Some Oscillation Problems for a Second Order Linear Delay Differential Equation

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For the delay differential equation

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad g_k(t) \leq t, \]

a connection between the following properties is established: non-oscillation of the differential equation and the corresponding differential inequality, positiveness of the fundamental function, and the existence of a nonnegative solution of a generalized Riccati inequality. Explicit conditions for non-oscillation and oscillation and comparison theorems are presented.

1. INTRODUCTION

This paper deals with a scalar linear delay differential equation of second order. Such equations attract the attention of many mathematicians due to their significance in applications. We mention here the

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We study properties of these equations concerning non-oscillation. The main result is that under some natural assumptions for a delay differential equation the following four assertions are equivalent: non-oscillation of solutions of this equation and the corresponding differential inequality, positiveness of the fundamental function, and the existence of a nonnegative solution of a generalized Riccati inequality.

The equivalence of oscillation properties of the differential equation and the corresponding differential inequality can be applied to obtain new explicit conditions for non-oscillation and oscillation and also to prove some well-known results in a different way.

We employ a generalized Riccati inequality to compare oscillation properties of two equations without comparing their solutions. These results can be regarded as a natural generalization of the well-known Sturm comparison theorem for a second order ordinary differential equation.

By applying the positiveness of the fundamental function we compare positive solutions of two non-oscillation equations. There are a lot of results of this kind for delay differential equations of first order and only a few for second order equations. A. D. Myshkis [1] obtained one of the first comparison theorems for second order equations. The result presented here is more general and is proved in a different way.

The paper also contains conditions on the initial function and initial values which imply that the corresponding solution is positive. Such conditions are well known for first order delay differential equations; however, we have not come across similar results for second order equations.

Some of the results presented here generalize the known non-oscillation tests to the case of discontinuous parameters (we only need the minimal requirement of measurability). A new technique based on a generalized Riccati inequality made it possible to deal with discontinuous parameters and to obtain explicit oscillation criteria.

The paper is organized as follows. Section 2 contains the relevant definitions and notations. In Section 3 the equivalence of the four above-mentioned properties is established. In Section 4 we present the comparison results. The next section includes some explicit conditions for non-oscillation and oscillation. In the last section we present a sufficient condition for a solution to be positive.

In this paper we follow the method employed in [6] for first order delay differential equations.
2. PRELIMINARIES

We consider a scalar delay differential equation of the second order

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = 0, \quad t \geq 0, \]

(1)

under the following conditions:

(a1) \( a_k, k = 1, \ldots, m, \) are Lebesgue measurable and locally essentially bounded functions on \([0, \infty)\);

(a2) \( g_k: [0, \infty) \to R \) are Lebesgue measurable functions, \( g_k(t) \leq t, \)

\( t \geq 0, \lim_{t \to \infty} g_k(t) = \infty, k = 1, \ldots, m. \)

Together with (1) consider for each \( t_0 \geq 0 \) an initial value problem

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = f(t), \quad t \geq t_0, \]

(2)

\[ x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = x'_0. \]

(3)

We also assume that the following hypothesis holds

(a3) \( f: [t_0, \infty) \to R \) is a Lebesgue measurable locally essentially bounded function, \( \varphi: (-\infty, t_0) \to R \) is a Borel measurable bounded function.

DEFINITION. A function \( x: R \to R \) with locally absolutely continuous on \([t_0, \infty)\) derivative \( \dot{x} \) is called a solution of problem (2), (3) if it satisfies Eq. (2) for almost every \( t \in [t_0, \infty), \) and equalities (3) for \( t \leq t_0. \)

DEFINITION. For each \( s \geq 0, \) the solution \( X(t, s) \) of the problem

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = 0, \quad t \geq s, \]

(4)

\[ x(t) = 0, \quad t \leq s, \quad \dot{x}(s) = 1 \]

is called a fundamental function of Eq. (1).

We assume \( X(t, s) = 0, \) \( 0 \leq t < s. \)

Let functions \( x_1 \) and \( x_2 \) be the solutions of the problems

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = 0, \quad t \geq t_0, \]

\[ x(t) = 0, \quad t < t_0, \]

with initial values \( x(t_0) = 1, \) \( \dot{x}(t_0) = 0 \) for \( x_1 \) and \( x(t_0) = 0, \) \( \dot{x}(t_0) = 1 \) for \( x_2, \) respectively.

By definition \( x_2(t) = X(t, t_0). \)
Lemma 1 [7]. Let (a1)–(a3) hold. Then there exists one and only one solution of problem (2), (3) that can be presented in the form
\begin{equation}
x(t) = x_1(t)x_0 + X(t,t_0)x'_0 + \int_{t_0}^{t}X(t,s)f(s)\,ds - \sum_{k=1}^{m}\int_{t_0}^{t}X(t,s)a_k(s)\varphi(g_k(s))\,ds,
\end{equation}
where \(\varphi(g_k(s)) = 0\) if \(g_k(s) > t_0\).

3. Non-Oscillation Criteria

Definition. We will say that Eq. (1) has a positive solution for \(t > t_0\) if there exist an initial function \(\varphi\) and numbers \(x_0, x'_0\) such that the solution of the initial value problem (2), (3) \((f \equiv 0)\) is positive for \(t > t_0\).

Together with Eq. (1) consider the following second order delay differential inequality
\begin{equation}
\tilde{y}(t) + \sum_{k=1}^{m}a_k(t)y(g_k(t)) \leq 0, \quad t \geq 0.
\end{equation}

The following theorem establishes non-oscillation criteria.

