Positive solutions for a scalar differential equation with several delays

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Abstract

For a scalar delay differential equation $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0$, $h_k(t) \leq t$, we obtain new explicit conditions for the existence of a positive solution.

Keywords: Linear delay equations; Several delays; Positive solutions; Nonoscillation

1. Introduction and preliminaries

Recently a close connection between nonoscillation and exponential stability for scalar linear differential equations with several delays has been revealed. Besides, exponentially stable nonoscillatory equations can be applied as comparison equations to obtain explicit stability conditions for general differential equations with several delays [1,2].

Thus it is important to know explicit conditions for the existence of a positive solution for equations with several delays. Unfortunately, only few such conditions are known [3,4]. The aim of this work is to review known nonoscillation conditions and to give some new ones, especially for equations with two delays.

We consider a scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq 0,$$

under the following conditions:

(a1) $a_k : [0, \infty) \to \mathbb{R}$, $k = 1, \ldots, m$, are Lebesgue measurable locally essentially bounded functions;

(a2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $k = 1, \ldots, m$. 

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Suppose the inequalities in (1) hold. The function Lemma 2 has an eventually positive solution.

We will say a solution is nonoscillating if it either is eventually positive or is eventually negative.

2. Explicit nonoscillation conditions

Lemma 1 ([3]). Suppose \( a_k(t) \geq 0, k = 1, \ldots, m \), and for sufficiently large \( t \)
\[
\int_{\min h_i(t)}^{t} \sum_{i=1}^{m} a_i(s)ds \leq \frac{1}{e}.
\]
Then Eq. (1) has an eventually positive solution.

This explicit condition is easily checked, constant \( \frac{1}{e} \) is the best possible, but (2) contains only “the worst delay”. To give another result, where all delays are included, define
\[
A_{ij} = \lim_{t \to \infty} \int_{h_i(t)}^{t} a_j(s)ds.
\]

Lemma 2 ([4]). Suppose \( a_k(t) \geq 0, k = 1, \ldots, m, A_{ij} < \infty \) and there exist positive numbers \( x_i, i = 1, \ldots, m \), such that
\[
\ln x_i > \sum_{j=1}^{m} A_{ij} x_j, \quad i = 1, \ldots, m.
\]
Then Eq. (1) has an eventually positive solution.

Unfortunately, Lemma 2 contains only implicit nonoscillation conditions. To derive from Lemma 2 explicit conditions, we consider first an equation with two delays
\[
\dot{x}(t) + a(t)x(h(t)) + b(t)x(g(t)) = 0, \quad a(t) \geq 0, \quad b(t) \geq 0, \quad h(t) \leq t, \quad g(t) \leq t.
\]

Similarly to (3), we define (and assumed that \( a, b, c, d \) are finite)
\[
a = \lim_{t \to \infty} \int_{h(t)}^{t} a(s)ds, \quad b = \lim_{t \to \infty} \int_{h(t)}^{t} b(s)ds,
\]
\[
c = \lim_{t \to \infty} \int_{g(t)}^{t} a(s)ds, \quad d = \lim_{t \to \infty} \int_{g(t)}^{t} b(s)ds.
\]

We look for positive solutions of the system
\[
\begin{align*}
\ln x_1 & > ax_1 + bx_2, \\
\ln x_2 & > cx_1 + dx_2.
\end{align*}
\]

Theorem 1. Suppose at least one of the following conditions holds:

1. \( 0 < a < \frac{1}{e}, b > 0 \), there exists a number \( y_0 > 0 \) such that \( y_0 \leq -\frac{1+lna}{b} , \frac{c}{a} + dy_0 < \ln y_0 \).
2. \( c > 0, 0 < d < \frac{1}{e} \), there exists a number \( x_0 > 0 \) such that \( x_0 \leq -\frac{1+lna}{c} \), \( \frac{b}{a} + ax_0 < \ln x_0 \).

Then Eq. (5) has an eventually positive solution.

Proof. Suppose the inequalities in (1) hold. The function \( y = (\ln x - ax) / b \) has the unique maximum \( y_{\text{max}} = -\frac{1+lna}{b} \) at the point \( x_{\text{max}} = \frac{1}{a} \). The inequality \( -(1+lna) \geq by_0 > 0 \) implies \( y_{\text{max}} > 0 \), while \( y_0 \leq -\frac{1+lna}{b} \) yields the point \( (x_{\text{max}}, y_0) \) satisfies the first inequality in (7) in the case \( y_0 < y_{\text{max}} \). Since \( \frac{c}{a} + dy_0 < \ln y_0 \), then this point also satisfies the second inequality in (7). If \( y_0 = y_{\text{max}} \), then there exists \( y_1 < y_0 \) for which still \( \frac{c}{a} + dy_1 < \ln y_1 \) holds. Then \( (x_{\text{max}}, y_1) \) is a solution of (7). If (2) holds the proof is similar. □
Corollary 1. Suppose at least one of the following conditions holds:

\begin{align}
0 < a < \frac{1}{e}, \quad b > 0, \quad & \frac{c}{a} - \frac{d(1 + \ln a)}{b} < \ln \left( \frac{1 + \ln a}{b} \right), \\
0 < d < \frac{1}{e}, \quad & \frac{b}{d} - \frac{a(1 + \ln d)}{c} < \ln \left( \frac{1 + \ln d}{c} \right).
\end{align}

Then Eq. (5) has an eventually positive solution.

