hold, \( X(t) \) is a solution of problem (2.3), (2.4), \( Y(t) \) is a solution of the same problem, where function \( F(t) \) is replaced by \( G(t) \). If \( G(t) \leq F(t), \ t \geq t_0 \), then \( Y(t) \leq X(t), \ t \geq t_0 \).

The proof follows from solution representation (2.6) and inequality \( C(t,s) \geq 0 \) for \( t \geq s \geq t_0 \).

**Corollary 4.2.** Suppose conditions of Theorem 3.1 hold, \( X(t) \) is a solution of (2.2), \( Y(t) \) is a solution of the differential inequality

\[
\dot{Y}(t) + \sum_{k=1}^{m} A_k(t) Y(g_k(t)) \leq 0, \quad t \geq t_0
\]

and \( X(t) = Y(t), \ t \leq t_0 \). Then \( Y(t) \leq X(t) \) for \( t \geq t_0 \).

Consider for \( t \geq 0 \) together with equation (2.2) the following system

\[
\dot{Y}(t) + \sum_{k=1}^{m} B_k(t) X(g_k(t)) = 0.
\]

Suppose that (a1)–(a2) hold for system (4.2). Denote by \( D(t,s) \) the fundamental matrix of system (4.2).

**Theorem 4.1.** Suppose \( a^i_j(t) \leq 0 \) for \( i \neq j \), the fundamental functions of scalar equations (3.1) are positive for \( t \geq s \geq t_0 \), \( A_k(t) \geq B_k(t) \) and \( g_k(t) \geq h_k(t) \) for \( t \geq t_0 \). Then \( D(t,s) \geq 0 \) for \( t \geq s \geq t_0 \).

**Proof.** By Lemma 2.2 the fundamental functions of the scalar equations

\[
\dot{y}(t) + \sum_{k=1}^{m} [b^k_{ij}(t)]^+ y(g_k(t)) = 0, \quad i = 1, \ldots, n
\]

are also positive for \( t \geq s \geq t_0 \). Inequality \( A_k(t) \geq B_k(t) \) implies \( b^k_{ij} \leq 0 \) for \( i \neq j \). Application of Theorem 3.1 completes the proof.

Let us compare solutions of differential equations with different matrices and right-hand sides. To this end consider together with (2.3), (2.4) the following initial value problem

\[
\dot{Y}(t) + \sum_{k=1}^{m} B_k(t) Y(h_k(t)) = G(t), \quad t \geq t_0,
\]

\[
Y(t) = \Phi(t), \quad t < t_0, \quad Y(t_0) = Y_0.
\]
Suppose that (a1)–(a3) hold for (4.3), (4.4). Denote by $X(t), C(t, s)$ the solution and the fundamental matrix of problem (2.3), (2.4), by $Y(t), D(t, s)$ the solution and the fundamental matrix of problem (4.3), (4.4), respectively.

**Theorem 4.2.** If $a_{ij}^k(t) \leq 0$ for $i \neq j$, the fundamental functions of scalar equations (3.1) are positive for $t \geq s \geq t_0$, $X(t) \geq 0$ and

$$A_k(t) \geq B_k(t) \geq 0, \quad G(t) \geq F(t), \quad Y_0 \geq X_0,$$

then $Y(t) \geq X(t) \geq 0$, $t \geq t_0$.