Theorem 1. Suppose \(a_k(t) \geq 0\) for \(k \geq 0, k = 1, \ldots, m\). Then the following statements are equivalent:

1. There exists \(t_0 \geq 0\) such that inequality (6) has a positive solution for \(t > t_0\).
2. There exists \(t_1 \geq 0\) such that the inequality
\begin{equation}
\tilde{u}(t) + u^2(t) + \sum_{k=1}^{m}a_k(t)\exp\left(-\int_{g_k(t)}^{t}u(s)\,ds\right) \leq 0
\end{equation}
has a nonnegative locally absolutely continuous for \(t \geq t_1\) solution, where the sum \(\sum'\) contains only those terms for which \(g_k(t) \geq t_1\).
3. There exists \(t_2 \geq 0\) such that \(X(t,s) > 0, t > s \geq t_2\).
4. There exists \(t_3 \geq 0\) such that Eq. (1) has a positive solution for \(t > t_3\).

Scheme of the Proof. \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)\).

(1) \(\Rightarrow\) (2). Let \(y(t)\) be a positive solution of inequality (6) for \(t > t_0\). Then there exists a point \(t_1\) such that \(g_k(t) \geq t_0\) if \(t \geq t_1\). We can assume without loss of generality that \(y(t_1) = 1\). Since \(y(t) > 0\) and \(\tilde{y}(t) \leq 0\), \(t \geq t_1\), then \(\tilde{y}(t) \geq 0\) for \(t \geq t_1\). Denote \(u(t) = \tilde{y}(t)/y(t)\), if \(t \geq t_1\) and \(u(t) = 0\), if \(t < t_1\). Then \(u\) is a nonnegative locally absolutely continuous function. Equalities \(\tilde{y}(t) – \begin{equation}
\tilde{y}(t) = \tilde{y}(t) + \sum_{k=1}^{m}a_k(t)y(g_k(t)) 
\end{equation}
for \(t > t_0\).
We substitute (8) into (6) and obtain after carrying the exponent out of the brackets an inequality
\[
\exp \left( \int_{t_1}^{t} u(s) \, ds \right) \left[ \dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp \left( - \int_{g_k(t)}^{t} u(s) \, ds \right) \right] \\
+ \sum_{k=1}^{m} a_k(t) y(g_k(t)) \leq 0,
\]
(9)
where the sum \( \Sigma^n \) contains only such terms for which \( t_0 \leq g_k(t) < t_1 \). Since \( y(t) \geq 0 \) for \( t \geq t_0 \) and \( a_k(t) \geq 0 \) then (9) implies inequality (7).

(2) \implies (3). Consider the initial value problem
\[
\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = f(t), \quad t \geq t_1, \\
x(t) = 0, \quad t < t_1, \quad x(t_1) = \dot{x}(t_1) = 0.
\]
(10)
Denote
\[
z(t) = \dot{x}(t) - u(t) x(t),
\]
(11)
where \( x \) is the solution of (10) and \( u \) is a nonnegative solution of (7). From (11) we obtain
\[
x(t) = \int_{t_1}^{t} \exp \left( \int_{s}^{t} u(\tau) \, d\tau \right) z(s) \, ds,
\]
\[
\dot{x}(t) = z(t) + u(t) \int_{t_1}^{t} \exp \left( \int_{s}^{t} u(\tau) \, d\tau \right) z(s) \, ds,
\]
\[
\ddot{x}(t) = \dot{z}(t) + \ddot{u}(t) \int_{t_1}^{t} \exp \left( \int_{s}^{t} u(\tau) \, d\tau \right) z(s) \, ds \\
+ u(t) \left[ z(t) + u(t) \int_{t_1}^{t} \exp \left( \int_{s}^{t} u(\tau) \, d\tau \right) z(s) \, ds \right] \\
= \ddot{z}(t) + u(t) z(t) \\
+ (\ddot{u}(t) + u^2(t)) \int_{t_1}^{t} \exp \left( \int_{s}^{t} u(\tau) \, d\tau \right) z(s) \, ds \quad \text{for } t \geq t_1.
\]
(12)
Substituting \( x, \ddot{x} \) into (10) we obtain

\[
\dot{z}(t) + u(t)z(t) + (\ddot{u}(t) + u^2(t))\int_{t_1}^t \exp \left( \int_s^t u(\tau) \, d\tau \right) z(s) \, ds \\
+ \sum_{k=1}^{m'} a_k(t) \int_{t_1}^t \exp \left( \int_s^{g_k(t)} u(\tau) \, d\tau \right) z(s) \, ds = f(t).
\] (13)

Equalities (10) and (11) imply \( z(t_1) = 0 \). Hence we can rewrite Eq. (13) in the form

\[
\dot{z}(t) + u(t)z(t) \\
= - \left( \ddot{u}(t) + u^2(t) + \sum_{k=1}^{m'} a_k(t) \exp \left( - \int_{g_k(t)}^t u(s) \, ds \right) \right) \\
\times \int_{t_1}^t \exp \left( \int_s^t u(\tau) \, d\tau \right) z(s) \, ds \\
+ \sum_{k=1}^{m'} a_k(t) \int_{t}^{g_k(t)} \exp \left( \int_s^{g_k(t)} u(\tau) \, d\tau \right) z(s) \, ds + f(t),
\]

\[z(t_1) = 0. \quad (14)\]

Then Eq. (14) is equivalent to the equation

\[z = H z + p, \quad (15)\]

where

\[(Hz)(t) = \int_{t_1}^t \exp \left( - \int_s^t u(\tau) \, d\tau \right) \\
\times \left[ - \left( u(s) + u^2(s) + \sum_{k=1}^{m'} a_k(s) \exp \left( - \int_{g_k(s)}^s u(\tau) \, d\tau \right) \right) \\
\times \int_{g_k(s)}^s \exp \left( \int_{g_k(s)}^\xi u(\xi) \, d\xi \right) z(\xi) \, d\xi \right] \, d\tau \\
+ \sum_{k=1}^{m'} a_k(s) \int_{g_k(s)}^s \exp \left( \int_{g_k(s)}^\xi u(\xi) \, d\xi \right) z(\xi) \, d\xi \right] \, ds, \quad (16)\]

\[p(t) = \int_{t_1}^t \exp \left( - \int_s^t u(\tau) \, d\tau \right) f(s) \, ds. \quad (17)\]
Inequality (7) yields that if $z(t) \geq 0$ for $t \geq t_1$ then $(Hz)(t) \geq 0$ for $t \geq t_1$ (i.e., operator $H$ is positive).

