Oscillation of equations with an infinite distributed delay
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**A R T I C L E I N F O**

Article history:
Received 25 March 2010
Accepted 24 August 2010

Keywords:
Oscillation
Distributed delay
Infinite delay
Comparison theorems
Integrodifferential equations

**A B S T R A C T**

For the equation with a finite or infinite distributed delay
\[
\dot{x}(t) + \int_{-\infty}^{t} x(s) R(t, s) \, ds = 0,
\]
the existence of nonoscillatory solutions is studied. A general comparison theorem is obtained which allows to compare oscillation properties of equations with concentrated delays to integrodifferential equations. Sharp nonoscillation conditions are deduced for some autonomous integrodifferential equations. Using comparison theorems, an example is constructed where oscillation properties of an integrodifferential equation are compared to equations with several concentrated delay which can be treated as its finite difference approximations.

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**1. Introduction**

In this paper we consider efficient nonoscillation conditions for a scalar equation with a distributed delay
\[
\dot{x}(t) + \int_{-\infty}^{t} x(s) R(t, s) \, ds = f(t), \quad t > t_0,
\]
with the initial function
\[
x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0.
\]

As particular cases, homogeneous equation (1.1) includes the following models: a delay differential equation
\[
\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(h_k(t)) = 0,
\]
if we assume
\[
R_1(t, s) = \sum_{k=1}^{m} a_k(t) \chi_{(h_k(t), \infty)}(s),
\]
where \( \chi_I \) is the characteristic function of interval \( I \), an integrodifferential equation
\[
\dot{x}(t) + \int_{-\infty}^{t} K(t, s) x(s) \, ds = 0,
\]
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doi:10.1016/j.camwa.2010.08.071
where
\[ R_2(t, s) = \int_{-\infty}^{s} K(t, \zeta) \, d\zeta, \quad (1.6) \]
a mixed equation
\[ \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) + \int_{-\infty}^{t} K(t, s)x(s) \, ds = 0, \quad (1.7) \]
where \( R_3(t, s) = R_1(t, s) + R_2(t, s), \) and \( R_1, R_2 \) are defined by (1.4), (1.6), respectively, and a mixed equation with an infinite number of delays:
\[ \dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x(h_k(t)) + \int_{-\infty}^{t} K(t, s)x(s) \, ds = 0 \quad (1.8) \]
which is obtained if
\[ R_4(t, s) = \sum_{k=1}^{\infty} a_k(t)x(h_k(t)) + \int_{-\infty}^{s} K(t, \zeta) \, d\zeta. \quad (1.9) \]

However, if \( R(t, \cdot) \) contains a component which is continuous but not absolutely continuous then (1.1) turns into an equation which differs from (1.8).

Among numerous publications on the oscillation of equations with a deviating argument we mention here papers [1–8], which are concerned with a distributed delay.

The paper is organized as follows. Section 2 contains relevant definitions and known results. In Section 3 we prove that the following four assertions are equivalent: nonoscillation of the equation and the corresponding differential inequality, positiveness of the fundamental function and the existence of a nonnegative solution of a certain nonlinear integral inequality which is constructed explicitly from the differential equation. Section 3 involves a comparison theorem which allows to compare oscillation properties of equations with concentrated and distributed delays. Next, in Section 4 sharp nonoscillation conditions for several classes of integrodifferential equations are considered. Further, using comparison theorems we obtain efficient nonoscillation conditions for various classes of nonautonomous delay equations. Finally, Section 5 involves some discussion and open problems.