**Proof.** By Theorem 4.1 the fundamental matrix $D(t, s)$ of vector equation (4.3) is positive for $t \geq s \geq t_0$. System (2.3) can be rewritten as

$$\dot{X}(t) + \sum_{k=1}^{m} B_k(t)X(h_k(t)) = \sum_{k=1}^{m} [B_k(t) - A_k(t)]X(h_k(t)) + F(t),$$

hence by solution representation (2.6) we have

$$X(t) = D(t, t_0)X_0 - \sum_{k=1}^{m} \int_{t_0}^{t} D(t, s)B_k(s)\Phi(h_k(s))ds$$

$$+ \int_{t_0}^{t} D(t, s)F(s)ds - \sum_{k=1}^{m} \int_{t_0}^{t} D(t, s)[A_k(s) - B_k(s)]X(h_k(s))ds$$

$$\leq D(t, t_0)Y_0 - \sum_{k=1}^{m} \int_{t_0}^{t} D(t, s)B_k(s)\Phi(h_k(s))ds$$

$$+ \int_{t_0}^{t} D(t, s)G(s)ds = Y(t),$$

where $\Phi(h_k(s)) = 0$ if $h_k(s) \geq t_0$ and $X(h_k(s)) = 0$ if $h_k(s) < t_0$. Thus $X(t) \leq Y(t)$, which completes the proof.

**Corollary 4.3.** If $a_{ij}^k(t) \leq 0$ for $i \neq j$, the fundamental functions of scalar equations (3.1) are positive for $t \geq s \geq t_0$ and $A_k(t) \geq B_k(t)$, then $D(t, s) \geq C(t, s) \geq 0, t \geq s \geq t_0$.

5. **High order scalar delay differential equations**

In this section we consider the linear scalar delay differential equation of the $n$-th order

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(h_{n-1}(t)) + \cdots + a_1(t)y'(h_1(t))$$

$$+ a_0(t)y(h_0(t)) = 0, \quad t \geq 0,$$
where for parameters of $(5.1)$ and other high order equations it is assumed that coefficients $a_k(t)$ are Lebesgue measurable locally essentially bounded functions, and delays $h_k(t) \leq t$ satisfy $\limsup_{t \to \infty} h_k(t) = \infty$, $k = 0, \ldots, n - 1$.

Together with equation $(5.1)$ consider the initial value problem

\begin{equation}
(5.2) \quad y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(h_{n-1}(t)) + \cdots + a_1(t)y'(h_1(t)) + a_0(t)y(h_0(t)) = f(t), \quad t \geq t_0,
\end{equation}

\begin{equation}
(5.3) \quad y^{(k)}(t) = \varphi_k(t), \quad t < t_0, \quad y^{(k)}(t_0) = y_k, \quad k = 0, \ldots, n - 1.
\end{equation}

**Definition.** A function $y : R \to R$ is called a solution of problem $(5.2), (5.3)$, if its $(n-1)$-th derivative $y^{(n-1)}$ is absolutely continuous on each finite interval and $y$ satisfies equation $(5.2)$ for almost all $t \in [t_0, \infty)$ and equalities $(5.3)$ for $t \leq t_0$.

**Definition.** For each $s \geq 0$ the solution $Y(t, s)$ of the problem

\begin{equation}
(5.4) \quad y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(h_{n-1}(t)) + \cdots + a_1(t)y'(h_1(t)) + a_0(t)y(h_0(t)) = 0, \quad t \geq s,
\end{equation}

\begin{equation}
(5.5) \quad y^{(k)}(t) = 0, \quad t < s, \quad k = 0, \ldots, n - 1,
\end{equation}

\begin{equation}
Y^{(k)}(s) = 0, \quad k = 0, \ldots, n - 2, \quad y^{(n-1)}(s) = 1
\end{equation}

is called the fundamental function of equation $(5.1)$.

Further, we will denote by $Y_k(t, s)$ a solution of $(5.4)$ with initial conditions

\begin{equation}
Y^{(j)}(t) = 0, \quad t < s, \quad j = 0, \ldots, n - 1,
\end{equation}

\begin{equation}
Y^{(j)}(s) = 0, \quad j \neq k, \quad y^{(k)}(s) = 1, \quad k = 0, \ldots, n - 1,
\end{equation}

instead of $(5.5)$.

We assume $Y(t, s) = 0$ for $0 \leq t < s$ and $Y_k(t, s) = 0$ for $0 \leq t < s, k = 0, \ldots, n - 1$; evidently $Y_{n-1}(t, s) = Y(t, s)$.

