BRAUER AND ETALE HOMOTOPY OBSTRUCTIONS ON OPEN COVERS

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Abstract

The Brauer-Manin obstruction can be used to explain the failure of the local-global principle for many algebraic varieties. In 1999, Skorobogatov gave the first example of a variety whose failure to satisfy that principle is not explained by the Brauer-Manin obstruction. He did so by applying the Brauer-Manin obstruction to étale covers of the variety, thus defining a finer obstruction. In 2010, Poonen gave the first example of failure of the local-global principle that cannot be explained by Skorobogatov's étale Brauer-Manin obstruction. Further obstructions such as the étale homotopy obstruction and the descent obstruction are unfortunately equivalent to the étale Brauer-Manin obstruction. However, Poonen's construction was not accompanied by a definition of a new, finer obstruction. In this paper we present an approach for a finer obstruction by applying the Brauer-Manin obstruction to each piece of every stratification of the variety. We prove that this obstruction is necessary and sufficient, over imaginary quadratic fields and totally real fields unconditionally, and over all number fields conditionally on the section conjecture. We then discuss the behavior of the étale homotopy obstruction in fibrations of varieties and use that discussion and other techniques to further analyze Poonen's example in light of our general results.

Contents

Part 1. Introduction and Setup

1. Introduction

Given a variety X over a global field k, a major problem is to decide whether $X(k) = \emptyset$. By [Poo09], it suffices to consider the case that X is smooth, projective, and geometrically integral. As a first approximation one can consider the set $X(\mathbb{A}_k) \supset X(k)$, where $\mathbb{A}_k$ is the adele ring of k. It is a classical theorem of Minkowski and Hasse that if X is a quadric, then $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$. When a variety X satisfies this implication, we say that it satisfies the Hasse principle, or local-global principle. In the 1940s, Lind and Reichardt ([Lin40] [Rei42]) gave examples of genus 1 curves that do not satisfy the Hasse principle. More counterexamples to the Hasse principle were given throughout the years, until in 1971 Manin [Man71] described a general obstruction to the Hasse principle that explained many of the counterexamples to the Hasse principle that were known at the time. The obstruction (known as the Brauer-Manin obstruction) is defined by considering a certain set $X(\mathbb{A}_k)^{\mathrm{Br}}$, such that $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. If X is a counterexample to the Hasse principle, we say that it is accounted for or explained by the Brauer-Manin obstruction if $\emptyset = X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k) \neq \emptyset$.

In 1999, Skorobogatov ([Sko99]) defined a refinement of the Brauer-Manin obstruction known as the étale Brauer-Manin obstruction and used it to produce an example of a variety X such that $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ but $X(k) = \emptyset$. Namely, he described a set $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ for which $X(k) \subset X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$ and found a variety X such that $\emptyset = X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

In his paper [Poo10], Poonen constructed the first example of a variety X such that $\emptyset = X(k) \subset X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \neq \emptyset$. However, Poonen's method of showing that $X(k) = \emptyset$ relies on the details of his specific construction and is not explained by a new finer obstruction. While it was hoped ([Pál10]) that an étale homotopy obstruction might solve the problem, it was shown ([HS13]) that this provides nothing new.

Therefore, one wonders if Poonen's counterexample can be accounted for by an additional refinement of $X(\mathbb{A}_k)^{\text{ét,Br}}$. Namely, can one give a general definition of a set

$$X(k) \subset X(\mathbb{A}_k)^{\text{new}} \subset X(\mathbb{A}_k)^{\text{ét,Br}}$$

such that Poonen's variety X satisfies $X(\mathbb{A}_k)^{\text{new}} = \emptyset$.

In this paper, we provide such a refinement and prove that it is necessary and sufficient over many number fields. The obstruction is essentially given by applying the finite abelian descent obstruction to open covers of X or decompositions of X as a disjoint union of locally closed subvarieties. We shall make this statement more precise in Section 3. As an example of our results, we can prove the following:

Corollary 1.1 (of Theorem 3.3)**.** *Let $X/\mathbb{Q}$ be a variety for which $X(\mathbb{Q}) = \emptyset$. Then there is a Zariski open cover $X = \bigcup_i U_i$ such that $U_i(\mathbb{A}_{\mathbb{Q}})^{\text{Br}} = \emptyset$ for all i.*

For a summary of all of the variants of this result, the reader may proceed to Section 3.

Part 2 contains the main proofs of these results, one method unconditional and one method assuming the section conjecture in anabelian geometry.

In Part 3, we introduce homotopy sections and use them to analyze what happens to the section conjecture and to the étale Brauer-Manin obstruction in fibrations. In particular, we derive an important result about the behavior of the étale-Brauer obstruction in fibrations:

Theorem 1.2 (Theorem 9.11)**.** *Let $f\colon X \to S$ be a geometric fibration (c.f. Definition 9.1), and suppose that $S(k) = S(\mathbb{A}_k^f)^{\text{ét,Br}}$ and that for all $a \in S(k)$, we have $X_a(k) = X_a(\mathbb{A}_k^f)^{\text{ét,Br}}$. Suppose furthermore that X and S satisfy the technical conditions 2-5 of Theorem 9.11. Then*

$$X(k) = X(\mathbb{A}_k^f)^{\text{ét,Br}}.$$

In Part 4, we discuss the example of [Poo10]. Using Theorem 9.11, we explain why Poonen's example required singular fibers:

Theorem 1.3 (Theorem 11.3)**.** *Let $f\colon X \to C$ be a smooth proper family of Châtelet surfaces over an elliptic curve C with $|C(k)| < \infty$. Suppose that for all $a \in C(k)$, we have $X_a(k) = \emptyset$. Suppose furthermore that the Tate-Shafarevich group of C has trivial divisible subgroup, and that for every real place v of k, every $a \in S(k)$, and every $b \in X_a(k_v)$, the map $\pi_1(X(k_v), b) \to \pi_1(S(k_v), a)$ is surjective (vacuous if k is totally imaginary). Then $X(\mathbb{A}_k)^{\text{ét,Br}} = \emptyset$.*

In Section 12, we find a Zariski open cover of the example of [Poo10] and prove that its pieces have empty étale-Brauer set. In fact, we introduce a few variants of this result, depending on different hypotheses, summarized at the beginning of Section 12. One of these uses Theorem 9.11.

Motivated by one of these examples, we also introduce the notion of quasi-torsors (Definition 13.1), which leads us to propose a new approach to computing obstructions for open subvarieties, known as the ramified étale-Brauer obstruction (Definition 13.2).

1.1. **Notation and Conventions.** Whenever we speak of a field k, we implicitly fix a separable closure k_s throughout, which produces a geometric point $\mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$ of k. We write X/k to mean that X is a scheme over $\mathrm{Spec}\, k$, and we denote by X^s the base change of X to k_s, following [Poo17]. We let G_k denote $\mathrm{Gal}(k_s/k)$. If X is a scheme, $\mathcal{O}(X)$ denotes $\mathcal{O}_X(X)$.

All cohomology is taken in the étale topology unless otherwise stated, and fundamental group always denotes an étale fundamental group. The same is true for higher homotopy groups, once they are defined in Definition 8.12. If k is a field, we write $H^*(k, -)$ for Galois cohomology of the field k.

For a global field k, we let $\mathbb{A}_k$ denote the ring of adeles of k. For a place v of k, we let k_v denote the completion of k at v, $\mathcal{O}_v$ the ring of integers in k_v, $\mathfrak{m}_v$ the maximal ideal in $\mathcal{O}_v$, π_v a generator of $\mathfrak{m}_v$, $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$, and q_v the size of $\mathbb{F}_v$. If S is a subset of the set of places of k, we let $\mathbb{A}_{k,S} = \prod_{v \in S}' (k_v, \mathcal{O}_v)$. This must not be confused with $\mathbb{A}_k^S = \prod_{v \notin S}' (k_v, \mathcal{O}_v)$. We let $\mathcal{O}_{k,S}$ denote the S-integers of k, i.e.,, in which primes in S are inverted. In general, we replace S by the letter "f" when S is the set of finite places of k. We also use $\mathbb{A}_k^f$ to denote $\mathbb{A}_{k,S}$ when S is the set of finite places, and we believe this does not lead to any confusion.

We use loc to denote the map from the version of some set over a global field to the adelic version (or the product over all places of the local version), e.g. for Galois cohomology or homotopy fixed points. We write loc_S when we use the $\mathbb{A}_{k,S}$ version, and we write loc_v to denote the map from the version over the global field k to that over k_v.

If X is a variety over a local field k_v, we let $X(k_v)_\bullet$ denote the set of connected components of $X(k_v)$ under the v-adic topology, and we similarly set $X(\mathbb{A}_{k,S})_\bullet = \prod_{v \in S}' (X(k_v)_\bullet, X(\mathcal{O}_v)_\bullet)$.

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2. Obstructions to Rational Points

2.1. **Generalized Obstructions.** Let k be a global field.

Definition 2.1. Let ω be a subfunctor of the functor $X \mapsto X(\mathbb{A}_k)$ from k-varieties to sets. We write $X(\mathbb{A}_k)^\omega$ instead of $\omega(X)$. We say that ω is a *generalized obstruction (to the local-global principle)* if $X(k) \subseteq X(\mathbb{A}_k)^\omega$ for every k-variety X.

If S is a subset of the places of k, we define $X(\mathbb{A}_{k,S})^\omega$ to be the projection of $X(\mathbb{A}_k)^\omega$ from $X(\mathbb{A}_k)$ to $X(\mathbb{A}_{k,S})$.

We note the trivial but important lemmas:

Lemma 2.2. *Let ω and ω' be two generalized obstructions. If $X(\mathbb{A}_k)^\omega \subseteq X(\mathbb{A}_k)^{\omega'}$, then $X(\mathbb{A}_{k,S})^\omega \subseteq X(\mathbb{A}_{k,S})^{\omega'}$.*

Proof. The containment $X(\mathbb{A}_k)^\omega \subseteq X(\mathbb{A}_k)^{\omega'}$ implies that the containment remains true when we project from $X(\mathbb{A}_k)$ to $X(\mathbb{A}_{k,S})$. $\square$

Lemma 2.3. *Let S, S' be two nonempty sets of places of k. Then $X(\mathbb{A}_{k,S})^\omega = \emptyset$ if and only if $X(\mathbb{A}_{k,S'})^\omega = \emptyset$.*

Proof. It suffices to suppose S' is the set of all places of k. The set $X(\mathbb{A}_{k,S})^\omega$ is the projection of $X(\mathbb{A}_{k,S'})^\omega$ from $X(\mathbb{A}_{k,S'})$ to $X(\mathbb{A}_{k,S})$, so one is empty if and only if the other is. $\square$

Lemma 2.4. *The association $X \mapsto X(\mathbb{A}_{k,S})^\omega$ is a subfunctor of the covariant functor from k-varieties to sets represented by* $\operatorname{Spec}(\mathbb{A}_{k,S})$.

Proof. Let $f\colon X \to Y$ be a map of k-varieties. It suffices to show that f maps $X(\mathbb{A}_{k,S})^\omega$ into $Y(\mathbb{A}_{k,S})^\omega$. Let $\alpha \in X(\mathbb{A}_{k,S})^\omega$. Then α is the projection of some $\alpha' \in X(\mathbb{A}_k)^\omega$. But $f(\alpha')$ is in $Y(\mathbb{A}_k)^\omega$ and projects to $f(\alpha)$, so $f(\alpha) \in Y(\mathbb{A}_{k,S})^\omega$. $\square$

2.1.1. *Very Strong Approximation.*

Definition 2.5. If k is a global field, S is a nonempty set of places of k, and ω is a generalized obstruction, then we call (ω, S, k) an *obstruction datum.*

When S is the set of finite places of k, we also write (ω, f, k) in this case.

We will sometimes leave out S and write (ω, k), in which case *we understand S to be the set of all places of k.* We often write (ω, S) when k is understood.

Definition 2.6. Let (ω, S, k) be an obstruction datum. We say that a variety X/k *is* or *satisfies very strong approximation* (VSA) for (ω, S) if $X(k) = X(\mathbb{A}_{k,S})^\omega$.

Lemma 2.7. *Let $W \subseteq X$ be be a (locally closed) subvariety of X over k. If X is VSA for (ω, S, k), then so is W.*

Proof. Let $\alpha \in W(\mathbb{A}_{k,S})^\omega$. Then $\alpha \in X(\mathbb{A}_{k,S})^\omega$ by functoriality. Since X is VSA, $\alpha \in X(k)$; i.e., $\operatorname{Spec} \mathbb{A}_{k,S} \xrightarrow{\alpha} W$ factors through a map $\operatorname{Spec} k \to X$.

Let $v \in S$. Then the v-component α_v of α is a k_v-point of X coming from a k-point of X. But because $\alpha \in W(\mathbb{A}_{k,S})$, this implies that $\alpha_v \in W(k_v)$. But if it is a k-point as a point of X, then we also have $\alpha_v \in W(k)$. $\square$

Remark 2.8. It is the need to prove Lemma 2.7 that prevents us from replacing the set of points at the real places by the set of their connected components, as the map from $W(\mathbb{A}_k)_\bullet$ to $X(\mathbb{A}_k)_\bullet$ need not be injective.

Lemma 2.9. *Suppose X and Y are k-varieties that are VSA for (ω, S, k). Then so is $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)(\mathbb{A}_{k,S})^\omega$. By functoriality of ω applied to the projections $X \times Y \to X, Y$, we have $x \in X(\mathbb{A}_{k,S})^\omega$ and $y \in Y(\mathbb{A}_{k,S})^\omega$. By VSA for X and Y, this implies $x \in X(k)$ and $y \in Y(k)$, so $(x, y) \in (X \times Y)(k)$. $\square$

2.2. **Functor Obstructions.** Let k be a global field. Following the formalism in Section 8.1 of [Poo17], let F be a contravariant functor from schemes over k to sets (a.k.a. a presheaf of sets on the category of k-schemes). For X a k-scheme and $A \in F(X)$, we have the following commutative diagram

$$\begin{tikzcd} X(k) \arrow[r] \arrow[d] & X(\mathbb{A}_k) \arrow[d] \\ F(k) \arrow[r, "\mathrm{loc}"] & \prod_v F(k_v), \end{tikzcd}$$

where the vertical arrows denote pullback of A from X to k or k_v. We define $X(\mathbb{A}_k)^A$ as the subset of $X(\mathbb{A}_k)$ whose image in the lower right object is in the image of loc. We then define the obstruction set

$$X(\mathbb{A}_k)^F = \bigcap_{A \in F(X)} X(\mathbb{A}_k)^A.$$

If $\mathcal{F}$ is a collection of such functors F, we define

$$X(\mathbb{A}_k)^{\mathcal{F}} = \bigcap_{F \in \mathcal{F}} X(\mathbb{A}_k)^F.$$

Lemma 2.10. *The functor $X \mapsto X(\mathbb{A}_k)^{\mathcal{F}}$ is a generalized obstruction.*

Proof. It is clear that $X(k) \subseteq X(\mathbb{A}_k)^{\mathcal{F}}$, so it suffices to verify functoriality, which is a simply diagram chase.

□

Lemma 2.11. *Let $\mathcal{F}' \subseteq \mathcal{F}$. Then $X(\mathbb{A}_{k,S})^{\mathcal{F}} \subseteq X(\mathbb{A}_{k,S})^{\mathcal{F}'}$ for all X.*

Proof. If $\alpha \in X(\mathbb{A}_k)^{\mathcal{F}}$, then $\alpha \in X(\mathbb{A}_k)^F$ for all $F \in \mathcal{F}$, hence for all $F \in \mathcal{F}'$. But this implies $\alpha \in X(\mathbb{A}_k)^{\mathcal{F}'}$, i.e. $X(\mathbb{A}_k)^{\mathcal{F}} \subseteq X(\mathbb{A}_k)^{\mathcal{F}'}$. The result for general S follows by Lemma 2.2. □

2.3. **Descent Obstructions.** Let G be a group scheme over k. We let F_G denote the contravariant functor from k-schemes to sets for which $F_G(X)$ is the set of isomorphism classes of G-torsors for the fppf topology over X. By Theorem 6.5.10(i) of [Poo17], we have $F_G(X) = H^1_{\mathrm{fppf}}(X, G)$ when G is affine. If G is also smooth, it is isomorphic to the étale cohomology, denoted $H^1(X, G)$ (c.f. Notation and Conventions).

Let $\mathcal{G}$ be a subset of the set of isomorphism classes of finite type group schemes over k. For $\mathcal{F} = \{F_G\}_{[G] \in \mathcal{G}}$, we set

$$X(\mathbb{A}_k)^{\mathcal{G}} = X(\mathbb{A}_k)^{\mathcal{F}}.$$

Usually, $\mathcal{G}$ will consist only of algebraic groups over k, i.e., finite type separated group schemes over k.

Definition 2.12. For $\mathcal{G}$ the set of smooth affine algebraic groups over k, we call this the *descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{descent}}$ (following [Poo17]).

Definition 2.13. For $\mathcal{G}$ the set of finite smooth (étale) algebraic groups over k, we call this the *finite descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{f-cov}}$ (following [Sto07],[Sko09]).

Definition 2.14. For $\mathcal{G}$ the set of finite solvable smooth algebraic groups over k, we call this the *finite solvable descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{f-sol}}$ (following [Sto07]).

Definition 2.15. For $\mathcal{G}$ the set of finite commutative smooth algebraic groups over k, we call this the *finite abelian descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{f-ab}}$ (following [Sto07]).

Remark 2.16. By the equivalence between (i) and (i') in Theorem 2.1 of [HS12], it suffices to take only finite *constant* algebraic groups over k in Definition 2.13.

When we wish to refer to the obstruction and specify a specific S, we write $(\mathrm{f}-\mathrm{ab}, S, k)$, etc.

Proposition 2.17. *We have* $X(\mathbb{A}_k)^{\mathrm{descent}} \subseteq X(\mathbb{A}_k)^{\mathrm{f-cov}} \subseteq X(\mathbb{A}_k)^{\mathrm{f-sol}} \subseteq X(\mathbb{A}_k)^{\mathrm{f-ab}}$.

Proof. This follows by Lemma 2.11 because the corresponding collections of isomorphism classes of group schemes become smaller as we progress from descent to $\mathrm{f}-\mathrm{cov}$ to $\mathrm{f}-\mathrm{sol}$ to $\mathrm{f}-\mathrm{ab}$. □

Corollary 2.18. *For fixed k and S, VSA for* $\mathrm{f}-\mathrm{ab}$ *implies VSA for* $\mathrm{f}-\mathrm{sol}$*, which implies VSA for* $\mathrm{f}-\mathrm{cov}$*, which implies VSA for* descent.

Proof. This follows immediately from Proposition 2.17 and Lemma 2.2. □

2.3.1. *Alternative Description in Terms of Images of Adelic Points.* Let G be a smooth affine algebraic group over k, and let $Z \in F_G(X)$ be given by, with abuse of notation, $f : Z \to X$. If $\tau \in F_G(k)$, one can define the twist $f^\tau \colon Z^\tau \to X$ as in Example 6.5.12 of [Poo17].

By Theorem 8.4.1 of [Poo17], if $x \in X(k)$, then $x \in f^\tau(Y^\tau(k))$ if and only if τ is the pullback of Z under $x\colon \operatorname{Spec} k \to X$.

It follows that

$$X(k) = \bigcup_{\tau \in F_G(k)} f^\tau(Z^\tau(k)) \subseteq \bigcup_{\tau \in F_G(k)} f^\tau(Z^\tau(\mathbb{A}_{k,S})).$$

Proposition 2.19. *We have*

$$\bigcup_{\tau \in F_G(k)} f^\tau(Z^\tau(\mathbb{A}_k)) = X(\mathbb{A}_k)^Z.$$

Proof. Suppose $\alpha \in X(\mathbb{A}_k)^Z$. We think of this point as a map $\alpha\colon \operatorname{Spec} \mathbb{A}_k \to X$, and we know by definition that the pullback of Z along this map is in the image of

$$H^1(k, G) \xrightarrow{\mathrm{loc}} \prod_v H^1(k_v, G),$$

say of $\tau \in H^1(k, G)$. One can easily check that twisting commutes with pullback (from k to $\mathbb{A}_k$). Thus the proof of 8.4.1 of [Poo17] tells us that the pullback of Z^τ under this adelic point is now the trivial element of $\prod_v H^1(k_v, G)$. But this means that the fiber of Z^τ over α is a trivial torsor and thus contains an adelic point; i.e., $\alpha \in f^\tau(Z^\tau(\mathbb{A}_k))$.

