Valuation Spectrum "~~like Spec~~" "topology"

plan:

Define space from vals. **Motiv** vals on $K(C)$, $\mathbb{Q}$

$\mathrm{Spv}(A) = \{\text{vals on } A\} \overset{1.28}{=}$ vals on residue fields

[dont erase]

$\mathrm{Spv}\,A$ $\boxed{v \to \mathrm{supp}\, v}$ fiber $\mathrm{Spv}\,K(\mathfrak{p})$

$\downarrow$

$\mathrm{Spec}\,A$

val spect. of scheme

Examples $\mathrm{Spv}\,\mathbb{Q}$

3-adic

2-adic $\mathrm{Spv}\,\mathbb{F}_2$ $\mathrm{Spv}\,\mathbb{F}_3$

triv

$U_6 = \{v \mid v(6) \geq 1\}$

[right board / Middle one]

$\mathrm{Spec}\,\mathbb{Z}$: 2, 3, 5, 7, ..., 613

$\mathrm{Spv}\,\mathbb{F}_p(t)$ $(\mathbb{F}_p(C))$ "remembers birat. info"

pts of $\mathbb{P}^1$

p_1

0

$\mathrm{Spec}\,\mathbb{F}_p[t]$: 0, 1

$\mathrm{Spv}\,\mathbb{C}$, $\mathrm{Spv}\,\mathbb{Z}_p$, $\mathbb{F}_p(t)(x)$

$\mathrm{Spv}\,K(S)$ S = surface

codim 1 pt (height 1 val.) **but** also on blowups, birat. ones

~~0~~? $\boxed{\text{but } \mathrm{Spv}\,\mathbb{P}^1/\mathbb{F}_p}$

Topology

Motivation $\mathrm{Spv}\,\mathbb{Q}$ like $\mathrm{Spec}\,\mathbb{Z}$

what we want:

1. $\mathrm{Spec}\,\mathbb{Z} \xrightarrow[\text{adic}]{} \mathrm{Spv}\,\mathbb{Q}$ cont.
2. $\mathrm{Spv}\,A \to \mathrm{Spec}\,A$ cont.

~~Open sets~~ $\boxed{A = \mathrm{Frac}\,R}$

$\underline{\mathrm{Spec}\,R}$ $a \in R$ $U_s = \{\mathfrak{p} \subseteq R \mid s \notin \mathfrak{p}\}$

$\underline{\mathrm{Spv}\,A}$ $U_s = \{v \in \mathrm{Spv}\,A \mid v(s) \geq 1\}$

"look at $\mathrm{Spv}\,\mathbb{Q}$, then $\mathrm{Spv}\,\mathbb{Z}$" $\boxed{U_6 \text{ would be draw on } \mathrm{Spv}\,\mathbb{Z} \text{ board}}$

Preim of $\mathrm{Spec}\,\mathbb{Z}[1/6]$ in $\mathrm{Spv}\,\mathbb{Z}$ not open

Right defn $U(f/s) := \{v \in \mathrm{Spv}\,A \mid |f| \leq |s| \neq 0\}$ "like $\left|\frac{f}{s}\right| \leq 1$" – not def. if $|s| = 0$

"$\mathrm{Spv}A(f/s)$"

so – $U(0/N)$ is preim of $\mathrm{Spec}\,\mathbb{Z}[1/N]$ ($U(0/s)$ preim of $\mathrm{Spec}\,A_s \subseteq \mathrm{Spec}\,A$)

(on a scheme, cover by affines, take topo on fiber of each affine)

$$\begin{array}{ccc} \mathrm{Spv}\,X & \supseteq & \mathrm{Spv}\,A \\ \downarrow & & \downarrow \\ X & \supseteq & \mathrm{Spec}\,A \end{array}$$

$\boxed{\text{Dont erase right away}}$

Consequences

$h^{-1}(\mathcal{U}(f/s)) = \mathcal{U}(h(f)/h(s))$ | p. 2

Prop ① $h: A \to B$, $h^*: \mathrm{Spv} B \to \mathrm{Spv} A$ conti.

Proof ~~exerc~~ ~~"explain"~~

② $A \to B$, $B = A/\mathfrak{a}$ or $S^{-1}A$

$$\begin{array}{ccc} \mathrm{Spv} B & \to & \mathrm{Spv} A \\ \downarrow & & \downarrow \\ \mathrm{Spec} B & \to & \mathrm{Spec} A \end{array}$$

cor $B = K(\mathfrak{p})$ get $\mathrm{Spv}\, K(\mathfrak{p})$ homeo to subset of $\mathrm{Spv} A$

pf exerc

③ $\mathrm{triv}(A) \subseteq \mathrm{Spv} A$

$$\begin{array}{ccc} \mathrm{triv}(A) & \subseteq & \mathrm{Spv} A \\ & \overset{\sim}{\searrow} & \downarrow \\ & & \mathrm{Spec} A \end{array}$$

Proof $\mathcal{U}(f/s) \cap \mathrm{Triv}(A) = \mathcal{U}_s$ in Zariski topo

More Conseq (**Defn** x is specialization, y generization if $x \in \mathrm{cl}(y)$ $\quad x \preceq y$
e.g. $(p) \in \mathrm{cl}(0)$ in $\mathrm{Spec}\,\mathbb{Z}_{(p)}$ not in vers $\quad (p) \preceq (0)$

Defn A topo space X is spectral if it is underlying top. space of affine scheme

Prop X Spectral $\Longleftrightarrow$
- X q.c
- X has basis of q.c. opens
- X sober (every irred. cl. has ! gen. pt)

("locally spectral if covered by spectral opens, $\Leftrightarrow$ topo space of scheme)

Thm $\mathrm{Spv} A$ spectral

(**Prop** $\mathrm{Spv} A \to \mathrm{Spec} A$, $\mathrm{Spv} B \to \mathrm{Spv} A$ q.c. like for affine schemes)

Specialization and Generization

Which pts in closure of others?