2. Preliminaries

We study Eqs. (1.1), (1.2) under the following assumptions:
(a1) \( R(t, \cdot) \) is a left continuous function of bounded variation and for each \( s \) its variation on the segment \([t_0, s]\)
\[ P(t, s) = \text{Var}_{\tau \in [t_0, s]} R(t, \tau) \quad (2.1) \]
is a locally integrable function in \( t; \)
(a2) \( R(t, s) = R(t, t+), \ t < s; \)
(a3) \( f : [t_0, \infty) \to \mathbb{R} \) is a Lebesgue measurable locally essentially bounded function, \( \varphi : (-\infty, t_0) \to \mathbb{R} \) is a Lebesgue measurable bounded function such that the Lebesgue–Stieltjes integral in (1.1) exists; in particular, if \( \varphi \) is continuous then (1.1) is well defined for any \( R(t, s). \)

Function \( x : \mathbb{R} \to \mathbb{R} \) is called a solution of Eqs. (1.1), (1.2) if it satisfies equation (1.1) for almost every \( t \in [t_0, \infty), \) and (1.2) holds for \( t \leq t_0. \)

Definition 1. For each \( s \geq t_0 \) denote by \( X(t, s) \) the solution of the problem
\[ \dot{x}(t) + \int_{-\infty}^{t} x(s) \, dR(t, s) = 0, \quad t \geq s, \quad x(t) = 0, \quad t < s, \quad x(s) = 1. \quad (2.2) \]
\( X(t, s) \) is called a fundamental function of Eq. (1.1). We assume \( X(t, s) = 0, \ 0 \leq t < s. \)

Lemma 1 ([9]). Let (a1)–(a3) hold. Then there exists one and only one solution of problem (1.1), (1.2) that can be presented in the form
\[ x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)f(s) \, ds - \int_{t_0}^{t} X(t, s) \, ds \int_{-\infty}^{s} \varphi(\tau) \, dR(s, \tau), \quad (2.3) \]
where \( \varphi(\tau) = 0, \) if \( \tau > t_0. \)
Together with (1.1) let us consider for \( t_1 \geq t_0 \) the homogeneous equation
\[
\dot{x}(t) + \int_{-\infty}^{t} x(s) \, d_i R(t, s) = 0, \quad t \geq t_1, \tag{2.4}
\]
and the differential inequality
\[
\dot{y}(t) + \int_{-\infty}^{t} y(s) \, d_i R(t, s) \leq 0, \quad t \geq t_1. \tag{2.5}
\]

We also study (1.1), (1.2) with a bounded aftereffect, i.e. the following hypothesis holds:

(a4) For each \( t_1 \) there exists \( s_1 = s(t_1) \leq t_1 \) such that \( R(t, s) = 0 \) for \( s < s_1, t > t_1 \) and \( \lim_{t \to -\infty} s(t) = \infty \).

If (a4) holds we can introduce the function
\[
h(t) = \inf_{s} \{ R(t, s) \neq 0 \} \tag{2.6}
\]
such that \( \lim_{t \to -\infty} h(t) = \infty \), then Eq. (2.4) can be rewritten as
\[
\dot{x}(t) + \int_{h(t)}^{t} x(s) \, d_i R(t, s) = 0, \quad t \geq t_1. \tag{2.7}
\]

**Definition 2.** Eq. (2.4) has a positive solution for \( t \geq t_1 \) if there exists initial function \( \varphi \) such that a solution of (2.4), (1.2) is positive for \( t \geq t_1 \).

### 3. General results and comparison theorems

**Theorem 1.** Suppose \( R(t, \cdot) \) is a nondecreasing function for any \( t \) for which (a1) and (a2) hold. Then the following hypotheses are equivalent:

1. There exists \( t_1 \geq 0 \) such that (2.5) has a positive solution for \( t \geq t_1 \), with \( y(t) = 0, t < t_1 \).
2. There exists \( t_1 \geq 0 \) such that the inequality
   \[
   u(t) \geq \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_i R(t, s) \tag{3.1}
   \]
   has a nonnegative locally integrable solution \( u(t) \) for \( t \geq t_1 \) (in (3.1) we assume \( u(t) = 0 \) for \( t < t_1 \)).
3. There exists \( t_1 \geq 0 \) such that \( X(t, s) > 0, t > s \geq t_1 \).
4. There exists \( t_1 \geq 0 \) such that (2.1) has a positive solution for \( t \geq t_1 \), with \( x(t) = 0, t < t_1 \).