**Lemma 5.1** ([2]). There exists a unique solution of problem $(5.2), (5.3)$, and it can be presented in the form

\begin{equation}
(5.6) \quad y(t) = \sum_{k=0}^{n-1} Y_k(t, t_0)y_k + \int_{t_0}^{t} Y(t, s)f(s)ds - \int_{t_0}^{t} Y(t, s) \sum_{k=0}^{n-1} a_k(s)\varphi_k(h_k(s))ds,
\end{equation}

where $\varphi_k(h_k(s)) = 0$, if $h_k(s) \geq t_0$.

Denote

\begin{equation}
x_1(t) = y(t), x_2(t) = y'(t), \ldots, x_n(t) = y^{(n-1)}(t),
\end{equation}
then
\[ x'_1 = x_2, \quad x'_2 = x_3, \ldots, \quad x'_{n-1} = x_n, \quad x'_n = -\sum_{k=1}^{n} a_{k-1}(t) x_k(h_{k-1}(t)). \]

Denote \( X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \),
\[
A_0(t) = \begin{pmatrix}
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad A_k(t) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{k-1}(t) & \cdots & 0
\end{pmatrix},
\]
\[ k = 1, \ldots, n, \]
where \( A_k(t) \) are \( n \times n \) matrices with a nonzero entry in the \( k \)-th column.

Hence equation (5.1) can be rewritten as the system
\[
(5.7) \quad \dot{X}(t) + A_0(t)X(t) + \sum_{k=1}^{n} A_k(t)X(h_{k-1}(t)) = 0,
\]
and for its fundamental system \( C(t, s) \) we easily deduce \( C_{1n}(t, s) = Y(t, s) \), where \( Y(t, s) \) is the fundamental function of (5.1).

As a corollary of Theorem 3.1 we can obtain the following result.

**Theorem 5.1.** Suppose that \( a_k(t) \leq 0, \ k = 0, \ldots, n-2 \) and the fundamental function \( X(t, s) \) of the scalar equation
\[
(5.8) \quad \dot{x}(t) + [a_{n-1}(t)]^+ x(h_{n-1}(t)) = 0
\]
is positive for \( t \geq s \geq t_0 \). Then for the fundamental function of equation (5.1) we have \( Y(t, s) \geq 0, \ t \geq s \geq t_0 \).

**Proof.** First let us check that all conditions of Theorem 3.1 hold for system (5.7). Evidently all non-diagonal entries of matrices \( A_k(t), \ k = 0, \ldots, n \) are nonpositive.

Scalar equations (3.1) have the form:
\[
\dot{x}_k(t) = 0, \quad k = 1, \ldots, n-1, \quad \dot{x}_n(t) + [a_{n-1}(t)]^+ x_n(h_{n-1}(t)) = 0.
\]
Fundamental functions of all these equations are positive. Theorem 3.1 implies that the fundamental matrix \( C(t, s) \) of system (5.7) is nonnegative for \( t \geq s \geq t_0 \). Since \( C_{1n}(t, s) = Y(t, s) \), the fundamental function of equation (5.1) is nonnegative for \( t \geq s \geq t_0 \).
Corollary 5.1. Suppose that \( a_k(t) \leq 0, \, k = 0, \ldots, n-2, \, t \geq t_0 \) and
\[
(5.9) \quad \sup_{t \geq t_0} \int_{\max\{t_0, h_{n-1}(t)\}}^{t} [a_{n-1}(s)]^+ ds \leq \frac{1}{e}.
\]
Then the fundamental function \( Y(t, s) \) of equation (5.1) is nonnegative for \( t \geq s \geq t_0 \).

Corollary 5.2. Suppose that \( a_k(t) \leq 0, \, k = 0, \ldots, n-1, \, t \geq t_0 \). Then the fundamental function of the equation (5.1) is nonnegative for \( t \geq s \geq t_0 \).