NOT interesting for rigid spaces, vars yes for schemes, adic spaces
"a little weirder b/c different from Spec"

$\mathrm{Spv} A \downarrow \mathrm{Spec} A$

Ⓘ vertical = w/in a fiber of $\quad$ = same support = on Spv(field)

Ⓘⓘ horizontal trickier

(I) Can work on $A = K$ field ("~~[illegible]~~ as Spv(field) homeo to subset")

2.14

K, v, A = val ring

$$\left\{ \begin{matrix} A \subseteq B \subseteq K \\ B \text{ val. ring} \end{matrix} \right\} \longleftrightarrow \left\{ \Delta \text{ convex subgrp of } \Gamma_v \right\}$$

$K \to \Gamma_v$

$A \to \Delta$

$B \to |B^\times|$ "send more things to 1, so val ring is larger"

$A_{\text{val}(\Gamma_v/\Delta)} \leftarrow \Delta$

Prop $\left\{ \begin{matrix} w \text{ generization of } v \\ \text{on } K \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} w \text{ s.t.} \\ A(v) \subseteq A(w) \end{matrix} \right\}$

Proof $\underbrace{A(v) \subseteq A(w)}_{\text{w val}} \iff v(g) \leq 1 \implies w(g) \leq 1$

$\iff \left(\forall f, 0 \neq s \in A : v(f) \leq v(s) \implies w(f) \leq w(s) \right)$

$\iff \left(\forall f, s \in A : v \in \mathrm{Spv}\, U(f/s) \implies w \in \mathrm{Spv}(f/s) \right)$

Cor $\left\{ \begin{matrix} w \text{ vert. generization} \\ \text{of } v \end{matrix} \right\} \longrightarrow \left\{ H \subseteq \Gamma_v \text{ convex} \right\}$

Rmk $ht(w) \leq ht(v)$ (just like hts of prime ideals in schemes)

~~Examples~~ $\boxed{2.5 \to \text{specializations } \mathrm{Spv}\, A(v)/m(v)}$

Examples (1) $\mathrm{Spv}\,\mathbb{Q}$ $\quad |\,|_p$ ht 1 $\quad \mathbb{Z}_{(p)}$

triv$\mathbb{Q}$ is generization $\quad \mathbb{Q}$

(2) ~~$K((y))((x))$~~ $(K(x,y) \subseteq)$ $K((y))((x))$

v: $x^i y^j \to (-i,-j) \in \mathbb{Z}^2$ (lexicographic 0 id $\neq 0$)

v(series) = look at min x power, look at min y w/ in that

"does not work on $K((x))((y))$"

$$A(v) = xK((y))[[x]] + K[[y]]$$

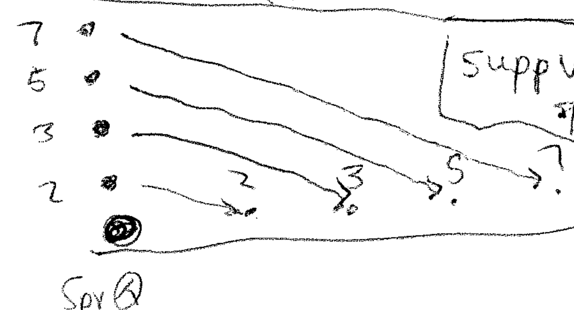
gen of v: $w(x^i y^j) = -i \in \mathbb{Z}$

$$A(w) = K((y))[[x]] \supseteq A(v)$$

($= dVR =$ localization at x of $K[[x,y]]$ didn't check)

$k(x,y) \cap A(v)$ not localization at (x,y) (which is Noeth but dim 2 so not valring)

(II) horizontal $\mathrm{Spv}\,\mathbb{Z}$



supp $v \geq$ supp w
special has larger supp
like for spec A

$p=5$ $|\;|_5 \to$ triv $\mathbb{F}_5$

Defn $c\Gamma_w = \langle \Gamma_{w,\geq 1} \cap \mathrm{im}(w) \rangle$ characteristic subgroup

Ex $c\Gamma_{|\,|_5} = \{0\} \subseteq \mathbb{Z}$ $\quad |5|_5 = -1$

Def Let $H \subseteq \Gamma_w$ be convex. Define

$$v_H(x) = \begin{cases} v(x) & \text{if } v(x) \in H \\ 0 & \text{if } v(x) \notin H \end{cases}$$

Convex ↑

Prop If $H \supseteq c\Gamma_w$, then $v|_H$ is a specialization of w

Proof

Example

2-adic

$x+1$-adic 3-adic

x-adic

Spv $\mathbb{Z}[x]$

x-adic 2-adic

triv triv triv triv

Spec $\mathbb{Z}[x]$ $(2, x)$ (2) (x) (0)

Thm All specializations are horiz of vert