**Proof.** Let us prove the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (1).

(1) \( \Rightarrow \) (2). Let \( y \) be a positive solution of (2.5) for \( t \geq t_1 \). Since \( \dot{y}(t) \leq 0, t \geq t_1 \) then \( y(t) \) is nonincreasing, hence \( u(t) \geq 0 \), where \( u \) is denoted by
\[
u(t) = -\frac{d}{dt} \ln \frac{y(t)}{y(t_1)},
\]
i.e.
\[
y(t) = y(t_1) \exp \left\{ -\int_{t_1}^{t} u(s) \, ds \right\}
\]
and assume \( u(t) = 0, t < t_1 \).

By substituting \( y \) into (2.5) we obtain
\[
-\dot{y}(t_1) \exp \left\{ -\int_{t_1}^{t} u(s) \, ds \right\} u(t) + y(t_1) \int_{-\infty}^{t} \exp \left\{ -\int_{t_1}^{s} u(\tau) \, d\tau \right\} \, d_i R(t, s) \leq 0,
\]
which implies
\[
y(t_1) \exp \left\{ -\int_{t_1}^{t} u(s) \, ds \right\} \left[ -u(t) + \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_i R(t, s) \right] \leq 0. \tag{3.2}
\]
The first factor is positive since \( y(t_1) > 0 \), consequently the expression in the brackets is nonpositive. Hence
\[
u(t) \geq \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_i R(t, s), \quad t \geq t_1. \tag{3.3}
\]
As was demonstrated above, which is equivalent to
Consequently we obtain an operator equation
Thus the left-hand side is equal to
Consequently we obtain an operator equation
which is equivalent to (3.4), where
Inequality (3.1) yields that if $z(t) \geq 0$, then $(Hz)(t) \geq 0$, i.e. $H$ is a positive operator ($t > t_1$). Besides, in each final interval $[t_1, b]$ operator $H$ is a sum of integral Volterra operators, which are compact in the space of integrable functions. Hence [10, p. 519] its spectral radius $r(H) = 0 < 1$ and consequently if in (3.6) right-hand side $f$ is nonnegative, then
We recall that the solution of (3.4) has the form (3.5), with $z$ being a solution of (3.6). Thus if in (3.4) $f \geq 0$, then $x(t) \geq 0$. On the other hand, the solution of (3.4) has the representation
As was demonstrated above, $f(t) \geq 0$ implies $x(t) \geq 0$. Hence the kernel of the integral operator is nonnegative, i.e. $X(t, s) \geq 0$ for $t \geq s > t_1$.
As a second step let us prove that $X(t, s)$ is strictly positive: $X(t, s) > 0$. To this end consider
\[ x(t) = x(t, t_1) - \exp \left\{ - \int_{t_1}^{t} u(s) \, ds \right\}, \quad x(t) = 0, \quad t < t_1, \]
and substitute $x(t)$ into the left-hand side of (3.4):

$$X'(t, t_1) + u(t) \exp \left\{ - \int_{t_1}^{t} u(s) \, ds \right\} + \int_{t_1}^{t} X(s, t_1) \, d_s R(t, s) - \int_{t_1}^{t} \exp \left\{ - \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_s R(t, s) = 0 + \exp \left\{ - \int_{t_1}^{t} u(s) \, ds \right\} \left[ u(t) - \int_{t_1}^{t} \exp \left\{ - \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_s R(t, s) \right] \geq 0.$$ 

Therefore $x(t)$ is a solution of (3.4) with a nonnegative right-hand side. As shown above, $x(t) \geq 0$, consequently,

$$X(t, t_1) \geq \exp \left\{ - \int_{t_1}^{t} u(s) \, ds \right\} > 0.$$ 

For any $s > t_1$ inequality $X(t, s) > 0$ is verified in a similar way.

(3) $\Rightarrow$ (4). Function $x(t) = X(t, t_1)$ is a positive solution of (2.1) for $t \geq t_1$.

Implication (4) $\Rightarrow$ (1) is obvious. \( \square \)

**Corollary 1.** Suppose $a_k \geq 0$, $k = 1, \ldots, m$, $K(t, s) \geq 0$ and the following condition is satisfied:

(b1) $K(t, s)$ is Lebesgue integrable over each finite square $[t_0, b] \times [t_0, b]$, $a_k(t)$ are locally essentially bounded, $k = 1, \ldots, m$.