Denote

$$c(t) = \dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp \left[ - \int_{g_k(t)}^{t} u(s) \, ds \right].$$

Since $u$ is absolutely continuous in each finite interval, $c \in L_{[t_1, b]}$ for every $b > t_1$, where $L_{[a, b]}$ is the space of all Lebesgue integrable functions on $[a, b]$ with the usual integral norm.

We have for $t \in [t_1, b]$

$$| (Hz)(t) | \leq \exp \left[ \int_{t_1}^{b} u(\tau) \, d\tau \right] \int_{t_1}^{t} \left| c(s) \right| + \sum_{k=1}^{m} \left| a_k(s) \right| \int_{t_1}^{s} \left| z(\tau) \right| \, d\tau \, ds$$

$$= \exp \left[ \int_{t_1}^{b} u(\tau) \, d\tau \right] \int_{t_1}^{t} \left| c(s) \right| + \sum_{k=1}^{m} \left| a_k(s) \right| \int_{s}^{t} \left| z(\tau) \right| \, d\tau \, ds.$$

The kernel of the Volterra integral operator $H$ is bounded in each square $[t_1, b] \times [t_1, b]$, hence [8] $H: L_{[t_1, b]} \to L_{[t_1, b]}$ is a compact operator. Therefore [8] the spectral radius of this operator $\rho(H) = 0$.

Thus if in (15), $p(t) \geq 0$ for $t \geq t_1$ then

$$z(t) = p(t) + (Hp)(t) + (H^2p)(t) + \cdots \geq 0 \quad \text{for } t \geq t_1.$$

If $f(t) \geq 0$ for $t \geq t_1$ then by (17), $p(t) \geq 0$ for $t \geq t_1$. Hence for Eq. (13) we have the following: if $f(t) \geq 0$ for $t \geq t_1$ then $z(t) \geq 0$ for $t \geq t_1$.

Therefore (12) implies that the solution of (10) is nonnegative for any nonnegative right-hand side.

The solution of this equation can be written in the form (5),

$$x(t) = \int_{t_1}^{t} X(t, s) f(s) \, ds. \quad (18)$$

As it was shown, $f(t) \geq 0, t \geq t_1$, implies $x(t) \geq 0, t \geq t_1$. Consequently, the kernel of the integral operator (18) is nonnegative. Therefore $X(t, s) \geq 0$ for $t \geq s \geq t_1$. A function $x(t) = X(t, s)$ is a nonnegative solution of (4) for $t \geq s$. Hence $x(t) \geq 0, \dot{x}(t) \leq 0, t \geq s$, and then $\dot{x}(t) \geq 0, t \geq s$.

Since $\dot{x}(s) = 1$ implies $\dot{x}(t) > 0$ on some interval $[s, s + \sigma]$, the strict inequality $x(t) = X(t, s) > 0, t > s \geq t_1$ holds.

(3) $\Rightarrow$ (4). The function $x(t) = X(t, t_1)$ is a positive solution of Eq. (1).

Implication (4) $\Rightarrow$ (1) is evident.
Corollary. Equation (1) is non-oscillatory if and only if inequality (6) is non-oscillatory.

Remark. (1) If there exists a nonnegative solution of inequality (7) for $t \geq t_0$ then statements (1), (3), (4) of the theorem are also valid for $t \geq t_0$. 
(2) The connection between assertions (1) and (3) of Theorem 1 was first studied in [7]. A generalized Riccati equation arose for the first time in [9].

4. COMPARISON THEOREMS

Theorem 1 can be employed for comparison of oscillation properties. To this end, together with Eq. (1) consider the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = 0, \quad t \geq 0. \quad (19)$$

Suppose (a1) and (a2) hold for Eq. (19) and denote by $Y(t, s)$ a fundamental function of this equation.

Theorem 2. Suppose $a_k(t) \geq 0, a_k(t) \geq b_k(t)$ for $t \geq t_0$ and inequality (7) has a nonnegative solution for $t \geq t_0$. Then Eq. (19) has a positive solution for $t > t_0$ and $Y(t, s) > 0, t > s \geq t_0$.

Proof. Consider the problem

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = f(t), \quad t \geq t_0,$$

$$x(t) = 0, \quad t < t_0, \quad x(t_0) = \dot{x}(t_0) = 0. \quad (20)$$

We will show that if $f(t) \geq 0$ for $t \geq t_0$ then the solution of (20) is positive.