Conversely, suppose $\alpha \in f^\tau(Z^\tau(\mathbb{A}_k))$ for some $\tau \in H^1(k, G)$. Then for each v, we have $\alpha_v \in f^\tau(Z^\tau(k_v))$, so Theorem 8.4.1 tells us that the pullback of Z under $\alpha_v\colon \operatorname{Spec} k_v \to X$

is $\mathrm{loc}_v(\tau)$. But this implies that the pullback of Z under α is in the image of loc; i.e., $\alpha \in X(\mathbb{A}_k)^Z$. $\square$

2.4. **Comparison with Brauer and Homotopy Obstructions.** There is the Brauer-Manin obstruction set, $X(\mathbb{A}_{k,S})^{\mathrm{Br}}$, which is the obstruction for $F(-) = H^2(-, \mathbb{G}_m)$.

There is also the étale Brauer-Manin obstruction

$$X(\mathbb{A}_k)^{\text{ét,Br}} = \bigcap_{\substack{\text{finite étale } G \\ \text{all } G-\text{torsors} f : Y \to X}} \bigcup_{\tau \in H^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathrm{Br}}).$$

By Section 2.3.1, this is contained in $X(\mathbb{A}_{k,S})^{\mathrm{f-cov}}$.

A series of obstructions is defined in [HS13] when k is a number field:

$$X(\mathbb{A}_k)^h \subseteq \cdots \subseteq X(\mathbb{A}_k)^{h,2} \subseteq X(\mathbb{A}_k)^{h,1} \subseteq X(\mathbb{A}_k)$$

and

$$X(\mathbb{A}_k)^{\mathbb{Z}h} \subseteq \cdots \subseteq X(\mathbb{A}_k)^{\mathbb{Z}h,2} \subseteq X(\mathbb{A}_k)^{\mathbb{Z}h,1} \subseteq X(\mathbb{A}_k)$$

such that $X(\mathbb{A}_k)^{h,n} \subseteq X(\mathbb{A}_k)^{\mathbb{Z}h,n}$ for all n, $X(\mathbb{A}_k)^{\mathbb{Z}h} = \bigcap_n X(\mathbb{A}_k)^{\mathbb{Z}h,n}$, and $X(\mathbb{A}_k)^h = \bigcap_n X(\mathbb{A}_k)^{h,n}$. The definition of $X(\mathbb{A}_k)^h$ will be discussed in more detail in Section 8.2. When discussing obstruction data, we write (h, S, k) and $(\mathbb{Z}h, S, k)$.

By [HS13, Theorem 9.136] and Lemma 2.2, we have for a smooth geometrically connected variety X that:

$$\begin{aligned} X(\mathbb{A}_{k,S})^h &= X(\mathbb{A}_{k,S})^{\text{ét,Br}} \\ X(\mathbb{A}_{k,S})^{\mathbb{Z}h} &= X(\mathbb{A}_{k,S})^{\mathrm{Br}} \\ X(\mathbb{A}_{k,S})^{h,1} &= X(\mathbb{A}_{k,S})^{\mathrm{f-cov}} \\ X(\mathbb{A}_{k,S})^{\mathbb{Z}h,1} &= X(\mathbb{A}_{k,S})^{\mathrm{f-ab}} \end{aligned}$$

In particular, we have the two inclusions:

$$X(\mathbb{A}_{k,S})^{\text{ét,Br}} \subseteq X(\mathbb{A}_{k,S})^{\mathrm{Br}} \subseteq X(\mathbb{A}_{k,S})^{\mathrm{f-ab}}$$
$$X(\mathbb{A}_{k,S})^{\text{ét,Br}} \subseteq X(\mathbb{A}_{k,S})^{\mathrm{f-cov}} \subseteq X(\mathbb{A}_{k,S})^{\mathrm{f-ab}}.$$

3. Statement of Our Main Abstract Results

For the rest of this section, we assume that k is a number field.

The main reason to care about these obstructions is that they help prove that a variety X/k has no rational points. The most powerful obstruction currently known is $X(\mathbb{A}_k)^{\text{ét,Br}} = X(\mathbb{A}_k)^{\mathrm{descent}}$. But, as stated in the introduction, even in this case, there is a variety X with

$\emptyset = X(k) \subseteq X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$, as found in [Poo10]. In the method of proof that $X(k) = \emptyset$, it is clear that the étale Brauer-Manin obstruction and its avatars still appear, but they are applied separately to different *pieces* of X. From this point of view, it's natural to ask the following question:

Question 3.1. Let X be a k-variety with $X(k) = \emptyset$. Does there exist a finite open cover or stratification of X for which each constituent part has empty étale-Brauer set? More strongly, is the same true for any of the other obstruction sets from Section 2.3?

If true, this proves that $X(k) = \emptyset$, because if each constituent part has no rational points, then X does not.

One could ask for the stronger statement that each constituent part satisfies VSA:

Question 3.2. Let X be a k-variety and (ω, S, k) an obstruction datum. Does there exist a finite open cover or stratification of X for which each constituent part satisfies VSA for (ω, S, k)?

In this paper, we obtain the following result.

Theorem 3.3 (Corollary 5.4 and Corollary 5.9)**.** *The answer to Question 3.2 is yes for* $(\mathrm{f}-\mathrm{ab}, f, k)$ *when k is a imaginary quadratic or totally real number field.*

This is actually a very strong result when it applies, because $\mathrm{f}-\mathrm{ab}$ is weaker than the Brauer-Manin, finite descent, or étale Brauer-Manin obstruction.

To see whether we expect this to hold for all number fields, we prove the following conditional result:

Theorem 3.4 (Corollary 6.27)**.** *Assuming Grothendieck's section conjecture (6.25), the answer to Question 3.2 is yes for* $(\mathrm{f}-\mathrm{cov}, S, k)$*, where S is a nonempty set of finite places.*

This leaves only the question for $(\mathrm{f}-\mathrm{ab}, f, k)$ and arbitrary number fields k. Here, we can only make a conjecture:

Conjecture 3.5 (Conjecture 7.2)**.** *For any number field k, the answer to Question 3.2 is true for* $(\mathrm{f}-\mathrm{ab}, f, k)$

The primary method of proof consists in reducing the problem to proving VSA for an open subset of $\mathbb{P}^1$, a condition we call $A(\omega, f, k)$. Theorem 4.3 then tells us in general that this answers Question 3.2 (and therefore Question 3.1) affirmatively. This is described in Part 2.

In Part 3, we introduce homotopy sections and analyze what happens to the section conjecture and to the étale Brauer-Manin obstruction in fibrations. We derive an important result about the behavior of the étale-Brauer obstruction in fibrations, Theorem 9.11. Our analysis of the section conjecture in fibrations also allows us to reprove Theorem 3.4 in Corollary 10.17 . While this might seem logically unnecessary if the theorem is already proven, the second proof provides a possibly different open cover than that of the first proof.

In Part 4, we review the original example of [Poo10] in terms of our results. The main results of that section are summarized in the introduction, and some of them are summarized at the beginning of Section 12. We also introduce the notion of quasi-torsors (Definition 13.1), which leads us to propose the following conjecture:

Conjecture 3.6 (Conjecture 13.4)**.** *For any number field k and variety X/k with $X(k) = \emptyset$, does there exist a stratification*[1] $X = \coprod_i X_i$ *and for each i, a quasi-torsor Y_i over the closure $\overline{X_i}$ of X_i such that Y_i restricts to a torsor over X_i and $Y_i^\sigma(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ for all twists Y_i^σ of Y_i.*

3.1. **A New Obstruction.** As described in the introduction, our results were motivated by the search for a new obstruction. Here, we explain how the aforementioned results may be rephrased as results about a new obstruction.

Let $\mathscr{X} = \{X_i\}$ be a finite collection of locally closed subvarieties of X whose set-theoretic union is X (e.g., a stratification or open cover) and (ω, S, k) an obstruction datum. We define the following two sets, where unions take place in $X(\mathbb{A}_{k,S})$:

$$X(\mathbb{A}_{k,S})^{\mathscr{X},\omega} = \bigcup_i X_i(\mathbb{A}_{k,S})^\omega$$

Functoriality then gives us the inclusion:

$$X(k) \subseteq X(\mathbb{A}_{k,S})^{\mathscr{X},\omega} \subseteq X(\mathbb{A}_{k,S})^\omega.$$

Finally, we must consider the case where $\mathscr{C}$ is a *collection of finite collections of locally closed subvarieties*. For example, $\mathscr{C}$ might be the collection of all open covers of X, denoted by $\mathcal{OPEN}$, or the collection of all finite stratifications, denoted by $\mathcal{STRAT}$.

We then define

$$X(\mathbb{A}_{k,S})^{\mathscr{C},\omega} = \bigcap_{\mathscr{X} \in \mathscr{C}} X(\mathbb{A}_{k,S})^{\mathscr{X},\omega}.$$

In fact, $\mathcal{OPEN}$ and $\mathcal{STRAT}$ are really rules that assign such a $\mathscr{C}$ to each X/k.

We can then rephrase Theorem 4.3:

[1] We are defining a finite stratification of X as a finite partially-ordered index set I along with a locally closed subset $S_i \subseteq X$ for every $i \in I$ such that X is the disjoint union of all S_i, and the closure of any given S_i in X is the union $\bigcup_{j \leq i} S_j$.

Theorem 3.7. *If $A(\omega, S, k)$ holds, then for any X/k, there is a finite Zariski open cover $\mathscr{X}$ such that*

$$X(k) = X(\mathbb{A}_{k,S})^{\mathscr{X},\omega}.$$

As a result,

$$X(k) = X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}.$$

Just for completeness, we note a couple of basic properties of these types of obstructions.

Proposition 3.8. *Let (ω, S, k) be an obstruction datum and $f : X \to Y$ a map of k-schemes. Then f maps $X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ into $Y(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ and $X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$ into $Y(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$, respectively.*

Proof. Let $\alpha \in X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$, and let $\{Y_i\}_i$ be a finite open cover of Y. Then $\{f^{-1}(Y_i)\}_i$ is a finite open cover of X, so $\alpha \in f^{-1}(Y_i)(\mathbb{A}_{k,S})^{\omega}$. Then functoriality tells us that $f(\alpha) \in Y_i(\mathbb{A}_{k,S})^{\omega}$. As $\{Y_i\}_i$ was arbitrary, $f(\alpha) \in Y(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$.

The other part goes nearly word-for-word. Let $\alpha \in X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$, and let $\{Y_i\}_i$ be a finite stratification of Y. Then $\{f^{-1}(Y_i)\}_i$ is a finite stratification of X, so $\alpha \in f^{-1}(Y_i)(\mathbb{A}_{k,S})^{\omega}$. Then functoriality tells us that $f(\alpha) \in Y_i(\mathbb{A}_{k,S})^{\omega}$. As $\{Y_i\}_i$ was arbitrary, $f(\alpha) \in Y(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$. □

The following proposition explains why answering Question 3.2 positively for open covers rather than stratifications is more powerful:

Proposition 3.9. *For any (ω, S, k), we have*

$$X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega} \subseteq X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$$

Proof. Suppose $\alpha \in X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$. We must show that for every finite open cover $X = \bigcup_{i=1}^{n} X_i$, we have $\alpha \in X_i(\mathbb{A}_{k,S})^{\omega}$ for some i.

We build a stratification out of this open cover as follows. For $i = 1, \cdots, n$, we let

$$S_i = X_i \setminus \bigcup_{j<i} X_j.$$

Then $\{S_i\}$ forms a stratification, with $\overline{S_i} = \bigcup_{j \geq i} S_j$ for all i. Thus $\alpha \in S_i(\mathbb{A}_{k,S})^{\omega}$ for some i. But $S_i \subseteq X_i$, so functoriality tells us that $\alpha \in X_i(\mathbb{A}_{k,S})^{\omega}$. □

If one wishes to define $X(\mathbb{A}_k)^{\mathrm{new}}$ as in the introduction, the natural candidates are $X(\mathbb{A}_k)^{\mathcal{OPEN},\mathrm{f-ab}}$ and $X(\mathbb{A}_k)^{\mathcal{STRAT},\mathrm{f-ab}}$. The real power in our results, however, is that one need take only *a single open cover.*

Remark 3.10. It is interesting to consider the constructions in this subsection applied when ω is such that $X(\mathbb{A}_k)^\omega = X(\mathbb{A}_k)$ for all X. This is discussed in Appendix A.

Remark 3.11. This refinement of obstructions via open covers is related to the notion of cosheafification. This connection is discussed in Appendix B.

3.2. Relationship to the Birational Section Conjecture.

The reader might think that the ideas in this paper can be used to prove the birational section conjecture. The idea is this: let X be a proper smooth hyperbolic curve over $\mathbb{Q}$. If there is a birational Galois section (over $\mathbb{Q}$), then the p-adic birational section conjecture ([Koe05]) shows that there is a $\mathbb{Q}_p$-point for every place p. This provides an adelic point of X whose associated birational section comes from a global section. In other words, it is in the finite descent set of every open subset of our curve, and by the results here, it is rational.

The flaw in this argument is that an adelic point on X might not be an adelic point of *any proper open subvariety*. In the language of Appendix B, this is the fundamental reason why $\mathcal{F}_{\mathbb{A}_k^f}$ is not a cosheaf. Therefore, one would have to use the fact that its birational section is comes from a birational section over a global field in a deeper way to make such an argument work. All of this is in fact implicit in [Sti15] and Theorems 3.2, 3.3, 4.2, and 4.3 of [HS12].

Part 2. Main Result via Embeddings

4. General Setup

We set up the basic formalism for the various unconditional and conditional results proven via embeddings.

Definition/Theorem 4.1. *Let (ω, S, k) be an obstruction datum. The following statements are equivalent:*

(i) *There is a nonempty open k-subscheme of $\mathbb{P}^1_k$ that is VSA for (ω, S, k).*
(ii) *There is a nonempty open k-subscheme of $\mathbb{A}^1_k$ that is VSA for (ω, S, k).*
(iii) *For each positive integer n, there is a nonempty open k-subscheme of $\mathbb{A}^n_k$ that is VSA for (ω, S, k).*
(iv) *For each positive integer n, there is a nonempty open k-subscheme of $\mathbb{P}^n_k$ that is VSA for (ω, S, k).*

If this is the case, we say that the property $A(\omega, S, k)$ is true.

Proof. If (i) holds, let this open subscheme be U. Then $V := U \cap \mathbb{A}^1_k$ is nonempty because any nonempty open is dense, and V is then VSA by Lemma 2.7, so (ii) holds.

If (ii) holds, then V^n is a nonempty open subscheme of $\mathbb{A}_k^n$ for all n, which is VSA by Lemma 2.9, so (iii) holds.

If (iii) holds, then the VSA nonempty open subscheme of $\mathbb{A}_k^n$ is also a VSA nonempty open subscheme of $\mathbb{P}_k^n$, so (iv) holds.

Finally, (iv) implies (i) by setting $n = 1$. $\square$

We note that $\mathrm{PGL}_2(k)$ acts on $\mathbb{P}_k^1$ by k-automorphisms, and hence $(\mathrm{PGL}_2(k))^n$ acts on $(\mathbb{P}_k^1)^n$ in the same way.

Lemma 4.2. *Let k be an infinite field and x a closed point of $(\mathbb{P}_k^1)^n$. Then the orbit of x under $(\mathrm{PGL}_2(k))^n$ is Zariski dense in $(\mathbb{P}_k^1)^n$.*

Proof. We choose an algebraic closure $\overline{k}$ of k and a point $\overline{x}$ of $(\mathbb{P}^1(\overline{k}))^n$ representing x. We equivalently wish to show that for any open $U \subseteq (\mathbb{P}_k^1)^n$, the orbit of $\overline{x}$ under $(\mathrm{PGL}_2(k))^n$ intersects $U(\overline{k})$.

We first prove this for $n = 1$. The points $(\overline{x} + a)_{a \in k}$ of $\mathbb{P}^1$ are distinct. As there are infinitely many of them, and $\mathbb{P}^1$ is one-dimensional, they are Zariski dense. But these points are all in the orbit of $\overline{x}$, so this orbit is Zariski dense.

Let $\overline{x} = (x_1, \cdots, x_n)$ with $x_i \in \mathbb{P}_k^1(\overline{k})$, and let S_i be the $\mathrm{PGL}_2(k)$-orbit of x_i. The previous paragraph implies that each S_i is Zariski-dense in $\mathbb{P}^1(\overline{k})$, so $\prod_{i=1}^{n} S_i$ is dense in $(\mathbb{P}^1(\overline{k}))^n$.

A less detailed and less general version of this argument is given in Lemma 6.3 of [SS16].

$\square$

Theorem 4.3. *Suppose that $A(\omega, S, k)$ holds. Let X be a k-variety. Then there exists a finite affine open cover $X = \bigcup_i V_i$ such that V_i is VSA for (ω, S, k) for every i.*

Proof. Let x be any closed point. By the definition of a scheme, there is an affine open neighborhood of X containing x, so we can assume that X is affine.

We now embed X into $\mathbb{A}^n$ for some sufficiently large n. As $\mathbb{A}^1$ is an open subscheme of $\mathbb{P}^1$, we have an open inclusion $\mathbb{A}^n = (\mathbb{A}^1)^n \hookrightarrow (\mathbb{P}^1)^n$, so we get an embedding $\phi : X \hookrightarrow (\mathbb{P}^1)^n$.

Assuming $A(\omega, S, k)$, there is an open subset $U \subseteq (\mathbb{P}^1)^n$ that is VSA for (ω, S, k). By Lemma 4.2, there is a k-automorphism g of $(\mathbb{P}^1)^n$ sending $\phi(x)$ into U. Then U contains $g(\phi(x))$. But then $U \cap g(\phi(X))$ is an open subscheme of X containing x and is a locally closed subscheme of U. It is therefore VSA for (ω, S, k) by Lemma 2.7. Choose an affine open neighborhood V_x of x in $U \cap g(\phi(X))$, and V_x is again VSA.

We do this for every x, and we obtain an affine open cover $X = \bigcup_{x \in X} V_x$ such that V_x is VSA for (ω, S, k) for every x. As varieties are quasi-compact, this has a finite subcover, which proves the theorem. $\square$

Corollary 4.4. *Let X be a k-variety with $X(k) = \emptyset$, and suppose $A(\omega, S, k)$. Then there exists a finite open cover of X such that each constituent open set has empty obstruction set for (ω, S, k). The same is true for any other nonempty set S' of places of k.*

Proof. Let $\bigcup_i V_i$ be an open cover as in Theorem 4.3. As $X(k) = \emptyset$, we have $V_i(k) = \emptyset$. By VSA, this implies $V_i(\mathbb{A}_{k,S})^\omega = \emptyset$.

By Lemma 2.3, we also have $V_i(\mathbb{A}_{k,S'})^\omega = \emptyset$ for every i. $\square$

5. Embeddings of Varieties into Tori

For the rest of Section 5, we assume that k is a number field.

Lemma 5.1. *Let T be an algebraic torus over k, let S be a nonempty set of places of k, and let $\alpha \in T(\mathbb{A}_{k,S})$. The first statement implies the second, and if S consists only of finite places of k, the second implies the third:*

(i) $\alpha \in T(\mathbb{A}_{k,S})^{\mathrm{f-ab}}$.
(ii) For every $n \in \mathbb{Z}_{\geq 0}$, there exists $a_n \in T(k)$ and $\beta_n \in T(\mathbb{A}_{k,S})$ for which $\alpha = a_n(\beta_n)^n$.
(iii) $\alpha \in \overline{T(k)}$, with the closure taken in $T(\mathbb{A}_{k,S})$.

Proof. (i) $\implies$ (ii): For every n, there is a standard torsor $T \xrightarrow{[n]} T$ under the n-torsion scheme $T[n]$ of T given by the nth power map $T \to T$. The Kummer map is the map sending $x \in T(k)$ to the pullback of $T \xrightarrow{[n]} T$ under x, the result of which is a torsor under $T[n]$ over k. In this case, it is given by the boundary map $T(k) \to H^1(k, T[n])$ in Galois cohomology coming from the short exact sequence $0 \to T[n] \to T \to T \to 0$ of étale sheaves. In particular, the image of the Kummer map is canonically $T(k)/nT(k)$. The same holds with k_v in place of k, and all maps respect the inclusions $k \to k_v$. If $\alpha \in T(\mathbb{A}_{k,S})^{\mathrm{f-ab}}$, it must be in the image of

$$T(k)/nT(k) \to \prod_{v \in S} T(k_v)/nT(k_v),$$

which amounts to saying that $\alpha = a_n(\beta_n)^n$ for $\beta_n \in \prod_{v \in S} T(k_v)$. As $a_n, \alpha \in T(\mathbb{A}_{k,S})$, so is β_n.