Then the following hypotheses are equivalent:

1. There exists $t_1 \geq 0$, such that the inequality

   $$\dot{y}(t) + \sum_{k=1}^{m} a_k(t) y(h_k(t)) + \int_{-\infty}^{t} K(t, s) y(s) \, ds \leq 0$$

   (3.7)

   has a positive solution for $t \geq t_1$, where $y(t) = 0$, $t < t_1$.

2. There exists $t_1 \geq 0$ such that the inequality

   $$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp \left\{ \int_{h_k(t)}^{t} u(s) \, ds \right\} + \int_{-\infty}^{t} K(t, s) \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, ds$$

   (3.8)

   has a nonnegative locally integrable solution $u(t)$ for $t \geq t_1$ (in (3.8) it is assumed that $u(t) = 0$, if $t < t_1$).

3. There exists $t_1 \geq 0$ such that the fundamental function of (1.7) is positive for $t > s \geq t_1$.

4. There exists $t_1 \geq 0$, such that (1.7) has a positive solution for $t \geq t_1$, where $x(t) = 0$, $t < t_1$.

Consider together with (2.4) the following equation:

$$\dot{x}(t) + \int_{-\infty}^{t} x(s) \, d_s T(t, s) = 0, \quad t \geq t_1.$$  

(3.9)

We compare the properties of (2.4) and (3.9) related to the existence of a nonnegative solution. Theorem 1 immediately implies the following result.

**Corollary 2.** Suppose $R, T$ satisfy (a1) and (a2), functions $R(t, \cdot)$, $T(t, \cdot)$ and the difference $R(t, \cdot) - T(t, \cdot)$ are nondecreasing for each $t \geq t_1$. If inequality (3.1) has a nonnegative solution for $t \geq t_1$ (with $u(t) = 0$, $t < t_1$), then (3.9) has a positive solution for $t \geq t_1$ and its fundamental solution $Y(t, s)$ is positive for $t \geq s \geq t_1$.

**Proof.** Let $u(t)$ be a positive nonnegative for $t \geq t_1$ solution of (3.1). Since

$$u(t) \geq \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_s R(t, s) = \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_s R(t, s) + \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_s [R(t, s) - T(t, s)] \geq \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, d_s T(t, s),$$

then (3.9) has a nonnegative solution for $t \geq t_1$, with $x(t) = 0$ for $t \leq t_1$ and thus by Theorem 1 its fundamental solution $Y(t, s)$ is positive for $t \geq s \geq t_1$. \( \square \)

In future we will need a more advanced comparison result.

**Theorem 2.** Suppose $R, T$ satisfy (a1) and (a2), functions $R(t, \cdot)$ and $T(t, \cdot)$ satisfy

$$\lim_{s \to -\infty} R(t, s) = \lim_{s \to -\infty} T(t, s) = 0, \quad R(t, s) \geq T(t, s) \geq 0$$

(3.10)

and are nondecreasing in $s$ for each $t \geq t_1$. If inequality (3.1) has a nonnegative solution for $t \geq t_1$, then (3.9) has a positive solution for $t \geq t_1$ and its fundamental solution $Y(t, s)$ is positive for $t \geq s \geq t_1$. 


Proof. Let (3.1) hold for \( t \geq t_1 \). Then (we recall that \( u(s) = 0 \) for \( s < t_1 \))

\[
\begin{align*}
    u(t) & \geq \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} R(t, \tau) \, d\tau \right\} \, ds \, \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, \left[ R(t, s) \right]_{s=-\infty}^{s=t} \\
    & \quad + \int_{-\infty}^{t} R(t, s) u(s) \exp \left\{ \int_{s}^{t} R(t, \tau) \, d\tau \right\} \, ds = R(t, t) - 0 + \int_{-\infty}^{t} R(t, s) u(s) \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, ds \\
    & \geq T(t, t) + \int_{-\infty}^{t} T(t, s) u(s) \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, ds = \int_{-\infty}^{t} \exp \left\{ \int_{s}^{t} u(\tau) \, d\tau \right\} \, ds, \\
\end{align*}
\]

