Adm Reps 2/3/14 BU LLC learning seminar

K p-adic $\mathcal{O}_K$ ints π_K uniformizer

$\mathbb{G}$ conn. red / K p-adic

$\to \mathbb{G}(K) = G$ is loc. prof. (loc. cmpct. Hausd. s.t. cmpct open subgrps form basis of nbhds of 1)

E.g. $\mathbb{G} = GL_n \Rightarrow \{1 + \pi_K^m M_n(\mathcal{O}_K)\}_{m \geq 1} \subseteq C_0 = GL_n(\mathcal{O}_K)$ maximal cpct

Defn A rep $\pi: G \to GL_n(V)$ $(V/\mathbb{C})$

is (a) smooth if

$$\bigcup_{\substack{K \subseteq G \\ \text{open cpct}}} V^K = V$$

f newform level N $\Rightarrow$ fcn on $G(\mathbb{A})$

$G_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / GL_2(\hat{\mathbb{Z}})$

$\to GL_2(\mathbb{A}^f)$ translates

(b) admissible if smooth & each V^K is fin. dim.

Defn $\mathcal{A}(G)$ = set of equiv. classes of irred. adm. reps of G

Note Does not depend on topo. of $\mathbb{C}$

(smooth irred. $\Rightarrow$ adm. will depend on non-countability of $\mathbb{C}$)

Defn A q.c. χ is a ~~smooth~~ smooth ($\Rightarrow$ adm.) rep of $G = K^\times$

$\chi: K^\times \to \mathbb{C}$ (i.e. $1 + (\pi_K)^m \subseteq \operatorname{Ker} \chi$ for $m \gg 0$)

equiv cont. for discrete or compl topo on $\mathbb{C}$

§2 Hecke Alg

Given Haar measure dg, vect. space $C_c^{lc}(G)$ of loc. const. cpctly. supp fcns ("describe") is an <u>algebra</u>

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1}) f_2(g)\, dg \left(= \int_G f_1(g) f_2(g^{-1}h)\, dg \right)$$

measure dg also determines isom $C_c^{lc}(G) \to \mathcal{H}(G)$ "tensor"

$f \longmapsto f\,dg$

$\mathcal{H}(G)$ = vect space of loc. const. cpct supp. <u>measures</u>

$\Longrightarrow$ alg. structure on $\mathcal{H}(G)$ is <u>indep</u> of dg

<u>Defn</u> $\mathcal{H}(G)$ is called the Hecke algebra ☆

<u>assoc. alg</u> <u>w/o unit if G non-cmpct</u>

<u>Defn</u> If $C \subseteq G$ is cmpct open, (e.g. $1 + (\pi_k)^m M_n(\mathcal{O}_k)$)

$\mathcal{H}(G/\!/C) =$ $\phi \in \mathcal{H}(G)$ s.t. $\phi(c_1 h c_2) = \phi(h)$ $\forall\, c_1, c_2 \in C$

$\cap|$

$\mathcal{H}(G)$ <u>Rmk</u> $\phi \in \mathcal{H}(G/\!/C)$ has <u>finite</u> support on

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If $C' \subsetneq C^\circ$, then $\mathcal{H}(G/\!/C^\circ) \subseteq \mathcal{H}(G/\!/C')$

Each has identity $1_{C'}$ (resp 1_{C°) but $1_{C'} \notin \mathcal{H}(G/\!/C^\circ)$

In fact, $\mathcal{H}(G) = \bigcup_{C \subseteq G} \mathcal{H}(G/\!/C)$ (by loc. const. cpct supp.)

§3 Reps of Hecke Algebras

Def. If (π, V) is a smooth rep of G

~~$\mathcal{H}(G)$ acts on V by $\pi(\phi)$~~ $\pi : \mathcal{H}(G) \to \mathrm{End}_{\mathbb{C}}(V)$

by $\pi(\phi) v = \int_G \pi(g)\, d\phi$ (Note integral is finite sum as smooth)

~~Say $C \ll G$ s.t.~~

Given $x \in V$, take $C \ll G$ s.t. $x \in V^C$.