(ii) $\implies$ (iii): Let K be any open subgroup of $T(\mathbb{A}_{k,S})$. By Theorem 5.1 of [Bor63] (also c.f. [Con06])[2], $T(\mathbb{A}_{k,S})/T(k)K$ is finite, say of order h. We know that $\alpha = a_h(\beta_h)^h$. But $(\beta_h)^h \in T(k)K$, and therefore so is α. In other words, the coset αK contains an element of $T(k)$. The set of all such K and their cosets form a basis for the topology of $T(\mathbb{A}_{k,S})$ because S contains only finite places. It follows that every open neighborhood of α contains an element of $T(k)$, or $\alpha \in \overline{T(k)}$.

□

Lemma 5.2. *Let T be an algebraic torus over k and K an open subgroup of $T(\mathbb{A}_{k,S})$ such that $T(k) \cap K$ is finite. Then $T(k)$ is closed in $T(\mathbb{A}_{k,S})$. For example, this happens for $S = f$ if T has a model $\mathcal{T}$ over $\mathcal{O}_k$ with $\mathcal{T}(\mathcal{O}_k)$ finite.*

Proof. As $T(k)$ is a subgroup, each coset of K also has finite intersection with $T(k)$. The topological space $T(\mathbb{A}_{k,S})$ is the disjoint union of the cosets of K. This space is T_0, so finite sets are closed. Thus $T(k)$ is closed in each coset of K. This implies that it is closed in all of $T(\mathbb{A}_{k,S})$.

By the definition of the adelic topology, the subgroup

$$\mathcal{T}(\widehat{\mathcal{O}_k}) := \prod_v \mathcal{T}(\mathcal{O}_v) \subseteq T(\mathbb{A}_k^f)$$

is open. We can then conclude because $\mathcal{T}(\mathcal{O}_k) = \mathcal{T}(k) \cap \mathcal{T}(\widehat{\mathcal{O}_k})$. □

5.1. **The Result When k is Imaginary Quadratic.** In this subsection, we prove $A(\mathrm{f}-\mathrm{ab}, f, k)$ when k has finitely many units.

Remark 5.3. By Dirichlet's unit theorem, k has finitely many units if and only if k is $\mathbb{Q}$ or imaginary quadratic.

Corollary 5.4. *For k as above, $\mathbb{G}_m$ is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$. In particular, $A(\mathrm{f}-\mathrm{ab}, f, k)$ holds.*

Proof. That k has finitely many units means that $\mathbb{G}_m(\mathcal{O}_k)$ is finite. By Lemma 5.2, this implies that $\mathbb{G}_m(k)$ is closed in $\mathbb{G}_m(\mathbb{A}_k^f)$. By Lemma 5.1, this implies that $\mathbb{G}_m(k) = \mathbb{G}_m(\mathbb{A}_k^f)^{\mathrm{f-ab}}$; i.e., $\mathbb{G}_m$ is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$. □

5.2. **The Result When k is Totally Real.** We now prove $A(\mathrm{f}-\mathrm{ab}, f, F)$ when k is totally real. The material in this subsection is inspired by [Sti15].

In fact, we can choose the subscheme of $\mathbb{P}^1_k$ as in Definition/Theorem 4.1(i) to be the complement in $\mathbb{P}^1_k$ of the vanishing scheme of any quadratic polynomial over k with totally negative discriminant.

[2] In fact, [Bor63] covers only the case that $\mathbb{A}_k = \mathbb{A}_{k,S}$, but the result holds in general by considering the continuous projection $\mathbb{A}_k \to \mathbb{A}_{k,S}$.

Definition 5.5. For k a totally real number field and E/k a totally imaginary quadratic extension, we define the *norm one torus relative to E/k*

$$T = T_{E/k} = \ker(\mathrm{N}_{E/k}\colon \mathrm{Res}_{E/k}\mathbb{G}_m \to \mathbb{G}_m),$$

which is a group scheme over k.

Proposition 5.6. *Let $\alpha \in E$ with minimal polynomial $t^2 + bt + c$ over k. Let U be the complement in $\mathbb{P}^1_k$ (with coordinate t) of the vanishing locus of t^2+bt+c. Then U is isomorphic to T.*

Proof. As $E = k(\alpha)$, and the norm of $x - y\alpha$ is $x^2 + bxy + cy^2$, we can express T as $\operatorname{Spec} k[x, y]/(x^2 + bxy + cy^2 - 1)$. This has projective closure $\mathrm{Proj}(k[x, y, z]/(x^2 + bxy + cy^2 - z^2))$, which is a smooth projective conic. This conic has a point $(x, y, z) = (1, 0, 1)$, so it is isomorphic to $\mathbb{P}^1_k$. The complement of T is given by $0 = z^2 = x^2 + bxy + cy^2$, which corresponds to the vanishing locus of $t^2 + bt + c$ by setting $t = x/y$. □

Proposition 5.7. *For some integral model $\mathcal{T}$ of T, the set $\mathcal{T}(\mathcal{O}_k)$ is finite. Thus T is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$ by Lemma 5.2.*

Proof. The torus T has an integral model $\mathcal{T} = \ker(\mathrm{N}_{\mathcal{O}_E/\mathcal{O}_k}\colon \mathrm{Res}_{Oc_E/\mathcal{O}_k}\mathbb{G}_m \to \mathbb{G}_m)$. Thus $\mathcal{T}(\mathcal{O}_k)$ is the kernel of $\mathrm{N}_{\mathcal{O}_E/\mathcal{O}_k}\colon \mathbb{G}_m(\mathcal{O}_E) \to \mathbb{G}_m(\mathcal{O}_k)$.

The composition $\mathbb{G}_m(\mathcal{O}_k) \hookrightarrow \mathbb{G}_m(\mathcal{O}_E) \xrightarrow{\mathrm{N}_{\mathcal{O}_E/\mathcal{O}_k}} \mathbb{G}_m(\mathcal{O}_k)$ is $x \mapsto x^2$. Its image in the finitely generated abelian group $\mathbb{G}_m(\mathcal{O}_k)$ therefore has full rank, and therefore so does the image of $N\colon \mathbb{G}_m(\mathcal{O}_E) \to \mathbb{G}_m(\mathcal{O}_k)$. But E and k have the same number of Archimedean places, so $\mathbb{G}_m(\mathcal{O}_E)$ and $\mathbb{G}_m(\mathcal{O}_k)$ have the same rank. But a map between finitely generated abelian groups of the same rank whose image has full rank must have finite kernel. Therefore $\mathcal{T}(\mathcal{O}_k)$ is finite. □

Remark 5.8. One may alternatively prove Proposition 5.7 by noting that $\mathcal{O}_K$ is discrete in $k \otimes \mathbb{R}$ and that $T(k \otimes \mathbb{R})$ is compact.

Corollary 5.9. *If U is the complement in $\mathbb{P}^1_k$ of the vanishing locus of a quadratic polynomial $t^2 + bt + c$ with totally negative discriminant, then U is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$. In particular, $A(\mathrm{f}-\mathrm{ab}, f, k)$ holds.*

Proof. Let E be the splitting field of $t^2 + bt + c$. By Proposition 5.6, U is isomorphic to $T_{E/k}$, so by Proposition 5.7, U is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$. □

6. Finite Descent and the Section Conjecture

We now prove $A(\mathrm{f}-\mathrm{cov}, f, k)$ for any number field k assuming the section conjecture for $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$.

6.1. **Grothendieck's Section Conjecture.**

6.1.1. *The Kummer Map.* For the remainder of this subsection, we let X denote a quasi-compact quasi-separated geometrically connected scheme over a field k. We fix a separable closure k_s of k, which produces a geometric point $\mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$ of k. We identify $G_k := \mathrm{Gal}(k_s/k)$ with the étale fundamental group of $\mathrm{Spec}\, k$ based at this point. As in [Poo17], X^s denotes the base change X_{k_s}.

Definition 6.1. Let $\overline{a} \in X(k_s)$, which we may also view as an element of $X^s(k_s)$ and $\mathrm{Spec}\, k(k_s)$. By [Sta17, Tag 0BTX], there is an exact sequence

$$1 \to \pi_1(X^s, \overline{a}) \to \pi_1(X, \overline{a}) \to G_k \to 1,$$

known as the *fundamental exact sequence*

Definition 6.2. The set of sections $\mathscr{S}_{\pi_1(X/k),\overline{a}}$ is defined to be the set of (multiplicative) sections of the surjection $\pi_1(X, \overline{a}) \to \pi_1(\mathrm{Spec}\, k, \overline{a})$ modulo the action of $\pi_1(X^s, \overline{a})$ by conjugation. If we fix a section, we get an action of G_k on $\pi_1(X^s, \overline{a})$, and $\mathscr{S}_{\pi_1(X/k),\overline{a}}$ is isomorphic to the nonabelian continuous Galois cohomology pointed-set $H^1(k, \pi_1(X^s, \overline{a}))$ (c.f. "Generalized Sections" in 1.2 of [Sti13]).

Fact 6.3. As in Definition 23 of [Sti13], the sets $\mathscr{S}_{\pi_1(X/k),\overline{a}}$ for different choices of $\overline{a}$ are in canonical bijection, so we may write $\mathscr{S}_{\pi_1(X/k)}$ without ambiguity.

If k'/k is a field extension and X a k-scheme, $\mathscr{S}_{\pi_1(X/k')}$ denotes $\mathscr{S}_{\pi_1(X_{k'}/k')}$.

Definition 6.4 (Profinite Kummer Map)**.** Given a point $b \in X(k)$, there is a unique geometric basepoint $\overline{b}$ lying over b and compatible with the basepoint of $\mathrm{Spec}\, k$. Then b induces a pointed map of schemes $\mathrm{Spec}\, k \to X$, which induces a map of fundamental groups $G_k \to \pi_1(X, \overline{b})$ compatible with the projection $\pi_1(X, \overline{b}) \to G_k$, and hence an element of $\mathscr{S}_{\pi_1(X/k),\overline{b}}$. By Fact 6.3, this gives us a well-defined map

$$\kappa = \kappa_{X/k} \colon X(k) \to \mathscr{S}_{\pi_1(X/k)},$$

which we call the *(profinite) Kummer map.*

As with Definition 6.2, $\kappa_{X/k'}$ denotes $\kappa_{X_{k'}/k'}$. As well, if S is a set of places of k, then we set $\kappa_S = \kappa_{X/\mathbb{A}_{k,S}} := \prod_{v \in S} \kappa_{X/k_v} \colon X(\mathbb{A}_{k,S}) \to \prod_{v \in S} X(hk_v)$, or $\kappa_{X/\mathbb{A}_k}$ when S is all places.

A map $f \colon X \to Y$ of quasi-compact quasi-separated geometrically connected k-schemes induces a map $f \colon \mathscr{S}_{\pi_1(X/k)} \to \mathscr{S}_{\pi_1(Y/k)}$ by choosing a compatible pair of geometric basepoints for X and Y (but is easily seen to be independent of that choice). This map is compatible under the Kummer map with $f \colon X(k) \to Y(k)$.

Remark 6.5. If we assume the existence of a (fixed) Galois-invariant basepoint $\overline{a}$, we can define $\kappa(b)$ as the class in $H^1(G_k, \pi_1(X^s, \overline{a}))$ of the G_k-equivariant torsor $\pi_1(X^s, \overline{a}, \overline{b})$.

Remark 6.6. In Section 8, we will reformulate the section conjecture in more homotopical language, which will have the advantage of obviating the need for basepoints.

Conjecture 6.7 (Grothendieck [Gro97])**.** *Let k be a number field and X/k a geometrically connected, smooth, proper curve of genus at least 2. Then the Kummer map $X(k) \to \mathscr{S}_{\pi_1(X/k)}$ is a bijection.*

Remark 6.8. This is false for all nontrivial abelian varieties, as shown in [CS15].

We will, however, need to extend this conjecture to the non-proper case, as in Chapter 18 of [Sti13]. To do so, we must introduce the notion of cuspidal sections.

6.1.2. *Cuspidal Sections.*

Definition 6.9. A *cuspidal datum* (X, C, k) is a smooth geometrically connected variety X over a field k and a subset $C \subseteq \mathscr{S}_{\pi_1(X/k)}$, called the set of *cuspidal sections*.

Definition 6.10. A cuspidal datum (X, C, k) is said to satisfy the *surjectivity in the section conjecture* if $\mathscr{S}_{\pi_1(X/k)} \setminus C \subseteq \kappa(X(k))$.

Definition 6.11. Let $\overline{a}$ be a geometric basepoint, and assume that the fundamental exact sequence for $(X, \overline{a})$ has a section s. We define the centralizer $Z(s)$ as the set of elements of $\pi_1(X^s, \overline{a})$ that commute in $\pi_1(X, \overline{a})$ with $s(g)$ for all $g \in G_k$. If two sections are conjugate, their centralizers are conjugate. In particular, the property of having trivial centralizer is a property of elements of $\mathscr{S}_{\pi_1(X/k)}$.

Remark 6.12. As in Definition 6.2, a section s gives an action of G_k on $\pi_1(X^s, \overline{a})$. The centralizer of s is then isomorphic to the cohomology group $H^0(k, \pi_1(X^s, \overline{a}))$.

Definition 6.13. A cuspidal datum (X, C, k) is said to satisfy the *injectivity in the section conjecture* if

(1) The map $\kappa\colon X(k) \to \mathscr{S}_{\pi_1(X/k)}$ is injective.
(2) The sets $\kappa(X(k))$ and C are disjoint.
(3) The centralizer of every element of $\kappa(X(k))$ is trivial

Remark 6.14. The reader might wonder what the last two conditions have to do with injectivity. Theorem 9.7 will show that this stronger definition of injectivity holds inductively in geometric fibrations, while the more naive version does not. We also note that this stronger version is known to hold for hyperbolic curves (Theorem 6.24).

Proposition 6.15. *Let k'/k be an extension of characteristic 0 fields and X as above. We choose a separable closure k'_s and an embedding of k_s in k'_s, which gives us a map $G_{k'} = \pi_1(\operatorname{Spec} k', k'_s) \to \pi_1(\operatorname{Spec} k, k'_s) \to \pi_1(\operatorname{Spec} k, k_s) = G_k$. Then there is a base change map*

$$\mathscr{S}_{\pi_1(X/k)} \to \mathscr{S}_{\pi_1(X/k')}$$

induced by precomposition with the map $G_{k'} \to G_k$ (defined by choosing a geometric basepoint for $X_{k'}$ but independent of basepoint). Furthermore, the diagram

$$\begin{tikzcd} X(k) \arrow[r] \arrow[d, "\kappa_{X/k}"'] & X(k') \arrow[d, "\kappa_{X/k'}"] \\ \mathscr{S}_{\pi_1(X/k)} \arrow[r] & \mathscr{S}_{\pi_1(X/k')} \end{tikzcd}$$

is commutative.

Proof. The base change map is Definition 27 of [Sti13]. Characteristic 0 is required to ensure that X_{k_s} and $X_{k'_s}$ have the same étale fundamental group. (Remark: This also works in positive characteristic if X is proper.)

Let $x \in X(k)$ and $\overline{x}$ an associated Galois-invariant basepoint of $X_{k'}$. We may apply $\pi_1(-, \overline{x})$ to the commutative diagram of schemes

$$\begin{tikzcd} \operatorname{Spec} k' \arrow[r] \arrow[d, "x"'] & \operatorname{Spec} k \arrow[d, "x"] \\ X_{k'} \arrow[r] & X \end{tikzcd}$$

to obtain the commutative diagram of profinite groups

$$\begin{tikzcd} G_{k'} \arrow[r] \arrow[d, "\kappa_{X/k'}(x)"'] & G_k \arrow[d, "\kappa_{X/k}(x)"] \\ \pi_1(X_{k'}, \overline{x}) \arrow[r] & \pi_1(X, \overline{x}) \end{tikzcd}$$

As the base change map is obtained by precomposition with the top horizontal arrow, the commutativity of the original diagram is clear. $\square$

Definition 6.16. If (X, C, k) is a cuspidal datum and k'/k an extension of fields of characteristic 0, we say that a cuspidal datum $(X_{k'}, C', k')$ is compatible with (X, C, k) if the base change map sends C into C'.

Lemma 6.17. *With the notation of the previous definition, if $(X_{k'}, C', k')$ satisfies the injectivity in the section conjecture, then so does (X, C, k).*

Proof. As $X(k) \hookrightarrow X(k')$ and $\kappa_{X/k'}$ are injective, Proposition 6.15 tells us that $\kappa_{X/k}$ is injective. As well, if there were $x \in X(k)$ for which $\kappa_{X/k}(x) \in C$, then Proposition 6.15 tells us that $\kappa_{X/k'}(x) \in C'$, which is not the case.

Finally, suppose there were an element s of $\kappa(X(k))$ with nontrivial centralizer. Then the image s' of s in $\mathscr{S}_{\pi_1(X/k')}$ lies in $\kappa(X(k'))$, so by injectivity for $(X_{k'}, C', k')$, we know that the centralizer of s' is trivial. But a nontrivial element of the centralizer of s is also a nontrivial element of the centralizer of s', so s must have trivial centralizer. $\square$

6.1.3. *Curves.*

Definition 6.18. Let U be a smooth geometrically connected curve over a field k of characteristic 0, and let $U \hookrightarrow X$ be its unique compactification. Let $Y = X \setminus U$ and $y \in Y(k)$. We fix a Henselization $\mathcal{O}^h_{X,y}$ and set $X_y = \operatorname{Spec} \mathcal{O}^h_{X,y}$. We let $U_y = X_y \times_X U$. By 18.3 of [Sti13], there is a short exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}(U_y)^\times/\mathcal{O}(X_y)^\times, \widehat{\mathbb{Z}}(1)) \to \pi_1(U_y) \to G_k \to 0,$$

and $\mathscr{S}_{\pi_1(U_y/k)}$ is the set of sections of the surjection $\pi_1(U_y) \to G_k$ modulo conjugation by the kernel. The map $U_y \to U$ induces a map $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}$, whose image C_y is known as the *packet of cuspidal sections at* y, originally defined by Grothendieck. The set C_U of all *cuspidal sections of the curve* U is defined to be

$$\bigcup_{y \in Y(k)} \operatorname{Im}\left(\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}\right) \subseteq \mathscr{S}_{\pi_1(U/k)}.$$

When we speak of cuspidal sections of a curve U/k, we always mean (U, C_U, k) unless otherwise stated.

Lemma 6.19. *Let U/k be as in Definition 6.18, and let k'/k be a field extension. Then (U, C_U, k) is compatible with $(U_{k'}, C_{U_{k'}}, k')$ in the sense of Definition 6.16.*

Proof. Let X be the compactification of U over k, and let $y \in (X \setminus U)(k)$. We can view y as an element of $(X \setminus U)(k')$. The map $\mathcal{O}(U_y)^\times/\mathcal{O}(X_y)^\times \to \mathcal{O}((U_{k'})_y)^\times/\mathcal{O}((X_{k'})_y)^\times$ is an isomorphism, so the map $(U_{k'})_y \to U_y$ induces an isomorphism on geometric fundamental groups. This produces a base change map $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1((U_{k'})_y/k')}$. A diagram chase as in Proposition 6.15 shows that the diagram

$$\begin{tikzcd}
\mathscr{S}_{\pi_1(U_y/k)} \arrow[r] \arrow[d] & \mathscr{S}_{\pi_1((U_{k'})_y/k')} \arrow[d] \\
\mathscr{S}_{\pi_1(U/k)} \arrow[r] & \mathscr{S}_{\pi_1(U_{k'}/k')}
\end{tikzcd}$$

commutes, which proves the desired compatibility.

□

Definition 6.20. Let U be a connected smooth curve over a separably closed field k. Let $U \hookrightarrow X$ be the unique embedding of X into a projective geometrically connected regular curve. Let $Y := X \backslash U$, which is a zero dimensional scheme of some degree n. Let g denote the genus of X. We define the *Euler characteristic* $\chi(U)$ of U to be $2 - 2g - n$.

Definition 6.21. Let U be a geometrically connected smooth curve over an arbitrary field k. We set $\chi(U) = \chi(U^s)$.

Definition 6.22. Let U/k be a geometrically connected smooth curve (not necessarily projective). Then we say U is *hyperbolic* if and only if $\chi(U) < 0$.

Remark 6.23. In characteristic 0, U is hyperbolic if and only if $\pi_1(U^s)$ is non-abelian.