Applying to \( \text{Theorem 1} \) completes the proof. \( \square \)

Let us compare (1.7) with the equation

\[
\dot{x}(t) + \sum_{k=1}^{m} b_k(t) x(g_k(t)) + \int_{-\infty}^{t} m(t, s)x(s) \, ds = 0. \tag{3.11}
\]

**Corollary 3.** Suppose \( a_k \) and \( b_k \), \( h_k \), and \( g_k \), \( K(t, s) \), and \( M(t, s) \) satisfy \( b1 \), \( a_k(t) \geq b_k(t) \geq 0 \), \( h_k(t) \leq g_k(t) \leq t \) and \( K(t, s) \geq M(t, s) \geq 0 \) for any \( t \geq t_1 \). If inequality (3.1) has a nonnegative solution for \( t \geq t_1 \), then (3.11) has a positive solution for \( t \geq t_1 \) and its fundamental solution \( Y(t, s) \) is positive for \( t \geq s \geq t_1 \).

**Proof.** Since

\[
0 \leq T(t, s) = \sum_{k=1}^{m} b_k(t) \chi(g_k(t), \infty)(s) + \int_{-\infty}^{t} M(t, \xi) \, d\xi \\
\leq \sum_{k=1}^{m} a_k(t) \chi(h_k(t), \infty)(s) + \int_{-\infty}^{t} K(t, \xi) \, d\xi = R(t, s),
\]

both \( T(t, s) \) and \( R(t, s) \) are nondecreasing in \( s \) for any \( t \geq t_1 \), then it is enough to demonstrate that the limit conditions in (3.10) are satisfied. Since \( \int_{-\infty}^{t} K(t, \xi) \, d\xi \) converges then \( \lim_{t \to -\infty} \int_{-\infty}^{t} K(t, \xi) \, d\xi = 0 \). Since \( R(t, s) = \int_{-\infty}^{t} K(t, \xi) \, d\xi \) for \( s < \min h_k(t) \), then \( \lim_{t \to -\infty} R(t, s) = 0 \), which due to the inequality \( 0 \leq T(t, s) \leq R(t, s) \) implies \( \lim_{t \to -\infty} T(t, s) = 0 \). This completes the proof. \( \square \)

4. Nonoscillation for some autonomous integral equations

Let us consider nonoscillation conditions for autonomous integro-differential equations

\[
\dot{x}(t) + \int_{-\infty}^{t} K(t, s)x(s) \, ds = 0, \tag{4.1}
\]

where \( K(t, s) = G(t - s) > 0 \).

**Example 1.** Let \( K(t, s) = A \chi[t-h, h](s), A > 0, h > 0 \), which corresponds to the equation

\[
\dot{x}(t) + \int_{t-h}^{t} A x(s) \, ds = 0. \tag{4.2}
\]

By \( \text{Theorem 1} \) it has a nonoscillatory solution as far as the inequality

\[
\lambda \geq \int_{t-h}^{t} A e^{\lambda(t-s)} \, ds = \frac{A}{\lambda} (e^{\lambda h} - 1)
\]

has a positive solution \( \lambda \). Thus, if there exists \( x > 0 \) such that

\[
f(x) = x^2 - A(e^{hx} - 1) > 0,
\]

then Eq. (4.2) has a nonoscillatory solution. For this \( x \), we have

\[
A \leq \frac{x^2}{e^{hx} - 1}.
\]

Thus, (4.2) is nonoscillatory if and only if

\[
A \leq B_1(h) := \sup_{x>0} \left( \frac{x^2}{e^{hx} - 1} \right). \tag{4.3}
\]
Fig. 1. Function $A = A(h) = B_1(h)$ (left) described in (4.3) and the graph of $Ah = hB_1(h)$ (right), it tends to infinity as $h \to 0$.

Obviously $B_1(h)$ is decreasing in $h$. For $h \approx 0.8047$ we have $A = B_1(h) = 1$, which is attained at $x \approx 1.98$. For $h \approx 0.569$ we have $A = B_1(h) = 2$ attained at $x \approx 2.8$ which illustrates the difference between equations with distributed and concentrated delay. For equations with concentrated delay $\dot{x} + ax(t - h) = 0$ the sharp nonoscillation condition $ah \leq 1/e$ implies that the delay boundary for $h$ decays twice as $a$ is doubled.