To this end rewrite (20) in the form

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t))$$

$$+ \sum_{k=1}^{m} [b_k(t) - a_k(t)]x(g_k(t)) = f(t), \quad t \geq t_0,$$

$$x(t) = 0, \quad t < t_0, \quad x(t_0) = \dot{x}(t_0) = 0.$$
Substitute into the equation
\[ x(t) = \int_{t_0}^{t} X(t, s) z(s) \, ds, \]
where \( X(t, s) \) is the fundamental function of (1). Then (20) is equivalent to the equation
\[ z - Tz = f, \quad (21) \]
where
\[ (Tz)(t) = \int_{t_0}^{t} X(g_k(t), s) \sum_{k=1}^{m} [a_k(t) - b_k(t)] z(s) \, ds, \quad t \geq t_0. \]
Since \( X(t, s) \) is a solution of (4), then for every \( b \geq t_0 \) [1]
\[ |X(t, s)| \leq \exp \left( \sum_{k=1}^{m} \int_{t_0}^{b} \max(|a_k(\tau)|, 1) \, d\tau \right), \quad t_0 \leq s \leq t \leq b. \]
Hence the Volterra integral operator \( T \) is a compact operator acting on the space of integrable functions \( L_{[t_0,b]} \) for every \( b > t_0 \). The spectral radius of this operator \( r(T) = 0 < 1 \). By Theorem 1, \( X(t, s) > 0, t > s \geq t_0 \), hence operator \( T \) is positive. Therefore for the solution of (21) we have
\[ z(t) = f(t) + (Tf)(t) + (T^2f)(t) + \cdots \geq 0, \quad \text{if } f(t) \geq 0 \text{ for } t \geq t_0. \]
Then as in the proof of Theorem 1 we conclude that \( Y(t, s) \geq 0, t > s \geq t_0 \) and we only need to prove that the strict inequality \( Y(t, s) > 0, t > s \geq t_0 \) holds.
Denote \( y(t) = Y(t, s), t \geq s \). Then \( y(t) \) is the solution of the problem
\[ \ddot{y}(t) + \sum_{k=1}^{m} b_k(t) y(g_k(t)) = 0, \quad t \geq s, \quad (22) \]
\[ y(t) = 0, \quad t \leq s, \quad y(s) = 1. \quad (23) \]
Rewrite (22) in the form
\[ \ddot{y}(t) + \sum_{k=1}^{m} a_k(t) y(g_k(t)) = \sum_{k=1}^{m} [a_k(t) - b_k(t)] y(g_k(t)). \]
Then for the solution of problem (22), (23) equality (5) implies
\[ y(t) = Y(t, s) = X(t, s) + \int_{s}^{t} X(t, \tau) \sum_{k=1}^{m} [a_k(\tau) - b_k(\tau)] Y(g_k(\tau), s) \, d\tau. \]
Then \( Y(t, s) \geq X(t, s) > 0 \), which completes the proof.
**Corollary 1.** Suppose \( a_k(t) \geq 0, a_k(t) \geq b_k(t) \) for \( t \geq t_0 \) and Eq. (1) has a positive solution for \( t > t_0 \). Then there exists \( t_1 \geq t_0 \) such that Eq. (19) has a positive solution for \( t > t_1 \).

Denote \( a^+ = \max(a, 0) \).

**Corollary 2.** (1) If the inequality
\[
\ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t)x(g_k(t)) \leq 0
\]
has a positive solution for \( t > t_0 \) then there exists \( t_1 \geq t_0 \) such that (1) has a positive solution for \( t > t_1 \).

(2) If the inequality
\[
\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k^+(t)\exp\left(-\int_{g_k(t)}^{t} u(s) \, ds\right) \leq 0, \quad t \geq t_0
\]
has a nonnegative absolutely continuous solution, where the sum contains only those terms for which \( g_k(t) \geq t_0 \), then Eq. (1) has a positive solution for \( t > t_0 \) and \( X(t, s) > 0, t > s \geq t_0 \).

**Proof.** Consider the equation
\[
\ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t)x(g_k(t)) = 0.
\]

Theorem 1 implies that for this equation all assertions of this theorem hold. Since \( a_k(t) \leq a_k^-(t) \), Theorem 2 implies the corollary.

Corollary 2 can be employed to obtain a comparison result which improves the statement of Theorem 2.

Consider the equation
\[
\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0, \quad t \geq 0.
\]

Suppose (a1), (a2) hold for (26) and denote by \( Y(t, s) \) the fundamental function of this equation.

**Theorem 3.** Suppose \( a_k(t) \geq 0 \) for \( t \geq 0 \) and there exists \( t_0 \geq 0 \) such that for Eq. (1) any of assertions (1)–(4) of Theorem 1 holds. If
\[
b_k(t) \leq a_k(t), \quad h_k(t) \leq g_k(t) \quad \text{for} \quad t \geq t_0,
\]
then there exists \( t_1 \geq t_0 \) such that Eq. (26) has a positive solution for \( t > t_1 \) and \( Y(t, s) > 0, t > s \geq t_1 \).
Proof. Theorem 1 implies that for some \( t_1 \geq t_0 \) there exists a nonnegative solution \( u \) of the inequality (7) for \( t \geq t_1 \). Inequalities (27) yield that this function is also a solution of the inequality

\[
\dot{u}(t) + u^2(t) + \sum_{k=1}^{m_0} b_k^+(t) \exp \left( - \int_{h_k(t)}^t u(s) \, ds \right) \leq 0, \quad t \geq t_0,
\]

where the sum contains only those terms for which \( h_k(t) \geq t_0 \). By Corollary 2 of Theorem 2, Eq. (26) has a positive solution for \( t > t_1 \) and the fundamental function of this equation is positive, which completes the proof.

Now let us compare the solutions of problem (2), (3) and the following one

\[
\ddot{y}(t) + \sum_{k=1}^{m} b_k(t) y(g_k(t)) = r(t), \quad t \geq t_0, \quad (28)
\]

\[
y(t) = \psi(t), \quad t < t_0, \quad y(t_0) = y_0, \quad \dot{y}(t_0) = y'_0. \quad (29)
\]

Denote by \( x(t) \) and \( y(t) \) the solution of (2), (3) and (28), (29), respectively, and let \( Y(t, s) \) be the fundamental function of Eq. (28).

**Theorem 4.** Suppose there exists a nonnegative solution of (7) for \( t \geq t_0, x(t) > 0 \) for \( t \geq t_0, \) and

\[
a_k(t) \geq b_k(t) \geq 0, \quad r(t) \geq f(t) \text{ for } t \geq t_0, \quad \psi(t) \geq \dot{\psi}(t) \text{ for } t < t_0, \quad y_0 = x_0, \quad y'_0 \geq x'_0.
\]

Then \( y(t) \geq x(t) \) for \( t \geq t_0. \)

Proof. Denote by \( u \) a nonnegative solution of inequality (7). The inequality \( a_k(t) \geq b_k(t) \), \( t \geq t_0 \), yields that the function \( u \) is also a solution of the inequality corresponding to (7) for Eq. (28). Hence by Theorem 1, \( Y(t, s) > 0, t > s \geq t_0. \)