Theorem 6.24. *Let k be a subfield of a finite extension of $\mathbb{Q}_p$, and U/k a geometrically connected smooth hyperbolic curve or a proper genus 1 curve. Then U satisfies the injectivity in the section conjecture.*

Proof. By Lemma 6.17 and Lemma 6.19, it suffices to prove the theorem for finite extensions of $\mathbb{Q}_p$.

The injectivity of the Kummer map follows from Corollary 74 and Proposition 75 of [Sti13]. The second part follows from Theorem 250 of [Sti13] for U hyperbolic and is vacuous for U proper. The fact that $H^0(k, \pi_1(U^s, \overline{a})) = 0$ is Proposition 104 of [Sti13].

□

The following is an extension of Conjecture 6.7 to non-proper curves:

Conjecture 6.25 (Grothendieck [Gro97])**.** *Let k be a number field and X/k a geometrically connected, smooth hyperbolic curve (not necessarily proper). Then the surjectivity in the section conjecture holds for (X, C_X, k).*

6.2. **VSA via the Section Conjecture.** We now explain the relationship between the section conjecture and very strong approximation:

Proposition 6.26. *Let S be a nonempty set of finite places of a number field k and X a variety over k. Suppose X satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for all $v \in S$ (and some compatible choice of cuspidal data). Then X satisfies VSA for* $(\mathrm{f-cov}, S, k)$

Proof. Let $P \in X(\mathbb{A}_{k,S})^{\mathrm{f-cov}}$, and let $P' \in X(\mathbb{A}_k)^{\mathrm{f-cov}}$ project to P. For each place v of k, let $s_v \colon = \kappa_{X/k_v}(P'_v) \in \mathscr{S}_{\pi_1(X/k_v)}$. Then P' satisfies (i) of Theorem 2.1 of [HS12] for U the trivial group and S (in the notation of [HS12]) the empty set. It therefore satisfies (iii) of the same theorem; i.e., (s_v) is image of some $s \in \mathscr{S}_{\pi_1(X/k)}$ under the bottom horizontal arrow of the following diagram:

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ {\scriptstyle \kappa_{X/k}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{X/\mathbb{A}_k}} \\ \mathscr{S}_{\pi_1(X/k)} & \longrightarrow & \prod_v \mathscr{S}_{\pi_1(X/k_v)} \end{array}$$

If s were cuspidal, s_v would be for each v by Lemma 6.19. But s_v is not cuspidal for every finite v by the second part of injectivity in the section conjecture, so s is not cuspidal. By surjectivity in the section conjecture, there is an element x of $X(k)$ mapping to s. It has the same image as P under $\prod_v \kappa_{X/k_v}$. But Theorem 6.24 tells us that κ_S is injective, so P' equals x at all $v \in S$. Thus, P equals x and therefore lies in $X(k)$. $\square$

This now allows us to answer Question 3.2.

Corollary 6.27. *Conjecture 6.25 implies that the property $A(\mathrm{f}-\mathrm{cov}, S, k)$ holds for any number field k and nonempty set S of finite places.*

Proof. By Conjecture 6.25 and Theorem 6.24, $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for all finite places v of k (which holds for a compatible choice of cuspidal data by Lemma 6.19). Proposition 6.26 then implies that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is VSA for $(\mathrm{f}-\mathrm{cov}, S, k)$. By Theorem 4.1, $A(\mathrm{f}-\mathrm{cov}, S, k)$ holds. $\square$

7. Finite Abelian Descent for all k

Let k be a number field. Lemma 5.1 states that the only reason tori do not satisfy VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$ is that the rational points are not closed in the adelic points. The following result therefore suggests that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ might play the role of tori over any number field k:

Proposition 7.1. *Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}_k$. The set $X(k)$ is closed in $X(\mathbb{A}_k^f)$.*

Proof. Let $\alpha \in X(\mathbb{A}_k^f) \setminus X(k)$. We find a neighborhood of α that does not intersect $X(k)$.

There must be some finite set S of places of k for which $\alpha \in U_S := \prod_{v \in S} X(k_v) \times \prod_{v \notin S} X(\mathcal{O}_v)$. But

$$X(k) \cap U_S = X(\mathcal{O}_{k,S})$$

is finite by Siegel's Theorem, so $U_S \setminus X(k)$ is an open set containing α. $\square$

This leads us to conjecture:

Conjecture 7.2. $\mathbb{P}^1 \setminus \{0, 1, \infty\}_k$ *is VSA for* $(\mathrm{f}-\mathrm{ab}, f, k)$*, so* $A(\mathrm{f}-\mathrm{ab}, f, k)$ *for any number field k.*

In fact, a weaker version of this follows from a conjecture of Harari and Voloch:

Proposition 7.3. *Conjecture 2 of* [HV10] *implies that the answer to Question 3.1 is yes for the finite abelian descent obstruction over any number field k.*

Proof. By the proof of Theorem 4.3, we can decompose any variety into affine open subsets that embed in some power of $(\mathbb{P}^1 \setminus \{0,1,\infty\})_k$. Therefore, it suffices to prove the result for U/k for which there exists $f\colon U \to X^n$ for some positive integer n and $X = (\mathbb{P}^1 \setminus \{0,1,\infty\})_k$. Suppose that $U(\mathbb{A}_k)^{\mathrm{f-ab}}$ is nonempty; we wish to show that U has a k-point.

Let $\alpha \in U(\mathbb{A}_k)^{\mathrm{f-ab}}$. We want to show that each coordinate of $f(\alpha)$ is in $X(\mathbb{A}_k)^{\mathrm{Br}}$. If X were projective, this would follow from Corollary 7.3 of [Sto07]. But X is not, so we must instead prove this using the étale homotopy obstruction of [HS13].

We $f(\alpha) \in X^n(\mathbb{A}_k)^{\mathrm{f-ab}} = X^n(\mathbb{A}_k)^{\mathbb{Z}h,1}$ by Theorem 9.103 of [HS13]. It follows by Lemma 2.10 that every coordinate $f_i(\alpha)$ $(1 \leq i \leq n)$ of $f(\alpha)$ is in $X(\mathbb{A}_k)^{\mathbb{Z}h,1}$. As X^s is connected, and $H_2(X^s) = 0$, we have $X(\mathbb{A}_k)^{\mathbb{Z}h,1} = X(\mathbb{A}_k)^{\mathbb{Z}h}$ by Theorem 9.152 of [HS13]. Theorem 9.116 of [HS13] tells us that $X(\mathbb{A}_k)^{\mathbb{Z}h} = X(\mathbb{A}_k)^{\mathrm{Br}}$. Therefore, $f_i(\alpha) \in X(\mathbb{A}_k)^{\mathrm{Br}}$. In particular, it pairs to 0 with every element of $\mathrm{Br}(X)$.

Let S be a finite set of places of k large enough so that X and f are defined over $\mathcal{O}_{k,S}$, and $f(\alpha)$ is integral over $\mathcal{O}_{k,S}$. Let $\mathcal{X} = (\mathbb{P}^1 \setminus \{0,1,\infty\})_{\mathcal{O}_{k,S}}$. By forgetting the v-components for $v \in S$, we get a point $f_i(\alpha)^S$ of $X(\mathbb{A}_k^S)^{\mathrm{f-ab}} \cap \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v)$ for each i. The pairing of any element of $B_S(X)$ (in the notation of [HV10]) with $f_i(\alpha)^S$ is the same as its pairing with $f_i(\alpha)$, for elements of $B_S(X)$ pair trivially with every element of $X(k_v)$; in particular, $f_i(\alpha)^S$ is orthogonal to all of $B_S(X)$. But the conjecture implies that $f_i(\alpha)^S$ comes from an element of $\mathcal{X}(\mathcal{O}_{k,S})$. It follows that $f(\alpha)_v$ is in $X^n(k)$ for every $v \notin S$. Thus for any $v \notin S$, we find that $f(\alpha)_v \in f(U(k))$. $\square$

Remark 7.4. It does not seem that Conjecture 2 of [HV10] implies Conjecture 7.2, for the former requires using only integral points, which in turn means that we must throw out a different set S of primes depending on the point in question.

Part 3. **The Fibration Method**

In this part, we describe an alternative proof of a positive answer to Question 3.2 for $(\mathrm{f-cov}, f, k)$ that depends on the section conjecture but works over any number field k. The proof essentially uses the fibration method, due originally to M. Artin and first used to prove many important properties of étale cohomology. We begin by reformulating the section conjecture in a form that works better in fibrations. We also prove some important auxiliary results on very strong approximation in fibration sequences (Proposition 9.11), which will be useful in Part 4 .

8. The Homotopy Section Conjecture

We will now reformulate the section conjecture in terms of homotopy fixed points before showing what happens to the section conjecture in fibration sequences in Section 9. There are three main reasons why homotopy fixed points work better than fundamental group sections for use in fibration sequences: (1) If the base has a nontrivial π_2, the ordinary section conjecture does not hold inductively in elementary fibrations (2) by working with homotopy types instead of fundamental groups, one need not deal with basepoints (3) the homotopy obstruction deals with connected components when the variety is not geometrically connected. However, in most of our applications, the space is an étale $K(\pi, 1)$, making (1) irrelevant. This begs the question why we did not use profinite groupoids instead of homotopy types, which would adequately deal with (2) while allowing us to dispose of Propositions 8.16 and 10.10. Aside from the natural inclination to prove results in the most general setup possible, we justify this choice by the fact that we also found it easier to cite references on homotopy fixed points of profinite spaces than of profinite groupoids.

Unless stated otherwise, we let k denote an arbitrary field. Let $\mathcal{S}$ denote the category of simplicial sets.

Definition 8.1. The category $\hat{\mathcal{S}}$ of *profinite spaces* is the category of simplicial objects in the category of profinite sets.

Definition 8.2. In Definition 4.4 of [Fri82], Friedlander defines the étale topological type $Et(X)$ of a scheme X, which is an object of $\mathrm{Pro}(\mathcal{S})$ and represents the étale homotopy type of Artin-Mazur ([AM69]). This is functorial in the scheme X.

Definition 8.3. In Section 2.7 of [Qui08], Quick defines the profinite completion functor $\mathrm{Pro}(\mathcal{S}) \to \hat{\mathcal{S}}$ and defines $\hat{E}t(X)$ to be the profinite completion of $Et(X)$. The profinite completion of pro-spaces is defined by first taking profinite completion level-wise to get a pro-object in the category of profinite spaces and then taking the inverse limit in the category of profinite spaces.

The cohomology of this profinite space with finite coefficients recovers the étale cohomology of X, and the fundamental group of this space is the profinite étale fundamental group when X is geometrically unibranch.

Definition 8.4. If X is a k-scheme, the *homotopy fixed points* $\hat{E}t(X^s)^{hG_k}$ (Definition 2.22 of [Qui11]) are defined as follows. If G is a profinite group, one may use the bar construction to define EG and BG as profinite spaces. Then $\hat{E}t(X^s)^{hG_k}$ is the space of G_k-equivariant maps from EG_k to $\hat{E}t(X^s)$. (Compare with 9.3.2 of [HS13], 8.2 of [BS16], and the introduction to [Pál15]). By the discussion immediately following Proposition 2.14 of [Qui08], this has the structure of a simplicial set. As is $\hat{E}t(X^s)$, this is functorial in the k-scheme X.

Definition 8.5. As in [HS13], [BS16], and [Pál15], we denote $\pi_0(\hat{Et}(X^s)^{hG_k})$ by $X(hk)$ and refer to it as the *homotopy sections* of X/k. This is again functorial in the k-scheme X.

If k'/k is a field extension and X a k-scheme, $X(hk')$ still denotes $\pi_0(\hat{Et}(X^s)^{hG_{k'}})$.

Definition 8.6. The *homotopy profinite Kummer map* $\kappa_{X/k} \colon X(k) \to X(hk)$ of Section 3.2 of [Qui11] is defined as follows. The set $\operatorname{Spec} k(hk)$ has one element, so by functoriality, an element of $X(k)$ gives rise to an element of $X(hk)$. Compare with 9.3.2 of [HS13], 8.2 of [BS16], and the introduction to [Pál15]. We write κ_X or even κ when k, X are understood. We also write κ_S, $\kappa_{X/\mathbb{A}_{k,S}}$, and $\kappa_{X/\mathbb{A}_k}$ as in the case of Definition 6.4.

The section conjecture has been reformulated in homotopy-theoretic terms; see Section 3.2 of [Qui11] and Theorem 9.7(b) of [Pál15], or Section 2.6 of [Sti13] for a summary. However, these sources lack an analogue of cuspidal sections, without which one can restrict only to projective curves or conjecture something like the "homotopy section property" (HSP) of [Pál15], a local-to-global statement. We will fix this gap. In fact, HSP follows from our versions of the section conjecture (although the former is expected to hold more generally), as proven below in Lemma 9.10. We now define the analogue of cuspidal sections:

Definition 8.7 (Analogue of Definition 6.9)**.** A *homotopy cuspidal datum* (X, C, k) is a smooth variety X over k and a subset $C \subseteq X(hk)$, called the set of *cuspidal fixed points*.

Remark 8.8. In practice (e.g. to reprove Corollary 6.27), we will use cuspidal fixed points only for varieties of dimension > 1 when they arise from good neighborhoods as in Definition 10.11. However, in [Cor20], we begin the study of a general notion of cuspidal sections for higher dimensional varieties similar to Definition 6.18.

Definition 8.9 (Analogue of Defintion 6.10)**.** A homotopy cuspidal datum (X, C, k) is said to satisfy the *surjectivity in the homotopy section conjecture* if $X(hk) \setminus C \subseteq \kappa(X(k))$.

Definition 8.10 (Analogue of Definition 6.13)**.** A homotopy cuspidal datum (X, C, k) is said to satisfy the *injectivity in the homotopy section conjecture* if

(1) The map $\kappa \colon X(k) \to X(hk)$ is injective.
(2) The sets $\kappa(X(k))$ and C are disjoint.
(3) Every connected component of $\hat{Et}(X^s)^{hG_k}$ in the image of the Kummer map is simply connected.

8.1. **Basepoints and Homotopy Groups.** While the statements of the homotopical versions of the section conjecture avoid the use of basepoints, we sometimes need to use them. In particular, in order to discuss homotopy groups, we must discuss basepoints for $\hat{Et}(X)$.

On p.37 of [Fri82], it is stated that a geometric point a of X gives rise to a basepoint of $Et(X)$, and this then gives rise to a basepoint of $\hat{Et}(X)$, which we call b_a.

If this geometric point is a rational point, it is invariant under G_k, so our basepoint of $\hat{E}t(X)$ is invariant under G_k, i.e. we have a pointed G_k-space. Sometimes X does not have any rational points, and yet we want a G_k-invariant basepoint. This is provided by the following construction:

Construction 8.11. Let $a \in X(hk)$. This is a homotopy class of G_k-equivariant maps $EG_k \to \hat{E}t(X^s)$. By taking a homotopy pushout of this map along the map from EG_k to a point, we obtained a new model of $\hat{E}t(X^s)$ as a profinite space with G_k-action with a G_k-invariant basepoint. Any such basepoint is denoted by b_a.

Definition 8.12. In Definition 2.15 of [Qui08] and Definition 2.12 of [Qui11], Quick defines the homotopy groups of a pointed profinite space. If X is a scheme, and a is a geometric point or a homotopy section of X, we denote $\pi_i(\hat{E}t(X), b_a)$ by $\pi_i^{ét}(X, a)$ or simply $\pi_i(X, a)$ (c.f. Notation and Conventions). If a is a Galois-invariant geometric point or a homotopy section of a scheme X/k, this basepoint is fixed by the action of G_k, so there is a natural action of G_k on $\pi_i(X^s, a)$.

Remark 8.13. Suppose that X is geometrically unibranch (e.g., normal), Noetherian, and connected. By Proposition 2.33 of [Qui08], the homotopy groups of $\hat{E}t(X)$ are isomorphic (as pro-groups) to the pro-homotopy groups of the profinite completion of the Artin-Mazur homotopy type of X. By Theorem 11.2 of [AM69], this profinite completion is already isomorphic to the (non-completed) Artin-Mazur homotopy type as a pro-homotopy type. But $Et(X)$ represents the Artin-Mazur homotopy type (c.f. Corollary 6.3 of [Fri82]), so its homotopy groups are isomorphic to all of the above.

More generally, if X is not assumed to be connected (but still Noetherian), then it is a finite disjoint union of connected schemes. Thus the same result holds, as all constructions behave well with respect to finite coproducts.

Remark 8.14. If X is connected, then $\pi_0(\hat{E}t(X))$ contains one element, so the homotopy groups are independent of the choice of basepoint up to isomorphism. This isomorphism is, however, unique only modulo the action of the fundamental group. This is also a subtlety in using different models for $\hat{E}t(X^s)$, as in Construction 8.11. Furthermore, any two basepoints of $\hat{E}t(X^s)$ arising from the same element of $X(hk)$ are homotopic in the G_k-equivariant category, so the resulting actions on $\pi_i(X^s)$ are equivalent. This latter fact is important in the statement of Theorem 8.18. The reader may check that this is does not pose a problem in any of the theorems we prove.

Definition 8.15. A k-variety X has the *étale* $K(\pi, 1)$*-property* or *is an étale* $K(\pi, 1)$ if X is geometrically connected, and $\pi_i(X^s, a) = 0$ for $i \geq 2$ and some (equivalently, any) geometric point a.

Proposition 8.16. *Let X be a geometrically connected variety over k. Then there is a map* $\mathrm{Trunc}_1 \colon X(hk) \to \mathscr{S}_{\pi_1(X/k)}$ *(functorial in X) whose composition with the homotopy profinite*

Kummer map is the ordinary profinite Kummer map. Furthermore, if X is an étale $K(\pi, 1)$, the following are true:

- *The map* Trunc_1 *is a bijection*
- *A cuspidal datum (X, C, k) is the same as a homotopy cuspidal datum (X, C, k)*
- *X satisfies the surjectivity in the section conjecture (6.10) if and only if it satisfies the surjectivity in the homotopy section conjecture (8.9)*
- *X satisfies the injectivity in the section conjecture (6.13) if and only if it satisfies the injectivity in the homotopy section conjecture (8.10)*

Remark 8.17. Related facts are discussed in Section 3.2 of [Qui11], Proposition 12.6(b) of [Pál15], and Section 2.6 of [Sti13], especially Equation (2.13).

Before proving Proposition 8.16, we must introduce the descent spectral sequence for homotopy fixed points. It is essentially Theorem 2.16 of [Qui11] for the pointed profinite G_k-space $\hat{E}t(X^s)$:

Theorem 8.18 (Theorem 2.16 of [Qui11])**.** *For each $a \in X(hk)$, there is a spectral sequence*

$$E_2^{p,q} = H^p(G_k; \pi_q(X^s, a)) \Rightarrow \pi_{q-p}(\hat{E}t(X^s)^{hG_k}, a),$$

known as the descent spectral sequence*. The basepoint a of $\hat{E}t(X^s)^{hG_k}$ naturally results from the G_k-invariant basepoint b_a of Construction 8.11, and the G_k-action on $\pi_q(X^s, a)$ is well-defined by Remark 8.14.*

Proof of Proposition 8.16. If $X(hk)$ is empty, then the map Trunc_1 automatically exists, and it is functorial because there is a unique map from an empty set. It follows that Trunc_1 is automatically a bijection. In this case, $X(k)$ and C are empty, so the surjectivity and injectivity in either version of the section conjecture always hold.

We now suppose that there is a homotopy fixed point $a \in X(hk)$. As $\pi_0(X_s, a)$ and hence $H^0(G_k; \pi_0(X^s, a))$ is trivial, the spectral sequence produces a map $X(hk) \to \mathscr{S}_{\pi_1(X/k)}$.

When $X = \operatorname{Spec} k$, this map is clearly an isomorphism. Functoriality of the spectral sequence and the definition of the Kummer map imply that for arbitrary X, the map $X(hk) \to \mathscr{S}_{\pi_1(X/k)}$ respects the Kummer map.

We now suppose that X is an étale $K(\pi, 1)$. As $\pi_i(X^s)$ contains one element for $i \neq 1$, the spectral sequence tells us that $\pi_{1-p}(\hat{E}t(X^s)^{hG_k}, a) \cong H^p(G_k; \pi_1(X^s, a))$. For $p = 1$, this tells us that Trunc_1 is a bijection. (Alternatively, we could avoid the spectral sequence by noting that an étale $K(\pi, 1)$ is the classifying space of its fundamental group and then applying Proposition 2.9 of [Qui15].)

The second bullet-point of the proposition follows from the first. The third bullet-point follows from the compatibility of the Kummer maps, as do the first two parts of injectivity (the fourth bullet-point).