The graph of $A(h) = B_1(h)$ is presented in Fig. 1, left. Let us note that $hA(h) = hB_1(h)$ is unbounded in contrast to the case of the constant concentrated delay (see Fig. 1, right).

**Example 2.** Let $A > 0$, $h > 0$ and $K(t, s) = \begin{cases} A(s + h - t), & t - s \leq h, \\ 0, & \text{otherwise,} \end{cases}$ then the equation is

$$\dot{x}(t) + \int_{t-h}^{t} A(s - t + h)x(s) \, ds = 0. \tag{4.4}$$

There exists a positive solution if for some positive $\lambda$

$$A \int_{t-h}^{t} (s - t + h)e^{\lambda(t-s)} \, ds = A \left( -\frac{1}{\lambda} (s - t + h)e^{\lambda(t-s)} - \frac{1}{\lambda^2} e^{\lambda(t-s)} \right) \bigg|_{s=t-h}^{s=t} = \frac{A}{\lambda^2} (e^{\lambda h} - 1 - \lambda h) \leq \lambda.$$

Thus, if

$$A \leq B_2(h) := \sup_{x>0} \left[ \frac{x^3}{e^{hx} - hx - 1} \right], \tag{4.5}$$

then (4.4) is nonoscillatory. For the graph of $A = A(h) = B_2(h)$; see Fig. 2.

Let us also note that supremum in (4.5) does not exceed $\sup_{x>0} \left[ \frac{x^3}{(h^2x^2 + 6hx + 1)^{1/2}} \right] \leq \frac{6}{h^3}$.
Example 3. Let \( K(t, s) = Ae^{\nu(s-t)}, A > 0, \nu > 0, \) then the equation is
\[
\dot{x}(t) + \int_{-\infty}^{t} Ae^{\nu(s-t)} x(s) \, ds = 0.
\] (4.6)

Eq. (4.6) has a nonoscillatory solution if for some \( \lambda > 0 \) and any \( t_1 < 0 \) we have
\[
A \int_{t_1}^{t} e^{\nu(s-t) + \lambda(t-s)} \, ds < \lambda.
\]
The equation is autonomous, so the supremum of the left-hand side in \( t \) is the integral with the lower bound \(-\infty\), which diverges for \( \nu \leq \lambda \), so we can consider \( \nu > \lambda \) only:
\[
A \int_{t}^{t_1} e^{\nu(s-t) + \lambda(t-s)} \, ds = A \int_{-\infty}^{t} e^{(\nu-\lambda)(s-t)} \, ds = \frac{A}{\nu - \lambda} [e^{(\nu-\lambda)s} - 1] \bigg|_{s=-\infty}^{s=t} = \frac{A}{\nu - \lambda} \leq \lambda,
\]
which means that the quadratic inequality
\[
x^2 - \nu x + A \leq 0
\]
has a positive root (which is valid if and only if the discriminant is nonnegative). Thus, (4.6) is nonoscillatory if and only if
\[
\nu^2 \geq 4A.
\] (4.7)

Example 4. Further, consider the truncated Gaussian kernel \( K(t, s) = Ae^{-\nu(s-t)^2}, A > 0, \nu > 0, \) then the equation is
\[
\dot{x}(t) + \int_{-\infty}^{t} Ae^{-\nu(s-t)^2} x(s) \, ds = 0.
\] (4.8)

Then we obtain (here we make a substitution \( \eta = s - t \)) the nonoscillation condition
\[
A \int_{-\infty}^{t} e^{-\nu(t-t')^2 + \lambda(t-t')} \, ds = A \int_{-\infty}^{0} e^{-\nu\eta^2 - \lambda\eta} \, d\eta \leq A \int_{-\infty}^{\infty} e^{-\nu\eta^2 - \lambda\eta} \, d\eta = Ae^{1/4\nu} \sqrt{\frac{\pi}{\nu}} \leq \lambda.
\]