Then $1_C x = x$, so action of $\mathcal{H}(G)$ on V is non-degenerate

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Fact $\{\text{smooth } G\text{-reps}\} \longrightarrow \left\{\begin{matrix}\text{non-deg.} \\ \mathcal{H}(G)\text{-modules}\end{matrix}\right\}$ is equiv. of cats

("think: G-reps, $\mathbb{C}[G]$-reps for finite G")

$\boxed{V \text{ adm}}$

If $C \leq G$, then $\mathcal{H}(G /\!/ C)$ leaves V^C invt ("as C smaller V^C bigger, $\mathcal{H}$ smaller")

(if time, as $v \in V^C$, $\phi \in \mathcal{H}(G/\!/C)$, $c \in C$, then

$c(\pi(\phi) v) = \int_G c(\pi(g) v)\, d\phi = \int_G \pi(cg) v\, d\phi$ but $d\phi$ C-invt.)

Prop Given C, $\left\{\begin{matrix}(\pi, V) \text{ s.t.} \\ V^C \neq 0\end{matrix}\right\} \longrightarrow \left\{\text{f.d. irred } \mathcal{H}(G/\!/C)\text{-modules}\right\}$

More Generally, if ~~all irred subquots of V~~

$\left\{\begin{matrix}(\pi, V) \text{ s.t.} \\ \text{all irred. subquot. } W \text{ of } V \\ \text{has } W^C \neq 0\end{matrix}\right\} \longrightarrow \left\{\text{non zero } \mathcal{H}(G/\!/C)\text{-modules}\right\}$

Cor (π, V) irred $\iff \mathcal{H}(G/\!/C)$-mod irred. $\iff \mathcal{H}(G/\!/C) \twoheadrightarrow \mathrm{End}_{\mathbb{C}}(V^C)$ classical

(**RmK** "As C gets smaller, class gets larger")

Cor $\mathcal{H}(G/\!/C)$ is commutative $\implies \dim V^C = 1$

RmK "this is space of actual newforms (non-norm.)"

(**Thm "Local Atkin-Lehner"** (Deligne) For $G = GL_2(K)$ (more gen?)
V irred. admissible, $\exists!\, C$ s.t. $\dim V^C = 1$)

E.g. $\mathbb{G} = GL_n$, $C = GL_n(\mathcal{O}_K)$
$\mathcal{H}(G/\!/C) \cong \mathbb{C}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]^{S_n}$

More generally $\mathbb{G}/\mathcal{O}_K$ flat affine grp scheme w/ both fibers reductive

(" $\mathcal{H}(\mathbb{G}(K)/\!/\mathbb{G}(\mathcal{O}_K)) = \mathbb{C}[\mathbb{G}_m \to \mathbb{T}]^{W_{\mathbb{G}_K}}$)
("Satake isom.")

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§3.5 Distr. Char

Given $\phi \in \mathcal{H}(G)$, $\phi \in \mathcal{H}(G/\!/C)$ for $C << G$ (by loc. const.)
$\implies \pi(\phi) V \subseteq V^C \implies \pi(\phi)$ has trace

Defn $\chi_\pi : \mathcal{H}(G) \to \mathbb{C}$ is called distribution char
$$\phi \longmapsto \mathrm{Tr}(\pi(\phi))$$

Fact 1 χ_π determines π if irred.

(Fact 2 χ_π rep. by locally integrable fcn
(loc. const. on ~~semisimple~~ reg. s.s. elts of $\mathcal{H}(G)$
invt under conj. $\chi_\pi : \{G\}^{reg} \to \mathbb{C}$))

§ 4 Rep Thy

Prop (Schur lemma)

If V smooth irred, ~~$End_G(V)$~~ $\dim End_G(V) = 1$ "hence scalars"

Proof G adm (general case uses $\mathbb{C}$ uncountable and used to prove any smooth irred is adm.)

Take $C \ll G$ s.t. $V^C \neq 0$

Then $f \in End_G(V)$ preserves V^C and is in $\mathbb{C}$

as V^C irred. as $\mathcal{H}(G/\!/C)$-mod (as we proved)

As b/c V^C is fin. dim. by ordinary Schur

As $V = \bigcup_{C \ll G} V^C$, we are done

~~Cor Irred reps of f.d. are~~

Prop (i) V irred. sm. $\Rightarrow$ adm.

(ii) G comm $\Rightarrow$ V 1-dim

Proof For $C \ll G$, $V^C \neq 0$
hence $= V$ as G-submod
but $\mathcal{H}(G/\!/C) \subseteq \mathcal{H}(G)$ comm.