Corollary 5.3. Suppose that the conditions of Theorem 5.1 hold, \( u(t) \) is a solution of problem (5.2), (5.3), \( v(t) \) is a solution of the problem, where \( f \) is replaced by \( g \). If \( g(t) \leq f(t) \) then \( v(t) \leq u(t), \, t \geq t_0 \).

Corollary 5.4. Suppose that the conditions of Theorem 5.1 hold, \( u(t) \) is a solution of equation (5.1), \( v(t) \) is a solution of the differential inequality
\[
y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(h_{n-1}(t)) + \cdots + a_0(t)y(h_0(t)) \leq 0.
\]
If \( v(t) = u(t), \, t \leq t_0 \), then \( v(t) \leq u(t), \, t \geq t_0 \).

Remark 5.1. Corollary 5.4 extends the famous result obtained by Chaplygin [9] for an ordinary differential equation of the second order
\[
y''(t) + a(t)y'(t) + b(t)y(t) = 0.
\]
Together with equation (5.1) consider the equation
\[
(5.10) \quad z^{(n)}(t) + b_{n-1}(t)z^{(n-1)}(g_{n-1}(t)) + \cdots + b_0(t)z(g_0(t)) = 0, \quad t \geq t_0,
\]
Denote by \( Z(t, s) \) the fundamental function of (5.10).

Theorem 5.2. Suppose that
\[
a_{n-1}(t) \geq b_{n-1}(t), \quad g_{n-1}(t) \geq h_{n-1}(t), \quad b_k(t) \leq 0, \quad k = 0, \ldots, n-2, \quad t \geq t_0
\]
and the fundamental function of (5.1) is positive for \( t \geq s \geq t_0 \). Then \( Z(t, s) \geq 0 \) for \( t \geq s \geq t_0 \).

Together with (5.2), (5.3) consider the initial value problem
\[
(5.11) \quad z^{(n)}(t) + b_{n-1}(t)z^{(n-1)}(h_{n-1}(t)) + \cdots + b_0(t)z(h_0(t)) = g(t), \quad t \geq t_0,
\]
\[
(5.12) \quad z^{(k)}(t) = \varphi_k(t), \quad t < t_0, \quad z^{(k)}(t_0) = z_k, \quad k = 0, \ldots, n-1.
\]
Further, let \( y(t) \), \( Y(t, s) \) be the solution and the fundamental function of problem (5.2), (5.3), \( z(t) \), \( Z(t, s) \) the solution and the fundamental function of problem (5.11), (5.12), respectively.
In the following theorem we compare positive solutions and fundamental functions of two nonoscillatory equations.

**Theorem 5.3.** Suppose that \( a_n(t) \geq b_{n-1}(t) \geq 0 \), for \( t \geq t_0 \), \( y(t) \geq 0 \), \( t \geq t_0 \), 
\[ a_k(t) \geq b_k(t), \quad k = 0, \ldots, n - 2, \quad g(t) \geq f(t), \quad y_k \leq z_k, \quad k = 0, \ldots, n - 1, \]
and the fundamental function of (5.1) is positive for \( t \geq s \geq t_0 \). Then \( Z(t,s) \geq Y(t,s) \geq 0 \) for \( t \geq s \geq t_0 \), and \( z(t) \geq y(t) \geq 0 \), \( t \geq t_0 \).

6. Positive solutions and stability

We begin this section with an analogue of Lemma 2.3 on existence of a positive solution.

**Theorem 6.1.** Suppose that \( a_i(t) \geq 0 \), \( i = 1, \ldots, n, \)
\[ a_i^k(t) \leq 0, \quad i \neq j, k = 1, \ldots, m, \quad F(t) \geq 0, \quad t \geq t_0, \]
\[ 0 \leq \Phi(t) \leq X_0, \quad X_0 > 0, \] and inequality (3.5) holds. Then for the solution \( X(t) \) of initial value problem (2.3), (2.4) we have \( X(t) \geq 0 \), \( t \geq t_0 \).