Thus for
\[
A \leq \sqrt{\frac{\nu}{\pi}} \sup_{x > 0} \left[ xe^{-x^2/(4\nu)} \right]
\]
Eq. (4.8) has a nonoscillatory solution. The supremum in the right-hand side is attained for \( x = \sqrt{2\nu} \) and equals \( \frac{2}{\sqrt{\pi}} \), so the sufficient nonoscillation condition is
\[
A \leq \sqrt{\frac{2}{\pi e\nu}}.
\] (4.9)

Remark 1. In Example 4, we have obtained a simple, but just a sufficient nonoscillation condition. Since
\[
A \int_{-\infty}^{0} e^{-\nu\eta^2 - \lambda\eta} \, d\eta = \frac{A}{2\sqrt{\nu}} \left[ \frac{\pi}{\nu} e^{1/4\nu} \right] \left[ 1 + \text{erf} \left( \frac{\lambda}{2\sqrt{\nu}} \right) \right] \geq \lambda,
\]
where \( \text{erf}(x) \) is the Gaussian error function [11, Chapter 7]
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \, dt,
\]
then this leads to the sharp nonoscillation condition
\[
A \leq \sup_{x > 0} \frac{2\sqrt{\nu}xe^{-x^2/(4\nu)}}{\sqrt{\pi} [1 + \text{erf}(x/(2\sqrt{\nu}))]}.
\] (4.10)

Using comparison Theorem 2 we can obtain several efficient nonoscillation conditions.

Theorem 3. Let (b1) hold.
If at least one of the following hypotheses holds for \( t \geq s \geq t_1 \) for some \( t_1 \geq 0 \):
Comparing Eq. 3 with notethat integral $\int_1^{0.568} (t, s) isnonoscillatory by Example 5.

\textbf{proof.} and prove that $t$, then it is enoughto check the inequality (4.4)

to deduce (5)–(6), we apply Theorem 2 and note that $T(t, s) = \sum_{k=1}^{m} a_k(t) \chi(h_k(t), \infty)(s)$ should be compared to $R(t, s) = \int_{-\infty}^{s} K(t, \zeta) d\zeta$ which equals

$$A \int_{t-h}^{t} (s-t+h) d\zeta = \frac{(s-t+h)^2}{2}$$

for (4.2) and (4.4), respectively, which implies (5) and (6).

To justify (7), let us first note that since $T(t, s)$ is a nondecreasing step function and $R(t, s)$ is nondecreasing in $s$ for any $t$, then it is enough to check the inequality $T(t, h_k(t)) \leq R(t, h_k(t))$ (where $R(t, h_k(t)) = \lim_{s \to 0^+} R(t, h_k(t) + s)$ only and prove $T(t, h_k(t)) \leq R(t, h_k(t))$ for $T(t, h_k(t)) = T(t, h_{k-1}(t))$). Since

$$R(t, h_k(t)) = A \int_{-\infty}^{h_k(t)} e^{(s-t)} d\zeta = \frac{A}{v} e^{-v(t-h_k(t))},$$

and $T(t, h_k(t)) = \int_{t-h}^{t} \zeta(t, s) d\zeta = \frac{A}{v} \left[ e^{-v(t-h_k)} - e^{-v(t-h_{k-1})} \right]$ and $T(t, h_k(t)) = T(t, h_k(t)) - T(t, h_{k-1}(t)) = a_k(t)$, this implies the statement of (7), which completes the proof.

\textbf{Example 5.} Consider differential equation (1.3) with coefficients compared to (4.2), where $A = 2$, $h \approx 0.569$. Then the delay equation

\begin{equation}
\bar{x}(t) + 55 a_k x(t - h_k) = \bar{x}(t) + 55 0.02x(t - 0.01k) = 0
\end{equation}

is nonoscillatory by Theorem 3, Part 5, while

$$a_k h_k = 55 0.002 0.01k = 0.308.$$
with several concentrated delays. This means that nonoscillation of the integrodifferential equation implies nonoscillation of its approximation by an equation with concentrated delays, under certain monotonicity conditions.