For third part of injectivity, we suppose that a is in the image of the Kummer map. Then the descent spectral sequence tells us that $\pi_1(\hat{E}t(X^s)^{hG_k}, a) \cong H^0(G_k; \pi_1(X^s, a))$. The group $\pi_1(X^s, a)$ has G_k-action arising from the section a, so $H^0(G_k; \pi_1(X^s, a))$ is simply the centralizer of $\mathrm{Trunc}_1(a)$.

The case of a geometrically connected, smooth hyperbolic curve follows by [SS16, Lemma 2.7(a)]. □

Proposition 8.19. *Let X be a finite scheme over k. Suppose that X_{red} is geometrically reduced (e.g., k is perfect). Then X satisfies the injectivity and surjectivity in the homotopy section conjecture.*

Proof. We first suppose that X is reduced. Such a scheme must be a disjoint union $\amalg_i \mathrm{Spec}(k_i)$, where k_i/k is a finite separable extension. Then $X^s = \amalg_i \amalg_{j=1}^{[k_i:k]} \mathrm{Spec}(k_s)$, and G_k acts transitively on $\amalg_{j=1}^{[k_i:k]} \mathrm{Spec}(k_s)$ for each i. Thus $\pi_0(X^s)^{G_k}$ is the set of all i for which $k_i = k$, which is the same as $X(k)$. But $\pi_n(X^s)$ is trivial for $n \geq 1$, so by Theorem 8.18, this is also $X(hk)$. We have $X(k) = X(hk)$, and it is easy to check that this equality comes from the Kummer map, so we are done in this case.

We now reduce to the reduced case by considering the map $X_{\mathrm{red}} \to X$. This map is a universal homeomorphism, and it remains so after base change to k_s, so the map $X^s_{\mathrm{red}} \to X^s$ induces an equivalence of étale sites. It follows that $\hat{E}t(X^s_{\mathrm{red}}) \to \hat{E}t(X^s)$ is an equivalence, so the induced map $X_{red}(hk) \to X(hk)$ is an isomorphism. Furthermore, the map $X_{\mathrm{red}}(k) \to X(k)$ is an isomorphism, so we are done. □

8.2. **Obstructions to the Hasse Principle.**

Proposition 8.20 (Analogue of Proposition 6.15)**.** *If k'/k is an extension of fields of characteristic* 0*, and X is a proper or smooth variety over k, then there is a base change map*

$$X(hk) \to X(hk')$$

Furthermore, the diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(k') \\ {\scriptstyle \kappa_{X/k}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{X/k'}} \\ X(hk) & \longrightarrow & X(hk') \end{array}$$

is commutative.

Proof. When X is smooth and connected, Proposition 5.4 in [Pál15] says that algebraically closed extension of base field in characteristic 0 induces an isomorphism on étale homotopy types. The same follows in the disconnected case by considering a finite disjoint union of connected smooth varieties. The same is also true in the proper case by [AM69, Corollary

12.12]. The result is then clear because a map of groups gives a map on homotopy fixed points.

□

Definition 8.21 (Analogue of Definition 6.16)**.** If (X, C, k) is a homotopy cuspidal datum and k'/k an extension of fields of characteristic 0, we say that a homotopy cuspidal datum $(X_{k'}, C', k')$ is compatible with (X, C, k) if the base change map of Proposition 8.20 sends C into C'.

Let X be a scheme over a field k. Then $\hat{E}t(X^s)$ can be represented as a pro-space $\{X_\alpha\}_\alpha$ with G_k-action, with each X_α π-finite (c.f. the proof of Lemma 9.5). As $\hat{E}t(X^s) = \operatorname{holim}_\alpha X_\alpha$, and homotopy limits commute with homotopy fixed points, we have $X(hk) = \pi_0(\operatorname{holim}_\alpha X_\alpha^{hG_k})$.

Definition 8.22. Let X be a variety over a number field k for which Proposition 8.20 applies (i.e., if X is smooth or proper). Then we define

$$\mathrm{loc}\colon X(hk) \to \prod_v X(hk_v)$$

as the product over all places v of k of the base change map $X(hk) \to X(hk_v)$.

We now recall that for a smooth variety X over a number field k there is an étale homotopy obstruction to rational points defined on p.314 of [HS13], denoted by $X(\mathbb{A}_k)^h$. These are defined using another version of homotopy fixed points ([HS13], Definition 3.3), which we denote by $X(hk)_2$. For a pro-space $\{X_\alpha\}$ with G_k-action, it is given by taking $\lim_\alpha \pi_0(X_\alpha^{hG_k})$ (while [HS13] uses the $X_\alpha^{hG_k}$ of [Goe95], it is easy to see that this is the same as that of [Qui11] by considering the descent spectral sequence). There is a natural map $X(hk) \to X(hk)_2$ and a map loc given by the product of the base change maps of Proposition 8.20 for each k_v/k, together giving us a diagram

$$\begin{tikzcd}
X(k) \arrow[r] \arrow[d, "\kappa_{X/k}"'] & X(\mathbb{A}_k) \arrow[d, "\kappa_{X/\mathbb{A}_k}"] \\
X(hk) \arrow[r, "\mathrm{loc}"] \arrow[d] & \prod_v X(hk_v) \arrow[d] \\
X(hk)_2 \arrow[r, "\mathrm{loc}"] & \prod_v X(hk_v)_2,
\end{tikzcd}$$

Then the $X(\mathbb{A}_k)^h$ of [HS13] is defined as the subset of the upper-right object of this diagram whose image in the lower-right object is contained in the image of the bottom horizontal arrow.

Lemma 8.23. *The map $X(hk) \to X(hk)_{HS}$ is surjective, and the map $X(hk_v) \to X(hk_v)_{HS}$ is an isomorphism for each v.*

Proof. As X is defined over a number field, we can express $\hat{Et}(X^s)$ as a pro-space $\{X_\alpha\}$, where α ranges over a countable cofiltering category. It follows that we may replace it by a tower, so we may apply Proposition VI.2.15 of [GJ09] to get an exact sequence

$$* \to \lim_{\alpha}{}^1 \pi_1 X_\alpha^{hG_k} \to X(hk) \to X(hk)_{HS} \to *$$

This implies the desired surjectivity.

Replacing k by k_v, we note that we can assume (by Theorem 11.2 of [AM69]) that each X_α has finite homotopy groups. As G_{k_v} is topologically finitely generated, this implies by Theorem 8.18 that $\pi_1 X_\alpha^{hG_k}$ is finite for each α. But this implies that our $\lim_{\alpha}{}^1$ vanishes, so we get the desired isomorphism. □

Proposition 8.24. *In the notation of the previous diagram, $X(\mathbb{A}_k)^h$ is $\kappa_{X/\mathbb{A}_k}^{-1}(\mathrm{Im}(\mathrm{loc}))$.*

Proof. This follows immediately from Lemma 8.23 and the definition of $X(\mathbb{A}_k)^h$. □

Proposition 8.25 (Analogue of Proposition 6.26)**.** *Let k be a number field, S a nonempty set of places of k, and X a smooth k-variety. If X satisfies the surjectivity in the homotopy section conjecture over k and the injectivity in the homotopy section conjecture over k_v for all $v \in S$ (for compatible choices of cuspidal data over k and all k_v), then it is VSA for (h, S, k). If X is geometrically connected, then the same is true for* $(\text{ét} - \mathrm{Br}, S, k)$.

Proof. Suppose $\alpha \in X(\mathbb{A}_{k,S})^h$ is the projection of $\alpha' \in X(\mathbb{A}_k)^h$. By the same argument as in Proposition 6.26, we find that α' comes from a rational point of X at every $v \in S$, i.e. α is a rational point. The last statement follows from [HS13, Theorem 9.136]. □

9. Homotopy Section Conjecture in Fibrations

In this section, "Kummer map," "cuspidal datum," and "section conjecture " will mean the homotopy versions as explained in Section 8. Of course, for an étale $K(\pi, 1)$, Proposition 8.16 says that these are equivalent to the corresponding definitions in Section 6.1.

Definition 9.1 (Definition 1.1 of [Fri82])**.** A *special geometric fibration* is a morphism $f : X \to S$ of schemes fitting into a diagram:

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Y \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying the following conditions:

(1) i is a closed embedding
(2) j is a open immersion which is dense in every fiber of $\overline{f}$, and $X = \overline{X} - Y$

(3) $\bar{f}$ is smooth and proper
(4) Y is a union of schemes $Y_1, \cdots, Y_m$, with Y_i of pure relative codimension c_i in $\overline{X}$ over S, with the property that every intersection $Y_{i_1} \cap \cdots \cap Y_{i_k}$ is smooth over S of pure codimension $c_{i_1} + \cdots + c_{i_k}$.

More generally, a *geometric fibration* is a map $f\colon X \to S$ admitting a Zariski covering $\{V_i \to S\}$ such that $f_{V_i}\colon X \times_S V_i \to V_i$ is a special geometric fibration for all i.

Lemma 9.2. *Let $f : X \to S$ be a geometric fibration and $U \to S$ a morphism of schemes. Then the base change*

$$f_U : X \times_S U \to U$$

is a geometric fibration.

Proof. This is clear because all of the properties in Definition 9.1 are stable under base extension. □

Definition 9.3. Let k be a field and S a k-scheme. Let $f\colon X \to S$ be a geometric fibration. Suppose we are given cuspidal data (S, C_S, k) for S and (X_s, C_{X_s}, k) for the fiber X_s above every $s \in S(k)$. Then we define the *cuspidal datum* $(X, C_{X,f}, k)$ *induced by* f as follows.

For $s \in S(k)$, let X_s denote the fiber of f above s and $\iota_s\colon X_s \to X$ the inclusion. The map f induces a map $f : X(hk) \to S(hk)$, and the map ι_s induces a map $\iota_s\colon X_s(hk) \to X(hk)$. We define $C_{X,f}$ by

$$C_{X,f} = f^{-1}(C_S) \bigcup \left(\bigcup_{s \in S(k)} \iota_s(C_{X_s}) \right) \subseteq X(hk).$$

In particular, if C_S and C_{X_s} are all empty, which is typical for smooth proper curves, then $C_{X,f}$ is empty.

Remark 9.4. Suppose k'/k is an extension of characteristic 0 fields. Suppose that we begin with compatible cuspidal data (S, C_S, k), (S, C'_S, k'), (X_s, C_{X_s}, k), and (X_s, C'_{X_s}, k'). Then by a simple diagram chase, $(X, C_{X,f}, k)$ is compatible with $(X, C'_{X,f}, k')$.

Lemma 9.5. *Let $f\colon X \to S$ be a map of k-schemes. Suppose that $f\colon X \to S$ is a geometric fibration of Noetherian normal schemes with connected geometric fibers, and k has characteristic* 0*. Then the sequence*

$$\hat{E}t(X_s^s) \to \hat{E}t(X^s) \to \hat{E}t(S^s)$$

is a fibration sequence of profinite etale homotopy types (in the model structure of [Qui08]*).*

Proof. It follows by Theorem 11.5 of [Fri82] (also c.f. Theorem 3.7 of [Fri73]) that the sequence $\hat{E}t(X_s) \to \hat{E}t(X) \to \hat{E}t(S)$ is a fibration sequence, or equivalently that

$$\begin{tikzcd}
\hat{E}t(X_s) \arrow[r] \arrow[d] & \hat{E}t(X) \arrow[d] \\
* \arrow[r] & \hat{E}t(S)
\end{tikzcd}$$

is a homotopy pullback diagram. We need to show that the profinite completion of this homotopy pullback diagram remains a homotopy pullback diagram.

We may assume this is a diagram of fibrant cofibrant objects in $\mathrm{Pro}(\mathcal{S})$, with the Isaksen model structure. By Theorem 11.2 of [AM69], each is isomorphic to a levelwise π-finite space (after possibly replacing each by its Postnikov tower, which is invariant under Quick's profinite completion). A π-finite space is fibrant in $L_{K^\pi}\,\mathrm{Pro}(\mathcal{S})$, in the notation of Chapter 7 of [BHH17], by Proposition 7.2.10 of loc.cit. As a limit of fibrant objects, it is itself fibrant, i.e., this is a diagram of objects fibrant for $L_{K^\pi}\,\mathrm{Pro}(\mathcal{S})$. In particular, this means that it is in the image of the inclusion $(L_{K^\pi}\,\mathrm{Pro}(\mathcal{S}))_\infty \hookrightarrow \mathrm{Pro}(\mathcal{S})_\infty$ of infinity categories, which is induced by the identity functor on model categories. This inclusion of infinity categories is conservative and preserves limits, so it detects limits. In particular, the above diagram is a pullback diagram in $(L_{K_\pi}\,\mathrm{Pro}(S))_\infty$, or equivalently, a homotopy pullback diagram in the model category $L_{K_\pi}\,\mathrm{Pro}(S)$. Therefore, by Theorem 7.4.8 of [BHH17], applying Ψ_{K_π} to this diagram yields a homotopy pullback diagram in $\mathrm{Pro}(S_\tau)$, which by Proposition 7.4.1 of the same paper is the category $\hat{S}$ of profinite spaces, with Quick's model structure.

We are done if we can identify the map $\Psi = \Psi_{K^\pi}$ (these are the same functor on ordinary categories, just refer to different model structures) with the profinite completion functor of Definition 8.3. There is an adjunction $\Psi\colon \mathrm{Pro}\,\mathcal{S} \rightleftarrows \hat{S} :\Phi$ as well as an adjunction $\mathrm{const}\colon \mathcal{S} \rightleftarrows \mathrm{Pro}(\mathcal{S}) :\lim$. By construction, $\lim \circ \Phi$ takes a profinite space X to an object of $\mathrm{Pro}(\mathcal{S}_\tau)$ whose limit is X, then takes the limit of this pro-object as a pro-space. In particular, this is the forgetful functor from profinite spaces to simplicial sets of p.587 of [Qui08], and therefore $\Psi \circ \mathrm{const}$ is Quick's profinite completion from $\mathcal{S}$ to $\hat{\mathcal{S}}$.

It remains to show that Ψ can be obtained by applying Ψ levelwise and taking the limit. We therefore wish to show that Ψ preserves filtered limits. For this, let $\{X_j\}$ be a filtered system in $\mathrm{Pro}(\mathcal{S})$. For T an object of $\mathcal{S}_\tau$, we have $\mathrm{Hom}(\lim_j \Psi(X_j), T) = \mathrm{colim}_j \mathrm{Hom}(\Psi(X_j), T)$ because T is cocompact. But this latter is $\mathrm{colim}_j \mathrm{Hom}(X_j, \Phi(T))$, which in turn is $\mathrm{Hom}(\lim_j X_j, \Phi(T)) = \mathrm{Hom}(\Psi(\lim_j X_j), T)$ because $\phi(T)$ is in $\mathcal{S}$ and therefore cocompact. As $\mathcal{S}_\tau$ cogenerates $\mathrm{Pro}(\mathcal{S}_\tau)$, this shows that $\lim_j \Psi(X_j) = \Psi(\lim_j X_j)$ in $\mathrm{Pro}(\mathcal{S}_\tau)$, and we are done. $\square$

Lemma 9.6. *Let $f\colon X \to S$ be a map of k-schemes. Let $a \in X(hk)$ and $s \in S(k)$ such that $f(a) = \kappa(s)$. Let X_s be the fiber of f above s. Suppose that the sequence $\hat{E}t(X_s^s) \to \hat{E}t(X^s) \to \hat{E}t(S^s)$ is a fibration sequence of profinite etale homotopy types (e.g., by Lemma*

9.2 and Lemma 9.5, if f is a geometric fibration of Noetherian normal schemes with connected geometric fibers).

Then a comes from an element of $X_s(hk)$, also denoted by a, and there is a long exact sequence

$$\pi_1(\hat{E}t(X_s^s)^{hG_k}, a) \to \pi_1(\hat{E}t(X^s)^{hG_k}, a) \to \pi_1(\hat{E}t(S^s)^{hG_k}, \kappa(s)) \to X_s(hk) \to X(hk) \to S(hk),$$

where the latter three are pointed sets based at a and $\kappa(s)$.

Proof. As homotopy fixed points commutes with homotopy limits, the sequence $\hat{E}t(X_s^s)^{hG_k} \to \hat{E}t(X^s)^{hG_k} \to \hat{E}t(S^s)^{hG_k}$ is a fibration sequence of simplicial sets. This implies that we get an element of $X_s(hk)$ mapping to a.

We then choose a basepoint b_a of $\hat{E}t(X_s^s)^{hG_k}$ in the connected component represented by $a \in X_s(hk)$, which makes $\hat{E}t(X_s^s)^{hG_k} \to \hat{E}t(X^s)^{hG_k} \to \hat{E}t(S^s)^{hG_k}$ a fibration sequence of pointed simplicial sets. The exact sequence then follows by the long exact sequence of homotopy groups for a pointed fibration. □

Theorem 9.7. *Let $f\colon X \to S$ satisfy the hypotheses of Lemma 9.6. If S and the fibers of f above k-points of S satisfy the injectivity in the section conjecture, then so does X.*

Proof. We refer to 1, 2, and 3 of Definition 8.10.

1 for X: Let $a, b \in X(k)$, and suppose $\kappa_X(a) = \kappa_X(b)$. This implies $f(\kappa(a)) = f(\kappa(b))$, so by 1 for S, $f(a) = f(b)$. Let $s = f(a)$. Then a, b are both in $X_s(k)$. By Lemma 9.6 and 3 for S, the map $\iota_s\colon X_s(hk) \to X(hk)$ is injective. Thus $\kappa_{X_s}(a) = \kappa_{X_s}(b)$, so 1 for X_s tells us that $a = b$.

2 for X: Let $a \in X(k)$ and $s = f(a)$. Then $f(\kappa_X(a)) = \kappa_S(s)$, so it is not in C_S by 2 for S, hence not in $f^{-1}(C_S)$. As in the previous paragraph, the map $\iota_s\colon X_s(hk) \to X(hk)$ is injective. As $a \in X_s(k)$, $\kappa_{X_s}(a) \notin C_{X_s}$ by 2 for X_s, so $\kappa_X(a) \notin \iota_s(C_{X_s})$. Suppose $s' \in S(k) \setminus \{s\}$. Then any element of $\iota_{s'}(F_{s'}(hk))$, in particular any element of $\iota_{s'}(C_{F_{s'}})$, maps to $\kappa_S(s')$ under f. By 1 for S, $\kappa_S(s') \neq \kappa_S(s)$, so $\kappa_X(a) \notin \iota_{s'}(C_{F_{s'}})$. Together, this implies that $\kappa(a) \notin C_{X,f}$.

3 for X: Let $a \in X(k)$ and $s = f(a)$, so $s \in X_s(k)$. We choose a basepoint of $\hat{E}t(X_s^s)^{hG_k}$ in the connected component $\kappa_{X_s}(a)$. Then 3 for X_s and S and Lemma 9.6 tell us that $\pi_1(\hat{E}t(X^s)^{hG_k})$ is trivial on the connected component $\kappa_X(a)$. □

Theorem 9.8. *Let $f\colon X \to S$ satisfy the hypotheses of Lemma 9.6. If S and the fibers of f above k-points of S satisfy the surjectivity in the section conjecture, then so does X.*

Proof. Let $a \in X(hk) \setminus C_{X,f}$. Then $f(a) \notin C_S$, so $f(a) = \kappa(s)$ for $s \in S(k)$. Then Lemma 9.6 gives us a long exact sequence. Thus a comes from $X_s(hk)$ and is not in C_{X_s}. By the surjectivity in the section conjecture for X_s, it comes from a k-point of X_s. □

We finally prove a general result about VSA for the étale homotopy obstruction that will be useful in Part 4. Before proving it, we need one definition.

Definition 9.9 (Notation 12.5 of [Pál15])**.** If X is a scheme over a number field k to which Definition 8.22 applies, the Selmer set $\mathrm{Sel}(X/k)$ is the subset of $X(hk)$ whose image under loc is in the image of $X(\mathbb{A}_k)$ in the following diagram:

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ \downarrow & & \downarrow \\ X(hk) & \xrightarrow{\mathrm{loc}} & \prod_v X(hk_v), \end{array}$$

Clearly $\kappa(X(k)) \subseteq \mathrm{Sel}(X/k)$.