Rewrite (2) in the form

\[
\ddot{x}(t) + \sum_{k=1}^{m} b_k(t) x(g_k(t)) = - \sum_{k=1}^{m} \left[ a_k(t) - b_k(t) \right] x(g_k(t)) + f(t).
\]
Hence (see (5)) for the solutions of (2), (3) and (28), (29) we have

\[ x(t) = y_1(t)x_0 + Y(t, t_0)x'_0 - \sum_{k=1}^{\infty} \int_{t_0}^{t} Y(t, s)[a_k(s) - b_k(s)]x(g_k(s)) \, ds \]

\[ - \sum_{k=1}^{\infty} \int_{t_0}^{t} Y(t, s)b_k(s)\varphi(g_k(s)) \, ds + \int_{t_0}^{t} Y(t, s)f(s) \, ds, \]

\[ y(t) = y_2(t)y_0 + Y(t, t_0)y'_0 - \sum_{k=1}^{\infty} \int_{t_0}^{t} Y(t, s)b_k(s)\psi(g_k(s)) \, ds \]

\[ + \int_{t_0}^{t} Y(t, s)r(s) \, ds, \]

where \( y_1 \) is the solution of (28), (29) with \( r = 0, \psi = 0, y_0 = 1, y'_0 = 0, \)

and \( \varphi(g_k(s)) = \psi(g_k(s)) = 0, \) if \( g_k(s) > t_0, x(g_k(s)) = 0, \) if \( g_k(t) < t_0. \)

Therefore \( y(t) \geq x(t) > 0, t \geq t_0, \) which completes the proof.

In the next statement the equality \( x_0 = y_0 \) is replaced by \( x_0 \geq y_0. \)

**Theorem 5.** Suppose there exists a nonnegative solution of (7) for \( t \geq t_0, x(t) > 0 \) for \( t \geq t_0,

\[ a_k(t) \geq b_k(t) \geq 0, \quad r(t) \geq f(t) \text{ for } t \geq t_0, \varphi(t) \geq \psi(t) \text{ for } t < t_0. \]

If \( x_0 \geq y_0 > 0 \) and, for some \( \alpha > 0, y'_0 + \alpha y_0 \geq x'_0 + \alpha x_0, \) then

\[ y(t) - x(t) \geq e^{-\alpha(t-t_0)}(y_0 - x_0) \text{ for } t \geq t_0. \]

Especially, if \( x_0 \geq y_0 > 0 \) and \( y'_0 > x'_0, \) then

\[ \liminf_{t \to \infty}(y(t) - x(t)) \geq 0. \]

**Proof.** First assume that \( f = r = 0. \) As in the proof of Theorem 4 we obtain that \( Y(t, s) > 0 \) for \( t \geq s. \) The inequalities \( x(t) > 0, x'(t) \leq 0, t \geq t_0 \)

imply that \( x(t) \geq 0, t \geq t_0. \) Then \( x(t) \geq x_0 \) and hence \( x(t) \geq x_0 e^{-\alpha(t-t_0)} \)

for \( t \geq t_0. \)

Denote

\[ u(t) = \begin{cases} x(t) - x_0 e^{-\alpha(t-t_0)}, & t \geq t_0, \\ \varphi(t), & t < t_0, \end{cases} \]

\[ v(t) = \begin{cases} y(t) - y_0 e^{-\alpha(t-t_0)}, & t \geq t_0, \\ \psi(t), & t < t_0, \end{cases} \]

\[ w(t) = \begin{cases} x(t) - x_0 e^{-\alpha(t-t_0)}, & t \geq t_0, \\ \varphi(t), & t < t_0, \end{cases} \]

\[ v(t) = \begin{cases} y(t) - y_0 e^{-\alpha(t-t_0)}, & t \geq t_0, \\ \psi(t), & t < t_0, \end{cases} \]
Then $u$ and $v$ are the solutions of the problems

$$
\ddot{u}(t) + \sum_{k=1}^{m} a_k(t)u(g_k(t)) = -x_0 \left[ x_0^2 e^{-a(t-t_0)} + \sum_{k=1}^{m} a_k(t)e^{-a(g_k(t)-t_0)} \right],
$$

(30)

$$
u(t) = \varphi(t), \quad t < t_0, \quad u(t_0) = 0, \quad \ddot{u}(t_0) = x_0^2 + \alpha x_0, \quad (31)
$$

$$
\ddot{v}(t) + \sum_{k=1}^{m} b_k(t)v(g_k(t)) = -y_0 \left[ y_0^2 e^{-a(t-t_0)} + \sum_{k=1}^{m} b_k(t)e^{-a(g_k(t)-t_0)} \right],
$$

(32)

$$
v(t) = \psi(t), \quad t < t_0, \quad v(t_0) = 0, \quad \ddot{v}(t_0) = y_0^2 + \alpha y_0, \quad (33)
$$

The assumptions of the theorem and Theorem 4 imply $v(t) \geq \alpha u(t)$ for $t \geq t_0$, i.e., $y(t) - y_0 e^{-a(t-t_0)} \geq x(t) - x_0 e^{-a(t-t_0)}$ for $t \geq t_0$, which can be rewritten in the form $y(t) - x(t) \geq e^{-a(t-t_0)}(y_0 - x_0)$ for $t \geq t_0$.

If $y_0 > x_0$ then, for a sufficiently small $\alpha > 0$, $y_0 + \alpha y_0 > x_0 + \alpha x_0$.

Therefore $y(t) - x(t) \geq e^{-a(t-t_0)}(y_0 - x_0)$ for $t \geq t_0$, which completes the proof in the case $f \equiv r \equiv 0$.

In the general case denote by $x_1$ and $y_1$ the solutions of problems (2), (3) and (28), (29), respectively, with $f \equiv r \equiv 0$ and by $x_2$ and $y_2$ the solutions of these problems with $x_0 = x_0' = y_0 = y_0' = 0$. By Theorem 4, $y_2(t) \geq x_2(t)$ for $t \geq t_0$. Clearly, $y(t) - x(t) = [y_1(t) - x_1(t)] + [y_2(t) - x_2(t)]$, and hence

$$y(t) - x(t) \geq e^{-a(t-t_0)}(y_0 - x_0) \quad \text{for} \quad t \geq t_0,$$

which concludes the proof.