By Theorem 2 if the equation
\[ \dot{x}(t) + a(t)x(h(t)) = 0, \quad a(t) \geq 0, \quad h(t) \leq t, \tag{5.1} \]
is nonoscillatory then so is the integrodifferential equation
\[ \dot{x}(t) + \int_{h(t)}^{t} K(t, s)x(s) \, ds = 0, \quad \text{where} \quad \int_{h(t)}^{t} K(t, s) \, ds = a(t), \quad K(t, s) \geq 0. \tag{5.2} \]
Really, then for \( s \in (h(t), t] \) the function
\[ R(t, s) = a(t) \chi_{(h(t), \infty)}(s) \geq T(t, s) = \int_{h(t)}^{t} K(t, \zeta) \, d\zeta \]
and both \( R(t, s) \) and \( T(t, s) \) are nondecreasing in \( s \) for any \( t \).

This is usually described in the following form: nonoscillation and stability properties of an equation with a distributed delay are better than that of an equation with concentrated delays. Above, we have demonstrated that we can deduce nonoscillation of equations with concentrated delays from nonoscillation of an integral equation.

Oscillation of integrodifferential and mixed equations was studied in [3,6]. For Eq. (4.2) and its modifications with a variable delay \( h(t) \) and also the upper bound \( \tau(t) \), where \( h(t) \leq \tau(t) \leq t \) sharp oscillation results were recently obtained in [7,8].

Finally, let us formulate some open problems.

1. If we have the same “total weight” \( a(t) \) and the maximal delay equals \( h(t) \), then nonoscillation of (5.1) implies nonoscillation of (5.2). If we assume the contrary that \( h(t) \) is the minimal delay then by Theorem 2 nonoscillation of the equation
\[ \dot{x}(t) + \int_{-\infty}^{h(t)} K(t, s)x(s) \, ds = 0, \quad \text{where} \quad \int_{-\infty}^{h(t)} K(t, s) \, ds = a(t), \quad K(t, s) \geq 0. \tag{5.3} \]
implies nonoscillation of (5.1).

Prove or disprove that the equation with a distributed delay is nonoscillating if the equation with the same total weight \( a(t) \) and the single delay concentrated at the “centre of mass” (expectation) of the delay is nonoscillatory. Is it true that nonoscillation of (5.1) implies nonoscillation of the equation
\[ \dot{x}(t) + \int_{g(t)}^{t} K(t, s)x(s) \, ds = 0, \tag{5.4} \]
where
\[ \int_{h(t)}^{t} K(t, s)(s - t) \, ds = a(t)(t - g(t)), \quad \int_{h(t)}^{t} K(t, s) \, ds = a(t)? \tag{5.5} \]

2. For the general integral equation there is a gap between nonoscillation and oscillation conditions. If
\[ \limsup_{t \to \infty} \int_{h(t)}^{t} K(t, s) \, ds > \frac{1}{e}, \quad \text{but} \quad \liminf_{t \to \infty} \int_{h(t)}^{t} K(t, s)(s - t) \, ds < \frac{1}{e}, \]
\[ \limsup_{t \to \infty} \int_{h(t)}^{t} K(t, s)(s - t) \, ds < 1, \]
then [3,6] the known tests do not allow to establish oscillation properties of the equation. Develop sharp nonoscillation conditions for integral equations, at least in the case when lim sup and lim inf coincide.

3. Most efficient nonoscillation tests for nonautonomous equations were obtained under the assumption that the delays are finite and tend to infinity as \( t \to \infty \). Deduce explicit nonoscillation and oscillation conditions for equations with a distributed delay in the case when (a4) is not satisfied, and the relevant kernels do not have an exponential estimate and are not bounded by a Gaussian function (which would allow to apply the comparison with Examples 3 and 4, respectively).

4. If \( R(t, s) \) is nondecreasing, obtain sufficient conditions under which any nonoscillatory solution of (2.4) is asymptotically and exponentially stable.

**Acknowledgements**

The first author was partially supported by Israeli Ministry of Absorption. The second author was partially supported by the NSERC Research Grant.
References