Lemma 9.10. *Suppose X is a variety over a number field k that satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for some place v of k (and compatible choices of cuspidal data, as in Proposition 8.25). Then $\kappa(X(k)) = \mathrm{Sel}(X/k)$.*

Proof. As mentioned in Definition 9.9, $\kappa(X(k)) \subseteq \mathrm{Sel}(X/k)$. Now suppose $s \in \mathrm{Sel}(X/k)$. It suffices by the surjectivity in the homotopy section conjecture to show that s is not cuspidal. But if it were cuspidal, its base change to k_v would be cuspidal (by compatibility), contradicting injectivity over k_v, so it is not. □

Theorem 9.11. *Let $f\colon X \to S$ be a geometric fibration of smooth varieties over a number field k with connected geometric fibers, let T be a nonempty set of places of k. Suppose that the follows conditions hold:*

(1) S is VSA for (h, T, k).
(2) One of the following is true:
  (a) S satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for some v.
  (b) The image of $S(k)$ in $S(hk)$ under κ is $\mathrm{Sel}(S/k)$, and S satisfies the injectivity in the section conjecture over k_v for some $v \in T$.
  (c) The localization map $\mathrm{loc}_T\colon S(hk) \to \prod_{v\in T} S(hk_v)$ is injective on $\mathrm{Sel}(S/k)$.
(3) For all finite $v \notin T$, κ_{S/k_v} is injective (vacuous if $T = f$).
(4) For all finite v, every connected component of $\hat{E}t(S^s)^{hG_{k_v}}$ in the image of the Kummer map is simply connected (e.g., by Proposition 104 of [Sti13] and Proposition 8.16, S is a smooth geometrically connected curve not isomorphic to $\mathbb{P}^1$).
(5) For every real place v of k, every $a \in S(k)$, and every $b \in X_a(k_v)$, the map $\pi_1(X(k_v), b) \to \pi_1(S(k_v), a)$ is surjective and $\pi_1(X_a((k_v)_s))$ is trivial (where π_1 denotes the topological fundamental group under the v-adic topology).

(6) For every $a \in S(k)$, the fiber X_a of f above a is VSA for (h, T, k).

Then X is VSA for (h, T, k). One may replace (h, T, k) by $(\text{ét} - \text{Br}, T, k)$ whenever the variety in question is smooth and geometrically connected.

Proof. We first note that 2a implies 2b, which implies 2c. The first implication is Lemma 9.10. For the second implication, suppose $\alpha, \beta \in \operatorname{Sel} S/k$ with $\operatorname{loc}_v(\alpha) = \operatorname{loc}_v(\beta)$. Then there are $a, b \in S(k)$ such that $\alpha = \kappa(a)$ and $\beta = \kappa(b)$. Therefore $a, b \in S(k_v)$ have the same image under κ_{X/k_v}, so injectivity tells us that $a = b$. This implies that $\alpha = \beta$, so 2c holds.

Now let $\alpha \in X(\mathbb{A}_k)^h$ project to $\alpha' \in X(\mathbb{A}_{k,S})$, and let $\beta \in X(hk)$ such that $\operatorname{loc}(\beta) = \kappa(\alpha)$. Then $\kappa(f(\alpha)) = \operatorname{loc}(f(\beta))$, so $f(\alpha) \in S(\mathbb{A}_k)^h$. By 1, there is $a \in S(k)$ such that $a = f(\alpha)_v$ for all $v \in T$.

For $v \in T$, we have $\operatorname{loc}_v(f(\beta)) = \kappa_{S/k_v}(f(\alpha)_v) = \kappa_{S/k_v}(a) = \operatorname{loc}_v(\kappa_{S/k}(a))$. Thus $\operatorname{loc}_T(f(\beta)) = \operatorname{loc}_T(\kappa_{S/k}(a))$. It's clear that $f(\beta), \kappa_{S/k}(a) \in \operatorname{Sel}(S/k)$, so 2c implies that $f(\beta) = \kappa_{S/k}(a)$.

This implies by Lemma 9.6 that there exists $\gamma \in X_a(hk)$ such that $i_a(\gamma) = \beta$, where i_a denotes the inclusion $X_a \hookrightarrow X$.

We now know that $\kappa_{S/k_v}(f(\alpha)_v) = \operatorname{loc}_v(\beta) = \kappa_{S/k_v}(a)$ for all v. By 3, we know that $a = f(\alpha)_v$ for all finite v. For v infinite, Theorem 1.2 of [Pál15] implies that a and $f(\alpha)_v$ are in the same connected component of $S(k_v)$. As f is an elementary fibration and therefore a fiber bundle for the v-adic topology, we can modify α so that $f(\alpha)_v = a$ and $\kappa(\alpha)$ remains the same. We do this at all infinite places, so that $f(\alpha) = a$, yet still $\kappa(\alpha) = \operatorname{loc}(\beta)$.

We now have $\alpha \in X_a(\mathbb{A}_k)$ such that $i_a(\operatorname{loc}(\gamma)) = i_a(\kappa_{X_a/\mathbb{A}_k}(\alpha))$. We need to show that $i_a : X_a(hk_v) \to X(hk_v)$ is injective for all v.

By Lemma 9.6, 4 implies injectivity for all finite v. At all complex v, the geometric connectedness assumption on X_a implies that $X_a(hk_v)$ is a point, so injectivity follows.

Let v be a real place. By Theorem 1.2 of [Pál10], the map $X_a(k_v)_\bullet \to X_a(hk_v)$ is a bijection, and by Theorem 1.2 of [Pál15], the map $X(k_v)_\bullet \to X(hk_v)$ is injective. It therefore suffices to prove that $X_a(k_v)_\bullet \to X(k_v)_\bullet$ is injective. But $X_a(k_v) \to X(k_v) \to S(k_v)$ is a fibration sequence of real analytic varieties, so the condition on topological fundamental groups guarantees the desired injectivity.

We conclude that $\operatorname{loc}(\gamma) = \kappa_{X_a/\mathbb{A}_k}(\alpha)$. We conclude by 6 that there is $c \in X_a(k)$ such that $c = \alpha'$.

□

10. Elementary Fibrations

This section gives us a tool for finding open subsets of arbitrary smooth geometrically connected varieties that satisfy the section conjecture.

The following definition is similar to one from [SGA4-3].

Definition 10.1 (Definition 11.4 of [Fri82])**.** An *elementary fibration* is a geometric fibration whose fibers are geometrically connected affine curves.

Remark 10.2. Assuming f is a special geometric fibration, Y defines a relatively ample divisor of $\overline{X}$, which means that $\overline{f}$ is automatically projective.

Remark 10.3. In Definition 9.3, if $f\colon X \to S$ is an elementary fibration, it is understood unless mentioned otherwise that C_{X_s} is taken as in Definition 6.18.

Definition 10.4. In the notation of the previous definition, if the geometric fibers of f are hyperbolic (Definition 6.22), we say that f is a *hyperbolic elementary fibration*.

Lemma 10.5. *Let k be a perfect field. Then a map $X \to \operatorname{Spec} k$ is a (hyperbolic) elementary fibration if and only if X is a (hyperbolic) smooth geometrically irreducible curve.*

Proof. It is clear that if $X \to \operatorname{Spec} k$ is a (hyperbolic) elementary fibration then X is a (hyperbolic) smooth geometrically irreducible curve. To show the converse, note that every smooth curve X/k has a unique smooth compactification $\bar{X}$. We take $\bar{f} : \bar{X} \to \operatorname{Spec} k$ and $Y := \bar{X} \backslash X$ and note that Y is smooth and finite and thus étale over $\operatorname{Spec} k$. ☐

Lemma 10.6. *Let $f : X \to S$ be a (hyperbolic) elementary fibration and $U \to S$ a map of schemes. Then the base change*

$$f_U : X \times_S U \to U$$

is a (hyperbolic) elementary fibration.

Proof. This is an immediate corollary of Lemma 9.2 and the fact that every geometric fiber of f_U is also a geometric fiber of f. ☐

Lemma 10.7. *Let X, S be two smooth geometrically connected varieties over an infinite field k, and let $f : X \to S$ be an elementary fibration over k. Let x be a closed point of X. Then there exist nonempty open subsets $U_X \subset X, U_S \subset S$, with $x \in U_X$, such that $f(U_X) \subset U_S$, and the restriction $f|_{U_X} : U_X \to U_S$ is an hyperbolic elementary fibration.*

Proof. The problem is local on S, so we may assume that S is quasi-projective and that f is a special geometric fibration with compactification $\overline{f}\colon \overline{X} \to S$. By Remark 10.2, $\overline{f}$ is projective, so $\overline{X}$ is quasi-projective by EGA II.5.3.4(ii) ([EGA]), meaning we can embed $\overline{X}$ into $\mathbb{P}^N_k$ for some $N \in \mathbb{N}$.

Let F denote the fiber of $\overline{f}$ over $f(x)$. We need to find a hyperplane section Y' of $\overline{X}$ that is smooth, intersects F transversely (i.e., with smooth scheme-theoretic intersection), and does not intersect $Y \cap F$ or x. The last condition defines a subscheme of the space of hyperplanes containing a dense open subset, and Bertini's Theorem (Théorème 6.3 of [Jou83]) implies that the first two conditions do as well. Since k is infinite, such a hyperplane exists over k, providing our Y'.

It follows that there is a neighborhood of $f(x)$ in S over which the map $Y \cup Y' \to S$ is étale. Since Y' is closed in $\overline{X}$, its projection to S is proper, so this map is finite étale.

We now replace Y by $Y \cup Y'$. Either the fibers of f are now hyperbolic, in which case we are done, or the fibers have two punctures, in which case we repeat our procedure to finally obtain a hyperbolic elementary fibration around x. □

Lemma 10.8. *Let x be a closed point of a smooth geometrically connected variety X over an infinite field k. Then there is an open subset U of X containing x and an elementary fibration $f\colon U \to S$ for some smooth geometrically connected variety S over k.*

Proof. The proof of Lemma 6.3 in [SS16] explains how to extend the argument of [SGA4-3] to the case of a closed point over an infinite field. □

10.1. **Good Neighborhoods.**

Definition 10.9. An S-scheme X is called a *(hyperbolic) good neighborhood* if there exists a sequence of S-schemes

$$X = X_n, ..., X_0 = S$$

and (hyperbolic) elementary fibrations

$$f_i : X_i \to X_{i-1}, i = 1, ..., n.$$

Following [Hos14], we refer to such a sequence of S-schemes as a *sequence of parametrizing morphisms.*

Proposition 10.10. *Let $f\colon X \to S$ be an elementary fibration over a field of characteristic 0. If S is an étale $K(\pi, 1)$, then so is X. In particular, a good neighborhood is an étale $K(\pi, 1)$. Under this condition, the results of Section 9 hold for the classical section conjecture.*

Proof. Lemma 2.7(a) of [SS16] implies that the geometric fibers are étale $K(\pi, 1)$'s. The result then follows from the long exact sequence of Theorem 11.5 of [Fri82], noting that Remark 8.13 implies that different notions of étale homotopy group coincide for normal schemes. □

Definition 10.11. Let $f\colon X \to S$ be a good neighborhood with a sequence of parametrizing morphisms. Suppose we are given a cuspidal datum (S, C, k) for S. Then we define the *cuspidal datum $(X, C_{X,f}, k)$ induced by f and the sequence of parametrizing morphisms* to be the cuspidal datum induced successively by the elementary fibrations.

Remark 10.12. By an abuse of notation, we do not include the sequence of parametrizing morphisms in the notation for $C_{X,f}$, but we believe this should not lead to too much confusion. Conjecture 6.4.1 of [Cor18] states that $C_{X,f}$ does not depend on the sequence of parametrizing morphisms, and the forthcoming paper [Cor20] will discuss this and other questions about cuspidal sections for higher dimensional varieties in more detail.

Corollary 10.13. *Let X be a hyperbolic good neighborhood over a sub-p-adic field k. Then $(X, C_{X,f}, k)$ satisfies the injectivity in the (homotopy) section conjecture (for any sequence of parametrizing morphisms).*

Proof. By Proposition 8.16 and repeated application of Proposition 10.10, the ordinary and homotopy section conjectures are equivalent. The result then follows by repeated application of Theorem 9.7 along with Theorem 6.24. □

Corollary 10.14. *Let X be a hyperbolic good neighborhood. If Conjecture 6.25 is true over k, then $(X, C_{X,f}, k)$ satisfies the surjectivity in the (homotopy) section conjecture (for any sequence of parametrizing morphisms).*

Proof. As in Corollary 10.13, the ordinary and homotopy section conjectures are equivalent. The result follows by repeated application of Theorem 9.8. □

Lemma 10.15. *Every smooth geometrically connected variety X over an infinite field k has an open cover by hyperbolic good neighbourhoods over k.*

Proof. Let x be a closed point of X. We proceed by induction on the dimension of X. If X has dimension 0, it is $\operatorname{Spec} k$, so we are done. Otherwise, Lemma 10.8 implies that there is a neighborhood U of x and an elementary fibration $f\colon U \to S$ over k. By Lemma 10.7, we can replace U and S so that f is now a hyperbolic elementary fibration. Finally, by the induction hypothesis, there is an open neighborhood V of $f(x)$ that is a hyperbolic good neighborhood over k. By Lemma 10.6, the pullback of f from S to V is still a hyperbolic elementary fibration, and this pullback is our desired hyperbolic good neighborhood. □

Remark 10.16. By Lemma 6.19 and Remark 9.4, the (ordinary or homotopy) cuspidal data of Definition 10.11 are compatible for varying k of characteristic 0.

We can now answer Question 3.2 for $(\mathrm{f-cov}, S, k)$ while circumventing Theorem 4.3:

Corollary 10.17. *Let X be a smooth geometrically connected variety over a number field k, and assume Conjecture 6.25. Then X has a Zariski open cover $X = \bigcup_i U_i$ such that U_i is VSA for $(\mathrm{f-cov}, S, k)$ for any nonempty set S of finite places of k.*

Proof. By Lemma 10.15, X has an open cover by hyperbolic good neighborhoods U_i. By Corollary 10.14, U_i satisfies the surjectivity in the section conjecture over k, and by Corollary

10.13, U_i satisfies the injectivity in the section conjecture over k_v for all finite places v of k. By Proposition 6.26, U_i satisfies VSA for $(\mathrm{f}-\mathrm{cov}, S, k)$. $\square$

Part 4. Examples

11. Poonen's Counterexample

In [Poo10], Poonen found the first example of a variety with no rational points that does not satisfy VSA for the étale-Brauer obstruction. In this section, we review the construction of this variety. We also use Theorem 9.11 to show, under certain conditions, that any example like Poonen's needs to have at least one singular fiber.

11.1. **Conic Bundles.**

We now present some general notions about conic bundles, as described in §4 of [Poo10]. We base our notation on [Poo10] and then add some notation of our own.

Let k be a field. From now on, let ε equal 1 if k has characteristic 2 and 0 otherwise.

Let B be a nice k-variety. Let $\mathcal{L}$ be a line bundle on B. Let $\mathcal{E}$ be the rank 3 locally free sheaf

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}$$

on B. Let $a \in k^\times$, and let $s \in \Gamma(B, \mathcal{L}^{\otimes 2})$ be a nonzero global section. Consider the section

$$\varepsilon \oplus 1 \oplus (-a) \oplus (-s) \in \Gamma(B, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \operatorname{Sym}^2 \mathcal{E}),$$

where the first $\mathcal{O}$ corresponds to the product of the first two summands of $\mathcal{E}$, and the last three terms $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}^{\otimes 2} \subset \operatorname{Sym}^2 \mathcal{E}$ are the symmetric squares of the three individual summands of $\mathcal{E}$. The zero locus of $\varepsilon \oplus 1 \oplus (-a) \oplus (-s)$ in $\mathbb{P}\mathcal{E}^v$ is a projective geometrically integral k-variety $X = X(B, \mathcal{L}, a, s)$ with a morphism $\alpha : X \to B$.

Definition 11.1. We call an element

$$(\mathcal{L}, s, a) \in \operatorname{Div} B \times \Gamma(B, \mathcal{L}^{\otimes 2}) \times k^\times,$$

where $s \neq 0$, a *conic bundle datum on* B and X *the total space of* $(\mathcal{L}, s, a)$. We denote $X = Tot_B(\mathcal{L}, s, a)$.

If U is a dense open subscheme of B with a trivialization $\mathcal{L}|_U \cong \mathcal{O}_U$, and we identify $s|_U$ with an element of $\Gamma(U, \mathcal{O}_U)$, then the affine scheme defined by $y^2 + \varepsilon yz - az^2 = s|_U$ in $\mathbb{A}^2_U$ is a dense open subscheme of X. We therefore refer to X as the conic bundle given by $y^2 + \varepsilon yz - az^2 = s$.

In the special case where $B = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(2)$, and the homogeneous form $s \in \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ is separable, X is called the Châtelet surface given by $y^2 + \varepsilon yz - az^2 = s(x)$, where $s(x) \in k[x]$ denotes a dehomogenization of s.

Returning to the general case, we let Z be the subscheme $s = 0$ of B. We call Z the degeneracy locus of the conic bundle $(\mathcal{L}, s, a)$. Each fiber of α above a point of $B - Z$ is a smooth plane conic, and each fiber above a geometric point of Z is a union of two projective lines crossing transversally at a point. A local calculation shows that if Z is smooth over k, then X is smooth over k.

11.2. **Poonen's Variety.** Let k be a global field, let $a \in k^\times$, and let $\tilde{P}_\infty(x), \tilde{P}_0(x) \in k[x]$ be relatively prime separable degree 4 polynomials such that the (nice) Châtelet surface $\mathcal{V}_\infty$ given by $y^2 + \varepsilon yz - az^2 = \tilde{P}_\infty(x)$ over k satisfies $\mathcal{V}_\infty(\mathbb{A}_k) \neq \emptyset$ but $\mathcal{V}_\infty(k) = \emptyset$. Such Châtelet surfaces always exist : see [[Poo10], Proposition 5.1 and 11] if the characteristic of k is not 2 and [Vir12] otherwise. If $k = \mathbb{Q}$, one may use the original example from [Isk71] with $a = -1$ and $\tilde{P}_\infty(x) := (x^2 - 2)(3 - x^2)$.

Let $P_\infty(w, x)$ and $P_0(w, x)$ be the homogenizations of $\tilde{P}_\infty$ and $\tilde{P}_0$. Let $\mathcal{L} = \mathcal{O}(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and define

$$s_1 := u^2 P_\infty(w, x) + v^2 P_0(w, x) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L}^{\otimes 2}),$$

where the two copies of $\mathbb{P}^1$ have homogeneous coordinates $(u : v)$ and $(w : x)$, respectively. Let $Z_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero locus of s_1. Let $F \subset \mathbb{P}^1$ be the (finite) branch locus of the first projection $Z_1 \to \mathbb{P}^1$. i.e.,

$$F := \left\{ (u : v) \in \mathbb{P}^1 | u^2 P_\infty(w, x) + v^2 P_0(w, x) \text{ has a multiple root} \right\}.$$

Let $\alpha_1 : \mathcal{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the conic bundle given by $y^2 + \varepsilon yz - az^2 = s_1$, a.k.a. the conic bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the datum $(\mathcal{O}(1, 2), a, s_1)$.

Composing α_1 with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ yields a morphism $\beta_1 : \mathcal{V} \to \mathbb{P}^1$ whose fiber above $\infty := (1 : 0)$ is the Châtelet surface $\mathcal{V}_\infty$ defined earlier.

Now let C be a nice curve over k such that $C(k)$ is finite and nonempty. Choose a dominant morphism $\gamma\colon C \to \mathbb{P}^1$, étale above F, such that $\gamma(C(k)) = \{\infty\}$. Define $X := \mathcal{V} \times_{\mathbb{P}^1} C$ to be the fiber product with respect to the maps $\beta_1\colon \mathcal{V} \to \mathbb{P}^1, \gamma\colon C \to \mathbb{P}^1$, and consider the morphisms α and β as in the diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"'] \arrow[dd, bend right=60, "\beta"'] & \mathcal{V} \arrow[d, "\alpha_1"'] \arrow[dd, bend left=60, "\beta_1"] \\
C \times \mathbb{P}^1 \arrow[r, "{(\gamma,1)}"] \arrow[d, "pr_1"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "pr_1"'] \\
C \arrow[r, "\gamma"] & \mathbb{P}^1
\end{tikzcd}$$

The variety X is the one constructed in §6 of [Poo10]. The same paper proves that $X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$ (Theorem 8.2) and $X(k) = \emptyset$ (Theorem 7.2). We present the proof that $X(k) = \emptyset$ because it is short and simple.