We obtain the most complete result if we compare two solutions $x$ and $y$ of the same Eq. (2). In this case we will not assume that $a_k(t) \geq 0$ and the solutions $x$ and $y$ are positive.

**Theorem 6.** Assume that inequality (25) has a nonnegative solution for $t \geq t_0 \geq 0$, $x$ and $y$ are two solutions of (2), (3) with right-hand sides $f$ and $r$ and initial functions $\varphi$ and $\psi$, respectively, moreover,

$$a_k(s)\varphi(g_k(s)) \geq a_k(s)\psi(g_k(s)) \quad \text{for} \quad s \text{ such that } g_k(s) < t_0;$$

$$r(t) \geq f(t) \quad \text{for} \quad t \geq t_0.$$

1. If $x_0 = y_0, y_0' \geq x_0'$, then $y(t) \geq x(t), t \geq t_0$.

2. If $x_0 \geq y_0 > 0$ and, for some $\alpha > 0, y_0' + \alpha y_0 \geq x_0' + \alpha x_0$, then

$$y(t) - x(t) \geq e^{-a(t-t_0)}(y_0 - x_0) \quad \text{for} \quad t \geq t_0.$$
(3) If \( x_0 \geq y_0 > 0 \) and \( y'_0 > x'_0 \), then
\[
\liminf_{t \to \infty} \left[ y(t) - x(t) \right] \geq 0.
\]

**Proof.** (1) Corollary 2 of Theorem 2 implies that the inequality\( X(t, s) > 0, t > s \geq t_0 \), holds for the fundamental function of (1).

For the solutions \( x \) and \( y \) we have
\[
x(t) = x_1(t)x_0 + X(t, t_0)x'_0 - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)a_k(s)\varphi(g_k(s)) \, ds
\]
\[
+ \int_{t_0}^{t} X(t, s)f(s) \, ds,
\]
\[
y(t) = y_1(t)y_0 + X(t, t_0)y'_0 - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)a_k(s)\psi(g_k(s)) \, ds
\]
\[
+ \int_{t_0}^{t} X(t, s)r(s) \, ds,
\]
where \( \varphi(g_k(s)) = \psi(g_k(s)) = 0 \), if \( g_k(s) \geq t_0 \). The theorem assumptions yield that \( y(t) \geq x(t) \) for \( t \geq t_0 \).

The proof of (2) and (3) is similar to the proof of the previous theorem.

**Remark.** Explicit constructions of solutions of inequality (25) will be presented in the last section.

### 5. Explicit Non-Oscillation and Oscillation Criteria

We will employ Corollary 2 of Theorem 2 to obtain explicit sufficient conditions for non-oscillation.

**Theorem 7.** Suppose the following condition holds
\[
\sup_{t \geq t_0} \sum_{k=1}^{m} a_k(t)\sqrt[3]{g_k(t)} \ln g_k(t) \ln t \leq 1/4 \quad \text{for some } t_0 \geq 1.
\]

Then there exists \( t_1 \geq t_0 \) such that Eq. (1) has a positive solution for \( t > t_1 \).

**Proof.** The statement of the theorem yields that \( x(t) = \sqrt{t} \ln t \) is a positive solution of inequality (24) for \( t > t_0 \), which completes the proof.

Now we turn to the oscillation problem.
Consider the equation
\[ \ddot{x}(t) + a(t)x(g(t)) = 0, \quad ct \leq g(t) \leq t \quad \text{for} \quad t \geq 0, 0 < c < 1, \]
\[ a(t) \geq 0 \quad \text{for} \quad t \geq 0, \]  
(34)
with continuation functions \( a \) and \( g \). In [10] the following result was obtained.

If the ordinary differential equation
\[ \ddot{x}(t) + ca(t)x(t) = 0 \]
is oscillatory then (34) is also oscillatory. We generalize this statement to Eq. (1) with several delays.

**Theorem 8.** Suppose \( a_k(t) \geq 0, g_k(t) \geq c_k t \) for \( t \geq 0, 0 < c_k < 1 \) and the ordinary differential equation
\[ \ddot{x}(t) + \sum_{k=1}^{m} c_k a_k(t)x(t) = 0 \]  
(35)
is oscillatory. Then (1) is oscillatory.

**Proof.** Suppose (1) is non-oscillatory. Theorem 3 implies that the equation
\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(c_k t) = 0 \]  
(36)
is non-oscillatory. Then by Theorem 1 there exists \( t_0 \geq 0 \) such that \( Y(t, s) > 0, t > s \geq t_0 \), where \( Y(t, s) \) is the fundamental function of (36). Hence \( y(t) = Y(t, t_0) \) is a nonnegative solution of the problem
\[ \ddot{y}(t) + \sum_{k=1}^{m} a_k(t)y(c_k t) = 0, \quad t \geq t_0, \]
\[ y(t) = 0, \quad t \leq t_0, \quad \dot{y}(t_0) = 1. \]  
(37)
Rewrite (37) in the form
\[ \ddot{y}(t) + \sum_{k=1}^{m} c_k a_k(t)y(t) + \sum_{k=1}^{m} a_k(t)[y(c_k t) - c_k y(t)] = 0. \]  
(38)
Since \( \dot{y} \) is a nonincreasing function then \( \dot{y}(c_k t) \geq \dot{y}(t) \). By integrating this inequality from \( t_0 \) to \( t \) we get
\[ (1/c_k)y(c_k t) - (1/c_k)y(c_k t_0) \geq y(t) - y(t_0), \]
where \( y(t_0) = y(c_k t_0) = 0 \). Thus, \( y(c_k t) \geq c_k y(t) \), therefore (38) implies
\[ \ddot{y}(t) + \sum_{k=1}^{m} c_k a_k(t)y(t) \leq 0 \quad \text{for} \quad t \geq t_0. \]
Theorem 1 yields that Eq. (35) is non-oscillatory. The contradiction obtained proves the theorem.