**Proof.** First, let \( F(t) = 0 \). We recall that \( \Phi(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T \), \( X_0 = [x_1^0, \ldots, x_n^0]^T \), where \( \varphi_i(t) \leq x_i^0 \), \( x_i^0 > 0 \). Hence for solutions \( y_i(t) \) of initial value problems
\[ \dot{y}(t) + \sum_{k=1}^{m} a_i^k(t) y(h_k(t)) = 0, \quad i = 1, \ldots, n, \]
\[ y(t) = \varphi_i(t), \quad t < t_0, \quad y(t_0) = x_i^0, \]
Lemma 2.3 implies \( y_i(t) > 0 \), \( t \geq t_0 \), \( i = 1, \ldots, n \).

Denote \( Y(t) = [y_1(t), \ldots, y_n(t)]^T \). Then
\[ \dot{Y}(t) + \sum_{k=1}^{m} A_k(t) Y(h_k(t)) \leq 0, \]
\[ Y(t) = X(t), \quad t \leq t_0. \]
Thus by Corollary 4.2 we have \( 0 \leq Y(t) \leq X(t) \).

For the case \( F(t) \geq 0 \) the proof follows from solution representation (2.6).
Corollary 6.1. If \( a_{n-1}(t) \geq 0, a_k(t) \leq 0, k = 0, \ldots, n - 2, t \geq t_0, (5.9) \) holds and
\[
0 \leq \varphi_k(t) \leq y_k, \quad y_k > 0, k = 1, \ldots, n, \quad f(t) \geq 0,
\]
then for the solution \( y(t) \) of initial value problem (5.2), (5.3) we have \( y(t) \geq 0, t \geq t_0. \)

Next, we establish in this section the connection between nonoscillation properties and stability for delay differential systems.

We need some auxiliary definitions and results.

Definition. Matrix \( A \) is called an \( M \)-matrix if \( a_{ij} < 0, i \neq j \) and there exists a nonnegative inverse matrix \( A^{-1} \geq 0. \)

We refer to [4, 8] for many equivalent forms of this definition.

Let \( k \in \mathbb{R}^n \) be an arbitrary vector norm in \( \mathbb{R}^n, \) by \( k \cdot k \) we will also denote the associated matrix norm
\[
k(A) = \sup_{\|x\| = 1, x \in \mathbb{R}^n} \|Ax\|.
\]

Definition. Equation (2.2) is asymptotically stable if for any \( s \geq 0 \) the fundamental matrix of this equation satisfies \( \lim_{t \to -\infty} C(t, s) = 0. \) Equation (2.2) is exponentially stable if there exist \( M > 0 \) and \( \alpha > 0 \) such that
\[
\|C(t, s)\| \leq Me^{-\alpha(t-s)}, \quad t \geq s \geq t_0.
\]

Lemma 6.1 ([2, Theorem 5.3.4]). Suppose for equation (2.2) functions \( a_{ij} \) are essentially bounded on \( [t_0, \infty), t - h_k(t) \leq H \) and for any essentially bounded \( [t_0, \infty) \) function \( F \) the solution of the initial value problem
\[
\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X(h_k(t)) = F(t), \quad t > t_0, \quad X(t) = 0, \quad t \leq t_0
\]
is bounded on \( [t_0, \infty). \) Then equation (2.2) is exponentially stable.

Corollary 6.2. Suppose that in vector equation (2.2) functions \( a_{ij} \) are essentially bounded on \( [t_0, \infty) \) and \( t - h_k(t) \leq H. \) Let there exist \( t_1 > t_0 \) such that for any function \( F \) which is essentially bounded on \( [t_0, \infty) \) and satisfies \( F(t) = 0, t \in [t_0, t_1] \) the solution of the initial value problem (6.2) is bounded on \( [t_0, \infty). \) Then equation (2.2) is exponentially stable.