Proof. If (8) holds then there exists \( \varepsilon > 0 \), such that for \( y_0 = -\frac{1 + \ln a}{b} - \varepsilon \) the first condition of Theorem 1 is satisfied. Similarly, (9) implies the second condition. \( \square \)

Remark 1. In Theorem 1 it is assumed that either \( a > 0, b > 0 \) or \( c > 0, d > 0 \). Including the cases when these conditions are not satisfied, by analyzing (7), we immediately obtain the following sufficient nonoscillation conditions:

1. \( b = 0, d > 0, a < 1/e, 1 + \ln d + c/e < 0 \),
2. \( c = 0, a > 0, d < 1/e, 1 + \ln a + b/e < 0 \),
3. \( a = 0, d > 0, ce^{b/d} + 1 + \ln d < 0 \),
4. \( d = 0, a > 0, be^{c/a} + \ln a + 1 < 0 \),
5. \( b = 0, c = 0, a < 1/e, d < 1/e \),
6. \( a = 0, c = 0, d < 1/e \),
7. \( b = 0, d = 0, a < 1/e \).

For \( a = d = 0 \) the situation is a little bit more complicated: there exists an eventually positive solution if the following condition is satisfied:

8. \( a = d = 0 \), there either exists \( x > 0 \) such that \( \ln x > be^{cx} \) or \( y > 0 \) such that \( \ln y > ce^{by} \).

Substituting in (7) specific values, say, \( x_1 = x_2 = e \), we can obtain simpler but more restrictive conditions than 8:

\( a = d = 0, b < 1/e, c < 1/e \).

Let us also note that conditions 1–4 are sharper than inequalities in Theorem 1 and Corollary 1 when one of the numbers \( a, b, c, d \) equals zero. Conditions 5–8 give a nonoscillation condition when Theorem 1 does not work.

Example 1. Consider the equation

\[ \dot{x}(t) + \frac{0.2}{\pi} \sin^2 t x(t - \pi) + \frac{0.2}{\pi} \cos^2 t x(t - 2\pi) = 0. \]

By simple calculations we have \( a = b = 0.1, c = d = 0.2 \). Condition (8) in Corollary 1 is not satisfied, but inequality (9) holds. Hence Eq. (10) has an eventually positive solution.

Fig. 1 illustrates the domain for \((x, y)\) where the inequalities of type (7) hold:

\[ \ln x > 0.1x + 0.1y, \quad \ln y > 0.2x + 0.2y. \]

We observe that the maximum of \( f(x) = 10 \ln(x) - x \) is not in the domain between the curves (thus, (8) is not satisfied), while the maximum of the function \( g(y) = 5 \ln(y) - y \) is in the intersection domain, so (9) holds. It should be noted that Lemma 1 fails for this equation.

Let us present different sufficient conditions for the existence of positive solutions.

Theorem 2. Suppose at least one of the following conditions holds:

1. there exists \( y_0 > 0 \), such that \( y_0 < (1 - ae)/b, ce + dy_0 < \ln y_0 \),
2. there exists \( x_0 > 0 \), such that \( x_0 < (1 - de)/c, ax_0 + be < \ln x_0 \).

Then Eq. (5) has an eventually positive solution.

Proof. Suppose (1) holds. Then \( ae < 1 \) and \( (e, y_0) \) is a solution of the system of inequalities (7). Similarly, if (2) holds, then \((x_0, e)\) is a solution of (7). \( \square \)

Remark 2. In Theorem 2 the value \( x = e \) was chosen to minimize the coefficient of \( x \) in the first inequality of the system

\[ \left( a - \frac{\ln x}{x} \right) x + by < 0, \quad cx + \left( d - \frac{\ln y}{y} \right) y < 0, \]

which is equivalent to (7); \( y = e \) minimizes the coefficient of \( y \) in the second inequality.
Corollary 2. If at least one of the following inequalities holds:

\[ ce + \frac{d}{b} (1 - ae) < \ln \left( 1 - ae \right) \]
\[ be + \frac{a}{c} (1 - de) < \ln \left( 1 - de \right) \]

then Eq. (5) has an eventually positive solution.

