Oscillation of a Second-Order Delay Differential Equation with Middle Term

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Abstract—For the second-order delay differential equation with the middle term

$$\ddot{x}(t) + b(t)\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad g_k(t) \leq t,$$

oscillation conditions are presented. The results are obtained by the comparison of the delay equation with the corresponding ordinary differential equations containing the middle term. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

This paper deals with a scalar linear delay differential equation of the second order. Such equations attract attention of many mathematicians due to their significance in applications. We mention here the monographs of Myshkis [1], Norkin [2], Ladde, Lakshmikantham and Zhang [3], Győri and Ladas [4], Erbe, Kong and Zhang [5] and references therein.

Most of the literature deals with equations not containing the term with the first derivative (the middle term). If the first derivative is included explicitly, such equations are much less studied. The following particular cases were considered: the middle term is not delayed (see, for example, papers [6–8]) and the delay is constant [9,10].

In this paper, we consider the oscillation problem for equations containing the middle term without deviating argument.

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In [7], oscillation criteria were obtained by comparison of delay differential equations with and without the middle term. We will obtain oscillation criteria by the comparison of delay differential equation with a middle term and ordinary differential equation with the same middle term. This result generalizes the corresponding theorems which were obtained in [11–15] for equations without the middle term.

This paper continues the investigations in [16], where nonoscillation criteria for delay differential equations with middle term were obtained.

2. PRELIMINARIES

We consider the scalar delay differential equation of the second order

\[ \ddot{x}(t) + b(t)\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq 0, \]

under the following conditions:

(a1) \( b, a_k, k = 1, \ldots, m, \) are Lebesgue measurable and locally essentially bounded functions on \([0, \infty)\);

(a2) \( g_k : [0, \infty) \to \mathbb{R} \) are Lebesgue measurable functions, \( g_k(t) \leq t, t \geq 0, \lim_{t \to \infty} g_k(t) = \infty, \ k = 1, \ldots, m. \)

Together with (1) consider for each \( t_0 \geq 0 \) an initial value problem

\[ \ddot{x}(t) + b(t)\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = f(t), \quad t \geq t_0, \]

\[ x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = x_0'. \]

We also assume that the following hypothesis holds:

(a3) \( f : [t_0, \infty) \to \mathbb{R} \) is a Lebesgue measurable locally essentially bounded function, \( \varphi : (-\infty, t_0) \to \mathbb{R} \) is a Borel measurable bounded function.

DEFINITION. A function \( x : \mathbb{R} \to \mathbb{R} \) with locally absolutely continuous derivative \( \dot{x} \) is called a solution of problem (2),(3) if it satisfies equation (2) for almost every \( t \in [t_0, \infty), \) and equalities (3) for \( t \leq t_0. \)

DEFINITION. For each \( s \geq 0 \) the solution \( X(t, s) \) of the problem

\[ \ddot{x}(t) + b(t)\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq s, \]

\[ x(t) = 0, \quad t < s, \quad x(s) = 0, \quad \dot{x}(s) = 1 \]

is called a fundamental function of equation (1).

We assume \( X(t, s) = 0, \ 0 \leq t < s. \)

Let functions \( x_1 \) and \( x_2 \) be solutions of the following problems:

\[ \ddot{x}(t) + b(t)\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq t_0, \quad x(t) = 0, \quad t < t_0, \]

with initial values \( x(t_0) = 1, \ \dot{x}(t_0) = 0 \) for \( x_1 \) and \( x(t_0) = 0, \ \dot{x}(t_0) = 1 \) for \( x_2, \) respectively.

By definition \( x_2(t) = X(t, t_0). \)

LEMMA 1. (See [16].) Let (a1)–(a3) hold. Then there exists one and only one solution of problem (2),(3) that can be presented in the form

\[ x(t) = x_1(t)x_0 + X(t, t_0)x_0' + \int_{t_0}^{t} X(t, s)f(s) \, ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)a_k(s)\varphi(g_k(s)) \, ds, \]

where \( \varphi(g_k(s)) = 0, \) if \( g_k(s) > t_0. \)
3. EXPLICIT OSCILLATION CONDITIONS

**Definition.** We will say that equation (1) has a positive solution for \( t > t_0 \) if there exist initial function \( \varphi \) and numbers \( x_0 \) and \( x_0' \) such that the solution of initial value problem (2),(3) \( f \equiv 0 \) is positive for \( t > t_0 \).

Consider together with equation (1) the following second-order delay differential inequality:

\[
\dot{y}(t) + b(t)y(t) + \sum_{k=1}^{m} a_k(t)y(g_k(t)) \leq 0, \quad t \geq 0.
\]

The following theorem establishes nonoscillation criteria.

**Theorem 1.** (See [16].) Suppose \( a_k(t) \geq 0, \; k = 1, \ldots, m, \)

\[
\int_0^\infty \exp \left( - \int_0^s b(r) \, dr \right) \, ds = \infty.
\]

Then the following statements are equivalent:

1. there exists \( t_0 \geq 0 \) such that inequality (6) has a positive solution for \( t > t_0 \),
2. there exists \( t_1 \geq 0 \) such that the inequality

\[
\dot{u}(t) + u^2(t) + b(t)u(t) + \sum_{k=1}^{m'} a_k(t) \exp \left\{ - \int_{g_k(t)}^t u(s) \, ds \right\} \leq 0
\]

has a nonnegative absolutely continuous in each interval \([t_1, b]\) solution, where the sum \( \sum' \) contains only such terms for which \( g_k(t) \geq t_1 \),
3. there exists \( t_2 \geq 0 \) such that \( X(t, s) > 0, \; t > s > t_2 \),
4. there exists \( t_3 \geq 0 \) such that equation (1) has a positive solution for \( t > t_3 \).