Proof. Since the solution of (2.2) with the zero initial conditions and the zero right hand side on \( [t_0, t_1] \) vanishes on \( [t_0, t_1], \) we can obtain by Lemma 6.1 that (2.2) with the initial point \( t_1 \) instead of \( t_0 \) is exponentially stable: \( \|C(t, s)\| \leq M_1e^{-\alpha(t-s)} \) for \( t \geq s \geq t_1. \)
Lemma 9 in [7] implies that there exists $M > 0$ (x is the same) such that inequality (6.1) holds for $t \geq s \geq t_0$.

For an essentially bounded on $[t_0, \infty)$ function define the norm $\|f\|_{L_x} = \text{ess sup}_{t \geq t_0} \|f(t)\|$.

**Theorem 6.2.** Suppose that for system (2.2) conditions of Theorem 3.1 hold, for some $t_0 \geq 0$ and $H > 0$ functions $a_{ij}^k$ are essentially bounded on $[t_0, \infty)$, $t - h_k(t) \leq H$, $a_{ii}^k \geq 0$, $i = 1, \ldots, n$, $k = 1, \ldots, m$,

$$\inf_{t \geq t_0} \sum_{k=1}^m a_{ii}^k(t) \geq a_i > 0,$$

and matrix $B = \{b_{ij}\}$ defined as

$$b_{ij} = \begin{cases} a_i, & i = j, \\ -\sum_{k=1}^m \|a_{ij}^k\|_{L_x}, & i \neq j, \end{cases}$$

is an $M$-matrix. Then system (2.2) is exponentially stable.

**Proof.** We apply Corollary 6.2. Consider the initial value problem

$$\dot{X}(t) + \sum_{k=1}^m A_k(t)X(h_k(t)) = F(t), \quad X(t) = 0, \quad t \leq t_0,$$  

where $F(t) = 0, \quad t_0 \leq t \leq t_0 + H$. Since $F(t) = 0, \quad t \leq t_0 + H$, then $X(t) = 0, \quad t \leq t_0 + H$.

Equation (6.4) can be rewritten as the system of scalar equations

$$\dot{x}_i(t) + \sum_{k=1}^m a_{ii}^k(t)x_i(h_k(t)) = -\sum_{j \neq i} \sum_{k=1}^m a_{ij}^k(t)x_j(h_k(t)) + f_i(t), \quad i = 1, \ldots, n.$$  

Denote by $X_i(t, s)$ the fundamental function of equation (3.1). By the assumption of the theorem $X_i(t, s) \geq 0, \quad t \geq s \geq t_0$. We can rewrite (6.4) in the form

$$x_i(t) = -\int_{t_0}^t X_i(t, s) \sum_{j \neq i} \sum_{k=1}^m a_{ij}^k(s)x_j(h_k(s))ds + g_i(t), \quad i = 1, \ldots, n,$$

where

$$g_i(t) = \int_{t_0}^t X_i(t, s)f_i(s)ds = \int_{t_0+H}^t X_i(t, s)f_i(s)ds.$$
By Lemma 2.4 we have

$$|g_i(t)| \leq \int_{t_0}^{t} X_i(t,s) \sum_{k=1}^{m} a_{ik}^k(s) \frac{\|f_i\|_{L_e}}{a_i} \leq \frac{\|f_i\|_{L_e}}{a_i}. $$

Hence \( \operatorname{ess\ sup}_{t \geq t_0} |g_i(t)| < \infty \).