Let us modify Example 1 to demonstrate that there are cases when one of Theorem 1 or Theorem 2 can be applied while the other one fails.

Example 2. Consider the following modified version of Eq. (10)

\[ \dot{x}(t) + \frac{0.5}{\pi} \sin^2 tx(t - \pi) + \frac{0.08}{\pi} \cos^2 tx(t - 2\pi) = 0. \]

Then \( a = 0.25, b = 0.04, c = 0.5, d = 0.08 \) and (12) becomes

\[ 0.5e + 2(1 - 0.25e) = 2 < 2.08 \approx \ln \left( \frac{1 - 0.25e}{0.04} \right), \]

i.e., (12) is satisfied and there exists an eventually positive solution of (14). Lemma 1 fails for (14), since \( 0.5 + 0.08 > 1/e \). Simple computations demonstrate that (8), (9) and (13) also fail for (14).

On the other hand, for the equation

\[ \dot{x}(t) + \frac{0.2}{\pi} \sin^2 tx(t - \pi) + \frac{0.25}{\pi} \cos^2 tx(t - 2\pi) = 0, \]

with \( a = 0.1, b = 0.125, c = 0.2, d = 0.25 \), inequality (9) is satisfied. This implies the existence of an eventually positive solution for (15), while Lemma 1, (8), (12) and (13) fail.

Now consider Eq. (1) with several delays.
Theorem 3. Suppose there exists $k \in \{1, 2, \ldots, m\}$ such that

$$b_i := \sum_{j \neq k} A_{ij} < \frac{1}{e}, \quad i = 1, 2, \ldots, m,$$

where $A_{ij}$ are defined in (3), and there exists $z > 0$ satisfying the following inequalities:

$$z < \min_{i \neq k} \frac{1 - b_i e}{A_{ik}}, \quad \sum_{j \neq k} A_{kj} e + A_{kk} z < \ln z. \tag{17}$$

Then Eq. (1) has an eventually positive solution.

Proof. Suppose such $k$ exists. Let $x_i = e, i \neq k$; $x_i = z, i = k$. Then the first inequality in (17) implies all inequalities in (4) but the $k$-th one, which is a corollary of the latter inequality in (17). Thus (4) has a positive solution, so Eq. (1) has an eventually positive solution, which completes the proof. \hfill \Box

Corollary 3. Suppose there exists $k \in \{1, 2, \ldots, m\}$ such that

$$e \sum_{j \neq k} A_{kj} + A_{kk} B < \ln B, \tag{18}$$

where $B = \min_{i \neq k} \frac{1 - b_i e}{A_{ik}}$. Then Eq. (1) has an eventually positive solution.

Proof. Due to the continuity of the function $\ln x - A_{kk}x$, there exists $\varepsilon > 0$ such that if we substitute $x = B - \varepsilon$ instead of $B$, the inequality (18) is still valid, i.e., the latter inequality in (17) is satisfied. Then $z < \frac{1 - b_i e}{A_{ik}}$ for any $i \neq k$, where the $b_i$ are defined in (16), so the first inequality in (17) is also satisfied. By Theorem 3 Eq. (1) has an eventually positive solution. \hfill \Box

Using the comparison theorem [5], Theorem 3, we can also apply Theorem 1 to general equations with several delays.

Theorem 4. Let $I_1 \subset I = \{1, \ldots, m\}$, $I_2 = I \setminus I_1$. Define $a(t) = \sum_{k \in I_1} a_k(t)$, $b(t) = \sum_{k \in I_2} a_k(t)$, $h(t) = \min_{k \in I_1} h_k(t)$, $g(t) = \min_{k \in I_2} h_k(t)$ where $h(t) \equiv t$ or $g(t) \equiv t$, if $I_1 = \emptyset$ or $I_2 = \emptyset$, respectively. If the hypotheses of Theorem 1 are satisfied, where $a, b, c, d$ are defined in (6), then Eq. (1) has an eventually positive solution.

Proof. Nonoscillation of Eq. (5) implies [3] nonoscillation of Eq. (1). \hfill \Box

Remark 3. Theorem 4 contains $2^m$ different nonoscillating conditions. In particular, if $I_1 = I$, $I_2 = \emptyset$, then Theorem 4 implies Lemma 2. Indeed, in this case we have $a(t) = \sum_{k=1}^m a_k(t)$, $b(t) \equiv 0$, $h(t) = \min_{k \in I} h_k(t)$, $g(t) \equiv t$. Then $a = \lim_{t \to -\infty} \int_{h(t)}^t \sum_{k=1}^m a_k(s) ds$, $b = c = d = 0$. If we take $x_1 = e, x_2 > 1$ then inequalities (7) have the form $a < \frac{1}{e}, \ln x_2 > 0$, which is equivalent to (2).

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References