Remark. As a consequence of Theorem 8 we obtain \((m = 1)\) that the condition \(\lim_{s \to +} a(s)^2 > 1/4c\) is sufficient for oscillation. Domshlak [15, 16] replaced the constant \(1/4c\) with a certain constant \(B_c\) and showed that this constant is strict.

The following statement is well known (see, for example, [5]): if for a certain \(k, 0 < k < 1\), the ordinary differential equation
\[
\dot{x}(t) + k \frac{g(t)}{t} a(t)x(t) = 0
\]
is oscillatory then the delay equation
\[
\dot{x}(t) + a(t)x(g(t)) = 0
\]
is also oscillatory.

The corollary of the following theorem generalizes this statement.

**Theorem 9.** Suppose \(a_k(t) \geq 0\) for \(t \geq 0\) and for every \(c > 0\) the ordinary differential equation
\[
\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) \frac{g_k(t) - c}{t - c} x(t) = 0, \quad t > c,
\]
is oscillatory. Then Eq. (1) is also oscillatory.

**Proof.** Suppose (1) is non-oscillatory. Then as in the proof of the previous theorem, there exists \(t_0 \geq 0\) such that the solution \(y(t)\) of the problem
\[
\begin{align*}
\ddot{y}(t) + \sum_{k=1}^{m} a_k(t) y(g_k(t)) &= 0, \quad t \geq t_0, \\
y(t) &= 0, \quad t \leq t_0, \quad \dot{y}(t_0) = 1
\end{align*}
\]
is positive for \(t > t_0\).

Inequality
\[
y(t) - y(t_0) \geq \dot{y}(t)(t - t_0), \quad t \geq t_0
\]
implies that \(y(t) - \dot{y}(t)(t - t_0) \geq 0\). Then the function \(f(t) = y(t)/(t - t_0), t > t_0\) is nonincreasing. Therefore
\[
\frac{y(t)}{t - t_0} \leq \frac{y(g_k(t))}{g_k(t) - t_0}
\]
and hence
\[ y(g_k(t)) \geq \frac{g_k(t) - t_0}{t(t - t_0)}y(t), \quad t > t_0. \]  

Equation (39) and inequality (40) imply
\[ \ddot{y}(t) + \sum_{k=1}^{m} a_k(t) \frac{g_k(t) - t_0}{t(t - t_0)}y(t) \leq 0, \quad t > t_0. \]

Then the corresponding equation
\[ \ddot{y}(t) + \sum_{k=1}^{m} a_k(t) \frac{g_k(t) - t_0}{t(t - t_0)}y(t) = 0, \quad t > t_0 \]
is non-oscillatory which contradicts the assumption of the theorem.

**Corollary.** Suppose \( a_i(t) \geq 0 \) and for some \( k_i, k = 1, 2, \ldots, m, 0 < k_i < 1 \), the ordinary differential equation
\[ \ddot{x}(t) + \sum_{i=1}^{m} k_i \frac{g_i(t)}{t} a_i(t) x(t) = 0 \]
is oscillatory. Then Eq. (1) is also oscillatory.

**Proof.** Suppose \( c > 0 \). The inequality
\[ k_i \frac{g_i(t)}{t} \leq \frac{g_i(t) - c}{t - c} \]  

is equivalent to
\[ ct - ck_i g_i(t) \leq (1 - k_i)tg_i(t). \]  

Since \( \lim g_i(t) = \infty \) then there exists \( t_i \geq 0 \) such that (42) holds for \( t \geq t_i \). Hence for \( t \geq t_0 = \max(t_i) \) inequalities (41) hold for all \( i, i = 1, 2, \ldots, m \).

Theorems 3 and 9 imply this corollary.

**Remark.** Explicit oscillation conditions different from the previous ones are presented in [11–14].

For ordinary linear differential equations of the second order the following oscillation criterion is well known: if an equation has an oscillatory solution then all its solutions are oscillatory. It is known that for delay differential equations this statement is not true.
We will show that if Eq. (1) has a slowly oscillatory solution, then all solutions of this equation are oscillatory. A similar result for delay differential equations of first order was obtained in [6].

**Definition.** A solution $x$ of (1) is said to be slowly oscillating if for every $t_0 \geq 0$ there exist $t_2 > t_1 > t_0$, such that

$$g_k(t) \geq t_1 \quad \text{for } t \geq t_2, x(t_1) = x(t_2) = 0, x(t) > 0, t \in (t_1, t_2).$$

and at the point $t_2$ the function $x(t)$ has a sign change.

**Theorem 10.** Suppose $a_k(t) \geq 0$ for $t \geq 0$. If there exists a slowly oscillating solution of Eq. (1) then all solutions of this equation are oscillatory.

**Proof.** Denote by $x$ a slowly oscillating solution of (1). Suppose this equation has a non-oscillatory solution. Then by Theorem 1, for a certain $t_0 \geq 0$, $X(t, s) > 0, t > s \geq t_0$.

There exist $t_1 > t_0, t_2 > t_0, \sigma > 0$ such that

$$g(t) \geq t_1 \quad \text{for } t \geq t_2, x(t_1) = x(t_2) = 0, x(t) > 0, t \in (t_1, t_2), x(t) < 0, t \in (t_2, t_2 + \sigma].$$

(43)

Due to (5) and (43) this solution for $t \geq t_2$ can be presented in the form

$$x(t) = X(t, t_2) \dot{x}(t_2) - \sum_{k=1}^{m} \int_{t_2}^{t} X(t, s) a_k(s) x(g_k(s)) \, ds, \quad (44)$$

where $x(g_k(s)) = 0$, if $g_k(s) > t_2$.

Inequality $g_k(t) \geq t_1$ for $t \geq t_2$ yields that the expression under the integral in (44) may differ from zero only if $t_1 < g_k(s) < t_2$ and therefore in (44), $x(g_k(s)) > 0$.