Since

$$|x_i(t)| \leq \int_{t_0}^{t} X_i(t,s) \sum_{k=1}^{m} a_{ik}^k(s) ds \sum_{j \neq i} \sum_{k=1}^{m} \frac{\|a_{jk}^k\|_{L_e}}{a_i} \sup_{0 \leq s \leq t} |x_j(s)| + \|g_i\|_{L_e},$$

then

$$\sup_{0 \leq s \leq t} |x_i(s)| \leq \sum_{j \neq i} \sum_{k=1}^{m} \frac{\|a_{jk}^k\|_{L_e}}{a_i} \sup_{0 \leq s \leq t} |x_j(s)| + \|g_i\|_{L_e}. $$

Denote

$$y_i(t) = \sup_{0 \leq s \leq t} |x_i(s)|, \quad Y(t) = [y_1(t), \ldots, y_n(t)]^T, \quad G(t) = [\|g_1\|_{L_e}, \ldots, \|g_n\|_{L_e}]^T, $$

$$c_{ij} = \begin{cases} 1, & i = j, \\ -\frac{1}{a_i} \sum_{k=1}^{m} \|a_{jk}^k\|_{L_e}, & i \neq j, \end{cases} \quad C = [c_{ij}].$$

We have \( CY(t) \leq G \) for any \( t > t_0 \). Evidently if \( B \) is an \( M \)-matrix then \( C \) is also an \( M \)-matrix. Thus \( C^{-1} \geq 0 \) and \( Y(t) \leq C^{-1} G \). Hence

$$\sup_{t \geq t_0} \|X(t)\| = \sup_{i \geq t_1} \|Y(t)\| \leq \|C^{-1} G\| < \infty.$$ 

By Corollary 6.2 equation (2.2) is exponentially stable. \( \square \)

**Corollary 6.3.** Suppose that for system (2.2) conditions of Theorem 3.1 hold, and for some \( t_0 \geq 0 \) and \( H > 0 \) functions \( a_{ik}^k \) are essentially bounded on \( [t_0, \infty) \) and \( t - h_k(t) \leq H \). Let matrix \( B = \{b_{ij} \} \) be defined in (6.3).

1) If the following condition holds:

(A) there exist two vectors \( z = [z_1, \ldots, z_n]^T \) and \( \varepsilon = [\varepsilon_1, \ldots, \varepsilon_n]^T \) such that \( z_i > 0 \) and \( \varepsilon_i \geq 1 \) for \( i = 1, \ldots, n \), and

$$Bz \geq \varepsilon, \quad (6.5)$$

then equation (2.2) is exponentially stable and

$$\lim_{t \to \infty} \int_{t_0}^{t} \sum_{j=1}^{n} C_{ij}(t,s) ds \leq z_i, \quad i = 1, \ldots, n. \quad (6.6)$$
2) If there exists a matrix $Y = \{y_{ij}\} \geq 0$ such that $y_{ii} > 0$ and the inequality
\[
BY \geq I,
\]
holds, then equation (2.2) is exponentially stable, and the entries of the Cauchy matrix satisfy the inequalities
\[
\lim_{t \to \infty} \int_{t_0}^{t} C_{ij}(t,s)ds \leq y_{ij}, \quad i, j = 1, \ldots, n.
\]

**Proof.** Let us note that condition (A) is equivalent to the hypothesis that $B$ is an $M$-matrix [8], as well as to the existence of matrix $Y$ such that (6.7) holds. For the estimates (6.6) and (6.8) see [3, Theorems 2 and 3].

**Remark 6.1.** It should be emphasized that the requirement that $B$ is an $M$-matrix in Theorem 6.2 is essential and becomes necessary in the case of constant coefficients $a_{ij}^k$. Note in this connection the following results obtained in [11, Theorem 1].

**Theorem 6.3** ([11, Theorem 1]). Suppose that all coefficients $a_{ij}^k$ are constants, $t - h_k(t) \leq H$ for some $t_0 \geq 0$ and $H > 0$, and for system (2.2) the conditions of Theorem 3.1 hold. Then system (2.2) is exponentially stable if and only if condition (A) is satisfied.