Now we turn to the problem of oscillation.

First, consider equation (1) with bounded delays.

**Theorem 2.** Suppose \( a_k(t) \geq 0, \) there exists \( \delta > 0 \) such that \( t - g_k(t) \leq \delta \) and condition (7) holds. Then equation (1) is oscillatory if and only if the ordinary differential equation

\[
\ddot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} a_k(t)x(t) = 0
\]

is oscillatory.

**Proof.** By inequalities \( t - \delta \leq g_k(t) \leq t \) and Theorem 3 [16], we only need to prove that the oscillation of (9) yields that the equation

\[
\ddot{z}(t) + b(t)z(t) + \sum_{k=1}^{m} a_k(t)x(t - \delta) = 0
\]

is oscillatory.

Suppose (10) is nonoscillatory. Then by Theorem 1 there exists \( t_0 \geq 0 \) such that \( Y(t, s) > 0, \; t > s \geq t_0 \), where \( Y(t, s) \) is the fundamental function of (10). Hence, \( y(t) = Y(t, s) \) is a nonnegative solution of the following problem:

\[
\dot{y}(t) + b(t)y(t) + \sum_{k=1}^{m} a_k(t)y(t - \delta) = 0, \quad t \geq t_0,
\]

\[
y(t) = 0, \quad t \leq t_0, \quad \dot{y}(t_0) = 1.
\]
Rewrite (11) in the following form:

\[ \dot{y}(t) + b(t)\dot{y}(t) + \sum_{k=1}^{m} a_k(t)y(t) + \sum_{k=1}^{m} a_k(t)[y(t-\delta) - y(t)] = 0. \]  \hspace{1cm} (12)

Inequalities \( y(t) > 0 \) and (7) imply [7] \( \dot{y}(t) \geq 0 \), then \( \dot{y}(t) \leq 0 \), hence, \( \dot{y} \) is nonincreasing. Then \( \dot{y}(t-\delta) \geq \dot{y}(t) \). By integrating this inequality from \( t_0 + \delta \) to \( t \), we obtain \( y(t-\delta) - y(t_0) \geq y(t) - y(t_0 + \delta) \), where \( y(t_0) = 0 \). Then \( y(t) - y(t-\delta) \leq y(t_0 + \delta) \), therefore, (12) implies the inequality

\[ \dot{y}(t) + b(t)\dot{y}(t) + \sum_{k=1}^{m} a_k(t)[y(t) - y(t_0 + \delta)] \leq 0. \]

Hence, the function \( z(t) = y(t) - y(t_0 + \delta) \) is a positive solution (for \( t > t_0 + \delta \)) of the inequality

\[ \dot{z}(t) + b(t)\dot{z}(t) + \sum_{k=1}^{m} a_k(t)z(t) \leq 0. \]

Theorem 1 yields that equation (9) is nonoscillatory. The contradiction obtained proves the theorem.

In a similar way, the following result can be proved.

THEOREM 3. Suppose \( a_k(t) \geq 0 \), there exist \( c_k, 0 < c_k < 1, k = 1, 2, \ldots, m \), such that \( g_k(t) \geq c_k t \), condition (7) holds and the ordinary differential equation

\[ \ddot{x}(t) + b(t)\dot{x}(t) + \sum_{k=1}^{m} c_k a_k(t)x(t) = 0 \]

is oscillatory.

Then (1) is also oscillatory.

REMARKS.

(1) Theorems 2 and 3 generalize some results from [11–14] obtained there for equation (1) without the middle term. Our proof is also different from those in [11–14].

(2) Explicit conditions of oscillation obtained in Theorems 2 and 3 are different from those in [7]. In particular, by Theorem 2 the equation

\[ \ddot{x}(t) + \frac{1}{t}\dot{x}(t) + \frac{\alpha}{t^2}x(t-\delta) = 0 \]  \hspace{1cm} (13)

is oscillatory for \( \alpha > 0 \). Using [7] one can obtain only the condition \( \alpha > 1/4 \) for oscillation of (13).

For ordinary linear differential equations of the second order the following oscillation criterion is well known. If an equation has an oscillatory solution, then all its solutions are oscillatory. As it is known, for delay differential equations this statement is not true.

We will show that if equation (1) has a slowly oscillating solution and condition (7) holds, then all solutions of this equation are oscillating. A similar result for the delay differential equation without the middle term was obtained in [15].

DEFINITION. A solution \( x \) of (1) is said to be slowly oscillating if for every \( t_0 \geq 0 \) there exist \( t_2 > t_1 > t_0 \), such that

\[ g_k(t) \geq t_1, \text{ for } t \geq t_2, \quad x(t_1) = x(t_2) = 0, \quad x(t) > 0, \quad t \in (t_1, t_2), \]

and at the point \( t_2 \) the function \( x(t) \) changes its sign.

THEOREM 4. Suppose \( a_k(t) \geq 0 \) and condition (7) holds. If there exists a slowly oscillating solution of equation (1), then all the solutions of this equation are oscillatory.

PROOF. This proof completely coincides with the proof of Theorem 7 in [15].

COROLLARY. Suppose \( a_k(t) \geq 0 \), condition (7) holds and equation (1) has a positive solution for \( t > t_0 \geq 0 \). Then (1) has no slowly oscillating solutions.
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