Proposition 11.2. *If X is the variety constructed above, then $X(k) = \emptyset$.*

Proof. Assume $x_0 \in X(k)$; we have $c_0 := \beta(x_0) \in C(k)$, but then $x \in \beta^{-1}(c_0)$. By the construction of X, $\beta^{-1}(c_0)$ is isomorphic to $\beta_1^{-1}(\gamma(c_0)) = \beta_1^{-1}(\infty) \cong \mathcal{V}_\infty$, but $\mathcal{V}_\infty(k) = \emptyset$ by construction. □

Note that X can also be considered as the variety corresponding to the datum $(O(1,2), a, s_1)$ pulled back via $(\gamma, 1)$ to $C \times \mathbb{P}^1$.

11.3. **The Necessity of Singular Fibers.** One of the original motivations for the étale homotopy obstruction of [HS13] was the fact that Poonen's counterexample was not a fibration (as it contained singular fibers). We complete this circle of reasoning to show that, assuming the Tate-Shafarevich conjecture and a technical condition on the real points, these singular fibers were necessary. Our main tool is Theorem 9.11, which relies on the étale homotopy obstruction.

Theorem 11.3. *Let $f\colon X \to C$ be a smooth proper family of Châtelet surfaces over an elliptic curve C with $|C(k)| < \infty$. Suppose that for all $a \in C(k)$, we have $X_a(k) = \emptyset$. Suppose furthermore that the Tate-Shafarevich group of C has trivial divisible subgroup, and that for every real place v of k, every $a \in S(k)$, and every $b \in X_a(k_v)$, the map $\pi_1(X(k_v), b) \to \pi_1(S(k_v), a)$ is surjective (vacuous if k is totally imaginary). Then $X(\mathbb{A}_k)^{\text{ét,Br}} = \emptyset$.*

Proof. We must first verify that the hypotheses of Theorem 9.11 hold for f.

By Corollary 8.1 of [Sto07], we have

$$C(k) = C(\mathbb{A}_k)^{\text{Br}}_{\bullet}.$$

In particular, we find that $C(k) = C(\mathbb{A}_k^f)^{\text{Br}}$, so C is VSA for (h, f, k), and hence 1 holds.

As $C(k)$ is finite, it equals the completed Selmer group of C (c.f. [Sto07], Section 2). But $C(hk) = H^1(k; H_1(C^s; \widehat{\mathbb{Z}})) = \varprojlim_n H^1(k; C[n])$, and the completed Selmer group is just $\mathrm{Sel}(C/k)$. This together with Theorem 6.24 implies that 2b holds.

3 and 4 follow immediately because $T = f$ and because S is a smooth geometrically connected curve, respectively.

Every geometric fiber is a Châtelet surface, so $\pi_1(X_a((k_v)_s))$ is trivial. By that and the hypothesis on real fundamental groups, 5 is verified.

Finally, every fiber X_a for $a \in C(k)$ is a Châtelet surface without rational points, so by [CTSSD87a, CTSSD87b], we have $X_a(\mathbb{A}_k)^{\mathrm{Br}} = X_a(\mathbb{A}_k)^{\text{ét,Br}} = \emptyset$. This shows that 6 holds.

By Theorem 9.11, X satisfies VSA for $(\text{ét} - \mathrm{Br}, f, k)$. By the argument of Proposition 11.2, we have $X(k) = \emptyset$, hence $X(\mathbb{A}_k)^{\text{ét,Br}} = \emptyset$. □

12. VSA Stratifications and Open Covers

In this section, we seek to positively answer Question 3.2 for Poonen's variety in an *explicit* fashion. That is, we want to prove that a *specific* open cover or stratification satisfies VSA. In fact, we present a few different results in this direction, all of which differ slightly in their exact hypotheses, conclusions, and methods. More precisely, the different results differ in:

(1) Whether they prove VSA for the Brauer-Manin or étale-Brauer obstructions,
(2) Whether they prove VSA for open covers or stratifications,
(3) Whether they apply over all number fields or not, and
(4) Whether they assume conjectures such as the Tate-Shafarevich conjecture or the section conjecture (Conjecture 6.25).

In both 1 and 2, the first condition is stronger than the second condititIon. In 4, for the sake of explicitness, assuming the Tate-Shafarevich conjecture is actually much more harmless, as there are specific elliptic curves for which the Tate-Shafarevich conjecture has been proven.

More specifically, in Proposition 12.3, we find a stratification that satisfies VSA for Brauer-Manin assuming Tate-Shafarevich. In Corollary 12.6, we find an open cover satisfying VSA for étale-Brauer assuming the section conjecture and crucially using the étale homotopy obstruction. Finally, in Theorem 13.5, we find a stratification that satisfies VSA for étale Brauer-Manin over totally imaginary number fields k assuming Tate-Shafarevich. While Theorem 13.5 is technically weaker than Proposition 12.3 with respect to our criteria listed above, it introduces the concept of the ramified étale-Brauer obstruction, which we believe is important in its own right. See also Remarks 12.4 and 13.6 for other variations on the hypotheses and conclusions.

12.1. **VSA Stratifications and Open Covers.**

Let U be an open subset of $C \setminus C(k)$, and let S be an open subset of the smooth locus containing $C(k)$ (we always assume that $C(k)$ is finite).

Lemma 12.1. *Suppose k is a number field. Let J be the Jacobian of C, and suppose that $J(k)$ is finite and that the Tate-Shafarevich group $Ш(J)$ has trivial divisible subgroup. Let $U = C \setminus C(k)$. Then U has empty Brauer set.*

Proof. By Corollary 8.1 of [Sto07], we have

$$C(k) = C(\mathbb{A}_k)^{\mathrm{Br}}_{\bullet}.$$

In particular, we find that $C(k) = C(\mathbb{A}_k^f)^{\mathrm{Br}}$, and therefore that $U(\mathbb{A}_k^f)^{\mathrm{Br}} = U(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. □

Lemma 12.2. *With S as above and $Z := C \setminus S$, we have $Z(\mathbb{A}_k)^{\mathrm{f-ab}} = \emptyset$.*

Proof. This follows immediately from Theorem 8.2 of [Sto07]. □

Proposition 12.3. *Under the conditions of Lemma 12.1, there is a stratification $X = X_1 \cup X_2$ such that $X_1(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ and $X_2(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.*

Proof. Let X_1 be $\beta^{-1}(U)$, and let X_2 be $\beta^{-1}(C(k))$. By Lemma 12.1, we know that $X_1(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.

We also know by construction that X_2 is a union of copies of $\mathcal{V}_\infty$. As $\mathcal{V}_\infty(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, we find that $X_2(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. □

Remark 12.4. One may get a similar result as Proposition 12.3 by assuming the section conjecture instead of $\text{Ш}(J)_{\mathrm{div}} = 0$, either by applying the section conjecture directly to U, or by applying Theorem 12.5 to S and Lemma 12.2 to $C \setminus S$.

Next, we present an explicit proof that X, as constructed in Section 11.2, has a Zariski open cover with empty étale-Brauer set using Theorem 9.11 and assuming the section conjecture. Section 13 will then prove a stronger result, namely that there is a finite ramified cover with empty étale-Brauer set, without assuming the section conjecture.

Theorem 12.5. *Suppose that every nonempty open subvariety of C satisfies Conjecture 6.25. Then there is an open subvariety $S \subseteq C$ such that $S \cup U = C$, and $X_S := S \times_C X$ satisfies VSA for $(\text{ét} - \mathrm{Br}, f, k)$.*

Proof. With the notation of Section 11.2, let $F' := \gamma^{-1}(F)$, and let $C' := C \setminus F'$. Then β is smooth over C'.

Let v be a real place of k. As C is smooth, $C'(k_v)$ is a disjoint union of components each homeomorphic to a line or circle. For each circle component containing a rational point, we wish to remove a closed non-rational point of C' that meets that component.

To do this, we need to show that each component contains a k_s-point that is not a k-point. Let k_v^{alg} denote the algebraic closure of k in k_v. By the completeness of the theory of real closed fields, any first-order formula true of k_v is also true of k_v^{alg}. By Theorems 2.4.4 and 2.4.5 of [BCR98], the property that a point lies in a given connected component is a semi-algebraic condition. As well, the property that a point is not equal to any element of $C'(k)$ is also a semi-algebraic condition. As each component has uncountably many k_v-points and hence at least one point not in $C'(k)$, each component has a k_v^{alg}-point not in $C(k)$. We choose one for each component with a rational point and remove the corresponding closed points of C'.

We let S denote the resulting open subvariety of C. It is clear that $S(k) = C(k)$, so that $S \cup U = C$. It suffices to verify the hypotheses of Theorem 9.11 for the projection $f = \beta\,|_{X_S} \colon X_S \to S$ and $T = f$.

By Lemma 2.7, S is VSA for (h, f, k), so 1 holds. By Conjecture 6.25 and Theorem 6.24, 2a holds. 3 and 4 follow immediately because $T = f$ and because S is a smooth geometrically connected curve, respectively.

By construction, every connected component of $S(k_v)$ is simply connected for each real place v. As well, every geometric fiber is a Châtelet surface, so $\pi_1(X_a((k_v)_s))$ is trivial, and hence 5 is verified.

Finally, every fiber over a rational point of S is isomorphic to $\mathcal{V}_\infty$. But we assumed that $\mathcal{V}_\infty(k) = \emptyset$, so by [CTSSD87a, CTSSD87b], we have $\mathcal{V}_\infty(\mathbb{A}_k)^{\mathrm{Br}} = \mathcal{V}_\infty(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. This shows that 6 holds. □

Corollary 12.6. *Under the hypotheses and notations of Theorem 12.5, the variety X has an open cover $X = X_S \cup X_U$ such that X_S and $X_U := U \times_C X$ both have empty étale-Brauer set.*

Proof. We have already shown that $X_S(k) \subseteq X(k) = \emptyset$, so by Theorem 12.5, we have $X_S(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. By Conjecture 6.25 and Proposition 6.26 applied to U, we find that $U(\mathbb{A}_k)^{\mathrm{f-cov}} = \emptyset$, which implies that $X_U(\mathbb{A}_k)^{\mathrm{f-cov}} = X_U(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. □

Remark 12.7. For the proof of Corollary 12.6, one may avoid using Conjecture 6.25 for U by assuming the hypotheses of Lemma 12.1. However, the argument in Theorem 12.5 already uses the section conjecture, so it makes sense to use it again.

13. The Brauer-Manin obstruction applied to ramified covers

In this section, we introduce the notion of the ramified étale-Brauer obstruction. In a sense, this lies between the obstructions from Section 3.1 and the ordinary étale-Brauer obstruction. We also formulate a conjecture analogous to Theorems 3.3 and 3.4. In Section 13.2, we then explain how to apply this to Poonen's example.

13.1. **Quasi-Torsors and the Ramified Etale-Brauer Obstruction.**

We shall now define slight generalizations of the concepts of torsors and the étale Brauer-Manin obstruction.

Definition 13.1. Let X be a variety over a field k, G a finite smooth k-group, and $D \subseteq X$ an a closed subvariety. A *G-quasi-torsor over X unramified outside D* is a map $\pi : Y \to X$ and a G-action on Y respecting π such that

(1) π is a surjective and quasi-finite morphism.
(2) The pullback of π from X to $X \setminus D$ is a G-torsor over $X \setminus D$.

We call $d = |G|$ the degree of Y.

Given $\rho\colon Z \to X$ an arbitrary morphism of k-varieties, the pullback $\rho^{-1}(Y)$ is a G-quasi-torsor unramified outside $\rho^{-1}(D)$.

Let $\pi : Y \to X$ be a G-quasi-torsor over X unramified outside D. As in the case of a usual G-torsor, one can twist $\pi : Y \to X$ by any $\sigma \in H^1(k, G)$ and get a G^σ-quasi-torsor $\pi^\sigma : Y^\sigma \to X$, also unramified outside D.

If we assume that $D(k) = \emptyset$, then as in Section 2.3.1, we get:

$$X(k) = \bigcup_{\sigma \in H^1(k,G)} \pi^\sigma(Y^\sigma(k))$$

If k is a global field, it follows that:

$$X(k) \subset X(\mathbb{A}_k)^{\pi,\mathrm{Br}} := \bigcup_{\sigma \in H^1(k,G)} \pi^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}}).$$

Definition 13.2. Letting π range over all quasi-torsors over X unramified outside D, we define the *(D-)ramified étale-Brauer obstruction* by

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D} := \bigcap_{\pi} X(\mathbb{A}_k)^{\pi,\mathrm{Br}} \subset X(\mathbb{A}_k)$$

It follows from Section 2.3.1 that $(X \setminus D)(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subseteq X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D}$.

Until now, we have left the fact that $D(k) = \emptyset$ as a black box. The idea behind this construction is that D has smaller dimension, and it therefore might be easier to show that D has empty étale-Brauer set. More generally, one might, e.g., find a subvariety E of D, prove that $D(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim E} = \emptyset$, and finally prove that $E(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. We formalize this as follows.

Definition 13.3. We say that *the absence of rational points is explained by the ramified étale-Brauer obstruction* if there exists a stratification $X = \coprod_i X_i$ and for each i, we have $X_i(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D_i} = \emptyset$, where $D_i := \overline{X_i} \setminus X_i$.

This is similar to the method of considering the étale-Brauer sets of dense open subsets, for $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D} = \emptyset$ implies that $(X \setminus D)(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. In particular, if the absence of rational points is explained by the ramified étale-Brauer obstruction, then the answer to Question 3.1 is yes. However, one crucial difference is that the ramified étale-Brauer obstruction requires proving only emptiness of Brauer sets of *proper* varieties.

Inspired by Theorem 3.3, we conjecture that the absence of rational points is always explained by the ramified étale-Brauer obstruction and that one may furthermore take a single quasi-torsor for each stratum:

Conjecture 13.4. *For any number field k and variety X/k with $X(k) = \emptyset$, there exists a stratification $X = \coprod_i X_i$ and for each i, a quasi-torsor Y_i over the closure $\overline{X_i}$ of X_i such that Y_i restricts to a torsor over X_i and $Y_i^\sigma(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ for all twists Y_i^σ of Y_i.*

This conjecture still does not imply that there is an algorithm to determine whether $X(k)$ is empty, as one may need to consider infinitely many twists of a quasi-torsor Y_i. Nonetheless, we now show that it is computable in the example of [Poo10] (in particular, see Remark 13.9 to see why we need only finitely many twists). Thus, we know of no counterexamples to Conjecture 13.4.

13.2. **Quasi-torsors in Poonen's Example.** In this subsection, we show (under some conditions) that for the variety X defined in Section 11.2, one can choose a divisor $D \subseteq X$ such that $D(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, and $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D} = \emptyset$.

More specifically, we assume as in Lemma 12.1 that the Jacobian J of C satisfies $|J(k)| < \infty$ and $\text{Ш}(J)_{\mathrm{div}} = 0$, which implies that

$$C(k) = C(\mathbb{A}_k)^{\mathrm{Br}}_\bullet$$

and

$$U(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset.$$

As before, let $F' := \gamma^{-1}(F) \subseteq C$ and $C' := C \backslash F'$. Note that C' is a non-projective curve. Now let $D := \beta^{-1}(F')$. Note that $\infty \notin F$, so that $C(k) \cap F' = \emptyset$. By Lemma 12.2, we have $F'(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, hence $D(\mathbb{A}_k)^{\mathrm{Br}}) = \emptyset$.

We will now spend the rest of Section 13.2 proving:

Theorem 13.5. *With notations as above, the absence of rational points on X is explained by the ramified étale-Brauer obstruction if k is a global field with no real places (i.e., a function field or a totally imaginary number field).*

Remark 13.6. One may in fact combine Theorem 13.5 with Lemma 12.1 to get a VSA open cover, rather than a stratification, without assuming the section conjecture as in Corollary 12.6.

Now X is a family over C of conic bundles over $\mathbb{P}^1$. The fiber over any element of $C(k)$ is isomorphic to the Châtelet surface $\mathcal{V}_\infty$. All the fibers over C' are smooth conic bundles.

Let $E' \subset (\mathbb{P}^1 \backslash F) \times (\mathbb{P}^1)^4$ be the curve defined by

$$u^2 P_\infty(w_i, x_i) + v^2 P_0(w_i, x_i) = 0, 1 \le i \le 4$$

$$(w_i : x_i) \neq (w_j : x_j), i \neq j, 1 \leq i, j \leq 4,$$

where $(u : v)$ are the projective coordinates of $\mathbb{P}^1 \backslash F$ and $(w_i : x_i), 1 \leq i \leq 4$ are the projective coordinates of the four copies of $\mathbb{P}^1$. Since $\tilde{P}_\infty(x)$ and $\tilde{P}_0(x)$ are separable and coprime, we have that E' is a smooth connected curve and that the first projection $E' \to \mathbb{P}^1 \backslash F$ gives E' the structure of an étale Galois covering of $\mathbb{P}^1 \backslash F$ with automorphism group $G = S_4$ that acts on the fibres by permuting

$$(w_i : x_i), 1 \leq i \leq 4.$$

Since every birationality class of curves contains a unique projective smooth member, one can construct an S_4-quasi-torsor over $E \to \mathbb{P}^1$ unramified outside F which gives E' when restricted to $\mathbb{P}^1 \backslash F$.

The k-twists of $E \to \mathbb{P}^1$ are classified by $H^1(k, S_4)$ which (since the action of Γ_k on S_4 is trivial) coincides with the set $\mathrm{Hom}(\Gamma_k, S_4)/\sim$ of homomorphisms up to conjugation. More concretely, for every homomorphism $\phi : \Gamma_k \to S_4$, define E_ϕ to be the k-form of E with Galois action that restricts to the action

$$\sigma : ((u : v), ((w_1 : x_1), (w_2 : x_2), (w_3 : x_3), (w_4 : x_4))) \mapsto$$

$$((u : v), ((w_{\phi_\sigma(1)} : x_{\phi_\sigma(1)}), (w_{\phi_\sigma(2)} : x_{\phi_\sigma(2)}), (w_{\phi_\sigma(3)} : x_{\phi_\sigma(3)}), (w_{\phi_\sigma(4)} : x_{\phi_\sigma(4)})))^\sigma$$

on E'.

For every $\phi : \Gamma_k \to S_4$, we set $C_\phi := C \times_{\mathbb{P}^1} E_\phi$ (relative to $\gamma : C \to \mathbb{P}^1$ and the first projection $\pi_\phi \colon E_\phi \to \mathbb{P}^1$) and $X_\phi := X \times_C C_\phi$ (relative to $\beta : X \to C$ and the projection $C_\phi \to C$).

Since the maps $\gamma : C \to \mathbb{P}^1$ and $E \to \mathbb{P}^1$ have disjoint ramification loci, all C_ϕ are geometrically integral, and so are all the X_ϕ.

Then X_ϕ is a complete family of twists of a quasi-torsor over X of degree 24 unramified outside D. As we already know that $D(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, it suffices for the proof of Theorem 13.5 to show that

$$X_\phi(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$$

for every $\phi \in H^1(\Gamma_k, S_4)$. We devote the rest of Section 13.2 to proving this fact.

13.2.1. *Reduction to X_{ϕ_∞}.*

Lemma 13.7. *For every $\phi \in H^1(k, S_4)$, we have $C_\phi(k) = C_\phi(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet$.*

Proof. By Corollary 7.3 of [Sto07], we have $C_\phi(\mathbb{A}_k)^{\mathrm{Br}}_\bullet = C_\phi(\mathbb{A}_k)^{\mathrm{f-ab}}_\bullet$ and $C(\mathbb{A}_k)^{\mathrm{Br}}_\bullet = C(\mathbb{A}_k)^{\mathrm{f-ab}}_\bullet$. Then Proposition 8.5 of [Sto07] and the fact that $C(k) = C(\mathbb{A}_k)^{\mathrm{Br}}_\bullet$ implies the result. $\square$

Denote by $\phi_\infty \in H^1(k, S_4)$ the map $\Gamma_k \to S_4$ defined by the Galois action on the four roots of P_∞.

Lemma 13.8. *Let $\phi \in H^1(\Gamma_k, S_4)$ be such that $\phi \neq \phi_\infty$. Then $C_\phi(k) = \emptyset$.*

Proof. Since $\phi \neq \phi_\infty$ we get that $E_\phi(k) \cap \pi_\phi^{-1}(\infty) = \emptyset$. Since $\gamma(C(k)) = \infty$, we get that $C_\phi(k) = \emptyset$. ◻

Let $\rho_\phi \colon X_\phi \to C_\phi$ denote the map defined earlier. For every $\phi \in H^1(k, S_4)$, we have

$$\rho_\phi(X_\phi(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet) \subseteq C_\phi(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet = C_\phi(k) = \emptyset,$$

so $X_\phi(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ for $\phi \neq \phi_\infty$.