**Theorem 6.4** ([11, Corollary 1]). Let the conditions of Theorem 6.3 be fulfilled, and $B$ be an $M$-matrix, then
\[
\lim_{t \to \infty} \int_{t_0}^{t} \sum_{j=1}^{n} C_{ij}(t,s)ds = z_i, \quad i = 1, \ldots, n,
\]
and
\[
\lim_{t \to \infty} \int_{t_0}^{t} C_{ik}(t,s)ds = y_{ik}, \quad i, k = 1, \ldots, n,
\]
where $z_i, i = 1, \ldots, n$ and $y_{ik}, i, k = 1, \ldots, n$ satisfy (6.5) and (6.7), respectively.

In the case of variable coefficients the following result can be obtained.

**Theorem 6.5.** Suppose for system (2.2) the conditions of Theorem 3.1 hold, $t - h_k(t) \leq H$ and
\[
\sum_{k=1}^{m} \sum_{j=1}^{n} a_{ij}^k(t) \leq 0, \quad t \geq t_0, \quad i = 1, \ldots, n.
\]
Then equation (2.2) is not asymptotically stable.
Proof. Denote \( Y(t) = [1, \ldots, 1]^T \). Then for \( t \geq t_0 + H \) vector function \( Y(t) \) is a solution of inequality (4.1). By Corollary 4.1 we have \( X(t) \geq Y(t) \), where \( X(t) = Y(t), t \leq t_1 \). Hence equation (2.2) is not asymptotically stable.

\[ \square \]

7. Discussion and open problems

Our main result is the generalization of the well-known Wazewski’s result [19] for the ordinary vector differential equation

\[ (7.1) \quad \dot{X}(t) + A(t)X(t) = 0. \]

to delay equations. By this result, equation (7.1) has a nonnegative fundamental matrix if and only if \( a_{ij} \leq 0, i \neq j \); for the proof of this theorem see [4].

In contrast with the classical Wazewski’s theorem, the condition \( a_{ij} \leq 0, i \neq j \) is not necessary for nonnegativity of all entries of the fundamental (Cauchy) matrix \( C(t,s) \) for equations with several delays, as was demonstrated in the present paper. Theorems 3.1 and 5.3 were proven in [11] by a different method. Paper [10] deals with nonnegativity of certain entries of the fundamental matrix. In [1] the authors considered nonoscillation problems for a general vector Volterra equation on a bounded interval. Positivity of all solutions with positive initial conditions and stability analysis for autonomous delay equations is given in [12, 18]. Our approach to positivity is different from one in [12, 18]. We study positivity of the fundamental matrix and not of all solutions. We also consider more general class of delay differential equations.

To the best of our knowledge, [3] is the first paper where connection between nonnegativity of the fundamental matrix and exponential stability was established, see also papers [14, 15, 16].

Finally, let us formulate some open problems.

1. Suppose that for equation (2.2) condition \( a_{ij} \leq 0, i \neq j \) holds and the fundamental matrix of this equation is nonnegative. Are the fundamental functions of scalar equations (3.1) necessarily positive?
2. Obtain explicit lower and upper estimates

\[ me^{-x(t-s)} \leq \|C(t,s)\| \leq Me^{-\beta(t-s)}, \quad 0 < \beta < x \]

for the fundamental matrix (in the case when it is nonnegative) of system (2.2) and for the fundamental function (again, when it is nonnegative) of high order equation (5.1) when the system/equation is exponentially stable.
3. Is there a connection between nonoscillation and asymptotic stability of (5.1)?

4. Establish connection between nonoscillation and asymptotic stability for equation (2.2) without the assumption that $a_{ii}^k \geq 0$.

5. Prove or disprove:

Suppose that $a_{ii}^k(t) \geq 0$, $a_{ij}^k(t) \leq 0$, $i \neq j$ in equation (2.2). Then the equation has a nonnegative solution if and only if its fundamental matrix is nonnegative.

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