Moreover, $x(t)$ has a sign change at the point $t_2$, thus $\dot{x}(t_2) \leq 0$. Hence (44) implies $x(t) \leq 0$ for each $t \geq t_2$. This contradicts the assumption that $x$ is an oscillatory solution.

**Corollary.** Suppose $a_k(t) \geq 0$ for $t \geq 0$ and Eq. (1) has a positive solution for $t > t_0 \geq 0$. Then (1) has no slowly oscillating solutions.

**Remark.** Yu. Domshlak [15, 16] demonstrated that if the $g_k$ are monotone increasing functions and an associated equation has a slowly oscillating solution, then every solution of (1) is oscillatory. He obtained several new explicit sufficient conditions for oscillation by explicit constructions of such slowly oscillating solutions.
6. EXISTENCE OF A POSITIVE SOLUTION

In this section assuming that Eq. (1) is non-oscillatory we will give a condition on the initial function and initial values which implies the positiveness of the solution of initial value problem (2), (3).

For delay differential equations of first order a few papers contain such results. In the recent paper [17] the most general result is presented. Our method is similar to the one used in [6] for an equation of first order.

**Theorem 11.** Suppose \( a_k(t) \geq 0, f(t) \geq 0 \) for \( t \geq 0 \), \( x \) is the solution of (2), (3), and \( u \) is a nonnegative solution of the inequality

\[
\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp\left\{-\int_{\max(t_0, g_k(t))}^{t} u(s) \, ds\right\} \leq 0, \quad t \geq t_0.
\]

(45)

If

\( x_0 > 0, \quad \varphi(t) \leq x_0, \quad t \leq t_0, \quad \text{and} \quad x'_0 \geq u(t_0)x_0, \)

then \( x(t) > 0, t \geq t_0. \)

**Proof.** First assume that \( f \equiv 0 \). Consider an auxiliary problem

\[
\ddot{z}(t) + \sum_{k=1}^{m} a_k(t) z(g_k(t)) = 0, \quad t \geq t_0,
\]

\[
z(t) = x_0, \quad t \leq t_0, \quad \dot{z}(t_0) = u(t_0)x_0.
\]

Denote

\[
u(t) = \begin{cases} x_0 e^{\int_{t_0}^{t} u(s) \, ds} & t \geq t_0, \\ x_0, & t < t_0, \end{cases}
\]

and for a fixed \( t \geq t_0 \) define the sets

\[ N_1(t) = \{k: g_k(t) \geq t_0\}, \quad N_2(t) = \{k: g_k(t) < t_0\}. \]

We obtain

\[
\ddot{\nu}(t) + \sum_{k=1}^{m} a_k(t) \nu(g_k(t))
\]
\[= x_0 \exp \left( \int_{t_0}^{t} u(s) \, ds \right) (\dot{u}(t) + u^2(t)) \]
\[+ x_0 \sum_{k \in N_k(t)} a_k(t) \exp \left( \int_{t_0}^{g_k(t)} u(s) \, ds \right) + x_0 \sum_{k \in N_k(t)} a_k(t) \]
\[= x_0 \exp \left( \int_{t_0}^{t} u(s) \, ds \right) \left[ \dot{u}(t) + u^2(t) \right] + \sum_{k \in N_k(t)} a_k(t) \exp \left( - \int_{g_k(t)}^{t_0} u(s) \, ds \right) \]
\[+ \sum_{k \in N_k(t)} a_k(t) \exp \left( - \int_{\max(t_0, g_k(t))}^{t_0} u(s) \, ds \right) \]
\[= x_0 \exp \left( \int_{t_0}^{t} u(s) \, ds \right) \left[ \dot{u}(t) + u^2(t) \right] + \sum_{k=1}^{m} a_k(t) \exp \left( - \int_{\max(t_0, g_k(t))}^{t_0} u(s) \, ds \right) \leq 0. \]

Therefore
\[\ddot{u}(t) + \sum_{k=1}^{m} a_k(t) v(g_k(t)) = r(t),\]

where \(r(t) \leq 0, t \geq t_0\). Inequality (45) implies (7). Since \(z(t) = v(t) = x_0\), \(t \leq t_0, z_0' = v_0' = u(t_0)x_0\), then Theorem 6 implies \(z(t) \geq v(t) > 0\) for \(t \geq t_0\). Hence the hypotheses of the theorem and Theorem 6 yield \(x(t) \geq z(t) > 0\) for \(t \geq t_0\).

In the case \(f = 0\) the theorem is proved. The general case is also a consequence of Theorem 6, since \(f(t) \geq 0\) for \(t \geq 0\).
Corollary. Suppose

\[ a_k(t) \geq 0, f(t) \geq 0 \text{ for } t \geq t_0 > 0; \]

\[ \varphi(t) \leq x_0 \text{ for } t \leq t_0; \quad x_0 > 0, x_0' \geq \frac{1}{2t_0}x_0 \]

and

\[ \sup_{t \geq t_0} \sum_{k=1}^{m} a_k(t) \sqrt{t^3 \max[t_0, g_k(t)]} \leq 1/4. \quad (46) \]

Then the solution of problem (2), (3) is positive.

Proof. It is sufficient to show that function \( u(t) = 1/2t \) is a solution of inequality (45). We have

\[
\begin{align*}
\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp \left\{ - \int_{\max[t_0, g_k(t)]} \frac{1}{2s} \, ds \right\} \\
&= - \frac{1}{2t^2} + \frac{1}{4t^2} + \sum_{k=1}^{m} a_k(t) \exp \left\{ - \int_{\max[t_0, g_k(t)]} \frac{1}{2s} \, ds \right\} \\
&= - \frac{1}{4t^2} + \sum_{k=1}^{m} a_k(t) \exp \left\{ \ln \left( \frac{\max[t_0, g_k(t)]}{t} \right)^{1/2} \right\} \\
&= - \frac{1}{4t^2} + \sum_{k=1}^{m} a_k(t) \left( \frac{\max[t_0, g_k(t)]}{t} \right)^{1/2} \leq 0,
\end{align*}
\]

since the last inequality is equivalent to (46).

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