Remark 13.9. It is Lemma 13.8 along with VSA for C coming from our condition on the Tate-Shafarevich group that lets us avoid all but one twist. In fact, this has a conceptual explanation. While infinitely many twists might have adelic points (as $\beta^{-1}(C')$ is not proper), all possible elements of the Brauer set lie in the fiber over $C(k)$, which is in fact proper. There are therefore finitely many twists that might have adelic points in this fiber.

13.2.2. *The proof that $X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.* In this subsection, we shall prove that if k has no real places, then $X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}} = X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$.

Let $p \in C_{\phi_\infty}(k)$. The fiber $\rho_{\phi_\infty}^{-1}(p)$ is isomorphic to the Châtelet surface $\mathcal{V}_\infty$. We shall denote by $\rho_p : \mathcal{V}_\infty \to X_{\phi_\infty}$ the corresponding natural isomorphism onto the fiber $\rho_{\phi_\infty}^{-1}(p)$. Recall that $\mathcal{V}_\infty$ satisfies $\mathcal{V}_\infty(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.

Lemma 13.10. *Let k be a global field with no real places. Let $x \in X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$. Then there exists $p \in C_{\phi_\infty}(k)$ such that $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A}_k)_\bullet)$.*

Proof. From functoriality and Lemma 13.7 we get

$$\rho_{\phi_\infty}(x) \in \rho_{\phi_\infty}(X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}}) \subset C_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet = C_{\phi_\infty}(k)$$

We denote $p = \rho_{\phi_\infty}(x) \in C'_{\phi_\infty}(k)$. Now it is clear that in all but maybe the infinite places $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A}_k))$. Hence it remains to deal with the infinite places, which by assumption are all complex. But since both X_{ϕ_∞} and $\mathcal{V}_\infty$ are geometrically integral, taking connected components reduces $X(\mathbb{C})$ and $\mathcal{V}_\infty(\mathbb{C})$ to a single point. ◻

Lemma 13.11. *Let $p \in C_{\phi_\infty}(k)$ be a point. Then the map*

$$\rho_p^* : \mathrm{Br}(X_{\phi_\infty}) \to \mathrm{Br}(\mathcal{V}_\infty)$$

is surjective.

We will prove Lemma 13.11 in Section 13.2.3.

Lemma 13.12. *Let k be global field with no real places. Then $X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$.*

Proof. Assume that $X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \neq \emptyset$. Let $x \in X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$. By Lemma 13.10 there exists a $p \in C_{\phi_\infty}(k)$ such that $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A}_k)_\bullet)$. Let $y \in \mathcal{V}_\infty(\mathbb{A}_k)_\bullet$ be such that $\rho_p(y) = x$. We shall show that $y \in \mathcal{V}_\infty(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

Indeed, let $b \in \mathrm{Br}(\mathcal{V}_\infty)$. By Lemma 13.11 there exists a $\tilde{b} \in \mathrm{Br}(X'_{\phi_\infty})$ such that $\rho_p^*(\tilde{b}) = b$. Now

$$(y, b) = (y, \rho_p^*(\tilde{b})) = (\rho_p(y), \tilde{b}) = (x, \tilde{b}) = 0$$

But by assumption $x \in X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$, so we have $(y, b) = (x, \tilde{b}) = 0$. Thus we have $y \in \mathcal{V}_\infty(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$ which is a contradiction. $\square$

13.2.3. *The surjectivity of ρ_p^*.* In this subsection, we shall prove the statement of Lemma 13.11. We switch gears for a moment and let $\alpha\colon X \to B$ be an arbitrary conic bundle given by datum $(\mathcal{L}, s, a)$.

Lemma 13.13. *The generic fiber X_η^s of $X^s \to B^s$ is isomorphic to $\mathbb{P}^1_{\kappa(B^s)}$, where $\kappa(B^s)$ is the field of rational functions on B^s .*

Proof. It is a smooth plane conic and it has a rational point since a is a square in $k_s \subset \kappa(B^s)$. $\square$

Lemma 13.14. *Denote the generic point of B by η. Let Z be the degeneracy locus. Assume that Z^s is the union of the irreducible components $Z^s = \bigcup_{1\le i\le r} Z_i$. Then there is a natural exact sequence of Galois modules.*

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus \mathbb{Z}Z_i \arrow[r, "\rho_1"] & \mathrm{Pic}\, B^s \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^- \arrow[r, "\rho_2"] & \mathrm{Pic}\, X^s \arrow[r, shift right, "\rho_3"'] \arrow[dr, "deg"'] & \mathrm{Pic}\, {X^s}_\eta \arrow[l, shift right, "\rho_4"'] \arrow[r] \arrow[d, equal] & 0 \\
 & & & & \mathbb{Z} &
\end{tikzcd}$$

where ρ_4 is a natural section of ρ_3.

Proof. Call a divisor of X^s vertical if it is supported on prime divisors lying above prime divisors of B^s, and horizontal otherwise. Denote by $Z_i^\pm$ the divisors that lie over Z_i and defined by the additional condition that $y = \pm\sqrt{a}z$. Now define ρ_1 by

$$\rho_1(Z_i) = (-Z_i, Z_i^+, Z_i^-)$$

and ρ_2 by

$$\rho_2(M, 0, 0) = \alpha^* M$$

$$\rho_2(0, Z_i^+, 0) = Z_i^+$$

$$\rho_2(0, 0, Z_i^-) = Z_i^-$$

Let ρ_3 be the map induced by $X^s_\eta \to X^s$. Each ρ_i is Γ_k-equivariant. Given a prime divisor D on X^s_η, we take $\rho_4(D)$ to be its Zariski closure in X^s. It is clear that $\rho_3 \circ \rho_4 = \mathrm{Id}$, so ρ_3 is indeed surjective.

The kernel of ρ_3 is generated by the classes of vertical prime divisors of X. In fact, there is exactly one above each prime divisor of B, except that above each $Z_i \in \mathrm{Div}\, B^s$ we have both $Z_i^+, Z_i^- \in \mathrm{Div}\, X^s$. This proves exactness at $\mathrm{Pic}\, X^s$.

Now, since $\alpha : X^s \to B^s$ is proper, a rational function on X^s with a vertical divisor must be the pullback of a rational function on B^s. Using the fact that the image of ρ_2 contains only vertical divisors, we prove exactness at

$$\mathrm{Pic}\, B^s \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^-$$

The injectivity of ρ_1 is then trivial. $\square$

We switch gears once again and let X be as in Poonen's example.

Lemma 13.15. *Let $p \in C_{\phi_\infty}(k)$ and $\rho_p : \mathcal{V}_\infty \to X_{\phi_\infty}$ be the corresponding map as above. Then the map of Galois modules*

$$\rho_p^* : \mathrm{Pic}(X^s_{\phi_\infty}) \to \mathrm{Pic}(\mathcal{V}^s_\infty)$$

has a section.

Proof. Consider the map $\phi_p : \mathbb{P}^1 \to \mathbb{P}^1 \times C_{\phi_\infty}$ defined by $x \mapsto (x, p)$. It is clear that the map $\rho_p : \mathcal{V}_\infty \to X_{\phi_\infty}$ comes from pulling back the conic bundle datum defining X_{ϕ_∞} over $\mathbb{P}^1 \times C_{\phi_\infty}$ by this map. Let $B = \mathbb{P}^1 \times C_{\phi_\infty}$, and consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus \mathbb{Z}Z_i \arrow[r] \arrow[d, shift left] & \mathrm{Pic}\, B^s \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^- \arrow[r] \arrow[d, shift left] & \mathrm{Pic}\, X^s_{\phi_\infty} \arrow[r, shift right, "deg"'] \arrow[d, "\rho_p^*"] & \mathbb{Z} \arrow[r] \arrow[l, shift right] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus \mathbb{Z}W_i \arrow[r] \arrow[u, bend left, "s_1"] & \mathrm{Pic}\, \mathbb{P}^1_{k_s} \oplus \bigoplus \mathbb{Z}W_i^+ \oplus \bigoplus \mathbb{Z}W_i^- \arrow[r] \arrow[u, bend left, "s_2"] & \mathrm{Pic}\, \mathcal{V}^s_\infty \arrow[r, shift right, "deg"'] & \mathbb{Z} \arrow[r] \arrow[l, shift right] & 0
\end{tikzcd}$$

where Z is the degeneracy locus of X_{ϕ_∞} over B, W is the degeneracy locus of $\mathcal{V}_\infty$ over $\mathbb{P}^1$, and $Z^s = \bigcup_{1 \le i \le r} Z_i$ and $W^s = \bigcup_{1 \le i \le r} W_i$ are decompositions into irreducible components. The existence of a section for ρ_p^* follows by diagram chasing and the existence of the compatible sections s_1 and s_2.

Every W_i $(1 \le i \le 4\ )$ is a point that corresponds to a different root $(w_i : x_i)$ of the polynomial $P_\infty(x, w)$. We can choose $Z_i \subset B^s$ to be Zariski closure of the zero set of $w_i x - x_i w$, and similarly $Z_i^\pm \subset X^s_{\phi_\infty}$ to be Zariski closure of the zero set of $y \pm \sqrt{a} z, w_i x - x_i w$.

Now we define: $Z_i = s_1(W_i)$ and $Z_i^{\pm} = s_2(W_i^{\pm})$ and the map $s_2 : \operatorname{Pic}\mathbb{P}^1_{k_s} \to \operatorname{Pic} B^s$ is defined by the unique section of the map $\phi_p : \mathbb{P}^1 \to \mathbb{P}^1 \times C_{\phi_\infty}$.

It is clear that s_1 and s_2 are indeed group-theoretic sections. To prove that s_1 and s_2 also respect the Galois action, we can write

$$p = (c, ((x_1^0 : w_1^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0))) \in C(k) \times_{\mathbb{P}^1(k)} E_{\phi_\infty}(k),$$

and since $\gamma(C(k)) = \{\infty\}$, the four points $\{(x_1^0 : w_1^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0)\}$ are exactly the four different roots of $P_\infty(x, w)$. ☐

Lemma 13.16 (Lemma 13.11)**.** *Let $p \in C_{\phi_\infty}(k)$. Then the map*

$$\rho_p^* : \operatorname{Br}(X_{\phi_\infty}) \to \operatorname{Br}(\mathcal{V}_\infty)$$

is surjective.

Proof. Denote by $s_p : \operatorname{Pic}\overline{(\mathcal{V}_\infty)} \to \operatorname{Pic}(X^s_{\phi_\infty})$ the section of

$$\rho_p^* : \operatorname{Pic}(X^s_{\phi_\infty}) \to \operatorname{Pic}(\mathcal{V}^s_\infty)$$

It is clear that s_p induces a section of the map

$$\rho_p^{**} : H^1(k, \operatorname{Pic}(X^s_{\phi_\infty})) \to H^1(k, \operatorname{Pic}(\mathcal{V}^s_\infty))$$

By the Hochschild–Serre spectral sequence for X, we have:

$$H^1(k, \operatorname{Pic}(X^s)) = \ker[\operatorname{Br}X \to \operatorname{Br}X^s]/\operatorname{Im}[\operatorname{Br}k \to \operatorname{Br}X]$$

Letting

$$\operatorname{Br}_1(X) := \ker[\operatorname{Br}X \to \operatorname{Br}X^s],$$

we get that the map $\rho_p^* : \operatorname{Br}_1(X_{\phi_\infty}) \to \operatorname{Br}_1(\mathcal{V}_\infty)$ is surjective. But since $\mathcal{V}^s_\infty$ is a rational surface (it is a Châtelet surface), we have $\operatorname{Br}\mathcal{V}^s_\infty = 0$, and thus $\operatorname{Br}_1(\mathcal{V}_\infty) = \operatorname{Br}(\mathcal{V}_\infty)$. So we get that $\rho_p^* : \operatorname{Br}(X_{\phi_\infty}) \to \operatorname{Br}(\mathcal{V}_\infty)$ is surjective. ☐

Part 5. **Appendices**

14. Appendix A: Obstructions Without Functors

As the main novelty of our methods is to introduce decompositions into subvarieties, one might wonder what happens if one tries to build local-global obstructions using only such decompositions. We extend the notation from above by omitting ω:

$$X(\mathbb{A}_{k,S})^{\mathscr{X}} = \bigcup_i X_i(\mathbb{A}_{k,S})$$
$$X(\mathbb{A}_{k,S})^{\mathscr{C}} = \bigcap_{\mathscr{X}\in\mathscr{C}} X(\mathbb{A}_{k,S})^{\mathscr{X}}.$$

For any ω, we then have:

$$X(k) \subseteq X(\mathbb{A}_{k,S})^{\mathscr{X},\omega} \subseteq X(\mathbb{A}_{k,S})^{\mathscr{X}}, X(\mathbb{A}_{k,S})^{\omega}.$$

The main point of this appendix is the following, open-ended question:

Question 14.1. For a variety X, describe $X(\mathbb{A}_{k,S})^{\mathcal{STRAT}}$ and $X(\mathbb{A}_{k,S})^{\mathcal{OPEN}}$.

In particular, for $X = \mathbb{A}^1$, this provides two interesting subsets of the adele ring $\mathbb{A}_{k,S}$. Is either a subring?

As a first step, we note that neither of these subsets is always equal to the set of rational points. We prove the following result modulo a lemma, then provide the lemma:

Proposition 14.2. *Let X/k be integral. If X/k has a generic k_v-point for every $v \in S$ (e.g., if X is a k-rational variety), we have $X(k) \not\subseteq X(\mathbb{A}_{k,S})^{\mathcal{STRAT}} \subseteq X(\mathbb{A}_{k,S})^{\mathcal{OPEN}}$.*

Proof. It suffices to find a k-algebra homomorphism $k(X) \to \mathbb{A}_{k,S}$, for such a homomorphism gives an $\mathbb{A}_{k,S}$-point of X that extends to every nonempty open in X. As the existence of such a homomorphism depends only on $k(X)$, it suffices to assume that X is affine and smooth. To do this, we need to find a point $\alpha \in X(\mathbb{A}_{k,S})$ such that $f(\alpha) \in \mathbb{A}_{k,S}^{\times}$ for every nonzero $f \in \mathcal{O}(X)$.

By spreading out, there exists a finite type model $\mathcal{X}$ of X over $\mathcal{O}_k$. As every element of $\mathcal{O}(X)$ is a $k^\times$-multiple of an element of $\mathcal{O}(\mathcal{X})$, it suffices to check $f(\alpha) \in \mathbb{A}_{k,S}^{\times}$ for every nonzero $f \in \mathcal{O}(\mathcal{X})$. This is equivalent to saying that $f(\alpha_v) \neq 0$ for all v, and $f(\alpha_v) \in \mathcal{O}_v \setminus \mathfrak{m}_v$ for all but finitely many v.

We choose an enumeration of $\mathcal{O}(\mathcal{X}) \setminus \{0\}$ by the positive integers, where to $i \in \mathbb{Z}_{>0}$ we associate $f_i \in \mathcal{O}(\mathcal{X})$. We thus wish to choose $\alpha \in \mathcal{X}(\mathbb{A}_{k,S})$ such that $f_i(\alpha) \in \mathbb{A}_{k,S}^{\times}$ for all $i \in \mathbb{Z}_{>0}$.

For $m \geq 1$ let $R_m = \{f_i\}_{1\leq i\leq m}$. By Lemma 14.4, there is some constant, which we denote by C_m, such that for all v of norm at least C_m, there is a smooth $\mathbb{F}_v$-point of $\mathcal{X}$ at which no element of R_m vanishes. We choose each C_m to be as small as possible for that m, so that $C_m \leq C_{m+1}$ for all m. We then set $D_m = m + C_m$, so that $D_m < D_{m+1}$, and $\lim_{m\to\infty} D_m = \infty$. Starting with $m = 1$, for each place v with norm in $[D_m + 1, D_{m+1}]$, we choose $a_v \in \mathcal{X}(\mathbb{F}_v)$ such that $f_i(a_v) \neq 0$ for all $1 \leq i \leq m$. It follows that for every m, the value of $f_m(a_v)$ is nonzero in $\mathbb{F}_v$ for v of norm greater than D_m.

For every $v \in S$ of norm $> D_1$, we choose an $\alpha_v \in \mathcal{X}(\mathcal{O}_v) \subseteq X(k_v)$ reducing to a_v modulo $\mathfrak{m}_v$ and which maps to the generic point of X. This exists by Lemma 14.3.

For every $v \in S$ of norm $\leq D_1$, we choose an arbitrary $\alpha_v \in X(k_v)$ that maps to the generic point of X.

We set $\alpha = \prod_{v \in S} \alpha_v$. Then for every nonzero $f \in \mathcal{O}(X)$, the value $f(\alpha_v)$ is nonzero for all v and invertible in $\mathcal{O}_v$ for all but finitely many v. That is, $f(\alpha)$ is invertible in $\mathbb{A}_{k,S}$, so α gives us our desired homomorphism from $k(X)$ to $\mathbb{A}_{k,S}$. $\square$

Lemma 14.3. *Let $\mathcal{X}$ be a finite type scheme over $\mathcal{O}_k$ with irreducible generic fiber X of dimension d. Let a be a smooth point in $\mathcal{X}(\mathbb{F}_v)$ for a finite place v of k. Then there is an $\mathcal{O}_v$-point α of $\mathcal{X}$ lifting a whose generic fiber is a generic point of $\mathcal{X}$.*

Proof. Let $x_1, \cdots, x_d$ be local parameters at a. Then Hensel's lemma says that there is a bijection between lifts of a to $\alpha \in \mathcal{X}(\mathcal{O}_v)$ and $(\mathcal{O}_v)^d$, given by

$$\alpha \mapsto (x_1(\alpha)/\pi_v, \cdots, x_d(\alpha)/\pi_v).$$

Let us choose an α associated to a set of d elements of $\mathcal{O}_v$ that are algebraically independent over k. Tensoring over $\mathcal{O}_k$ with k, we get a homomorphism $\mathcal{O}(X) \to \mathcal{O}_v \otimes_{\mathcal{O}_k} k = k_v$ whose image in k_v generates a field of transcendence degree d. This implies that the map $\mathcal{O}(X) \to k_v$ is injective; i.e., the point is generic. $\square$

Lemma 14.4. *Let $\mathcal{X}$ be a finite type integral scheme over $\mathcal{O}_k$ with generic fiber of dimension $d \geq 1$ that is geometrically integral and affine. Let R be a finite subset of $\mathcal{O}(\mathcal{X}) \setminus \{0\}$. Then for almost all places v of k, there is a smooth $\mathbb{F}_v$-point of $\mathcal{X}$ at which no element of R vanishes.*

Proof. By replacing R with the one-element set containing the product of all elements of R, we may assume that R has a single element, call it f.

By the chart in Appendix C of [Poo17], the properties "flat," "affine," and "geometrically integral fibers" satisfy spreading out. We can therefore find a set T of places of k so that the fiber of $\mathcal{X}$ over $\mathcal{O}_{k,T}$ has geometrically integral fibers and is flat and affine. By flatness, all of its fibers have the same dimension, which must be r. We replace $\mathcal{X}$ by its fiber over $\mathcal{O}_{k,T}$ at the expense of modifying finitely many places. The zero set of f is a closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ with generic fiber of dimension $d-1$. We can enlarge T so that $\mathcal{Y}$ is flat, ensuring that the dimension is constant. Let $\mathcal{Z}$ denote the open subscheme of $\mathcal{X}$ obtained by removing $\mathcal{Y}$. Its fibers are geometrically irreducible of dimension d. We need to show that $\mathcal{Y}(\mathbb{F}_v) \nsubseteq \mathcal{X}(\mathbb{F}_v)$, or that $\mathcal{Z}(\mathbb{F}_v) \neq \emptyset$, for almost all v.

Let us apply the Lang-Weil bounds, i.e., Theorem 7.7.1(iv) of [Poo17], with $Y = \operatorname{Spec} \mathcal{O}_{k,T}$ and $X = \mathcal{Z}$. Noting that $\mathcal{Z}_{\mathbb{F}_v}$ is geometrically integral, $\mathcal{Z}$ has a smooth $\mathbb{F}_v$-point for q_v sufficiently large, so it must be true for almost all v. □

15. Appendix B: Reformulation in Terms of Cosheaves

Definition 15.1. Let C be a site. Then a *precosheaf* $\mathcal{F}$ on C is a functor $\mathcal{F}\colon C \to \mathbf{Set}$.

Definition 15.2. Let C be a site. Then a *cosheaf* $\mathcal{F}$ on C is a precosheaf such that for any $U \in \operatorname{ob}(C)$ and covering $\{U_i\}_i$ of U, the natural map

$$\operatorname{colim}\left(\coprod_{i,j} \mathcal{F}(U_i \times_U U_j) \rightrightarrows \coprod_i \mathcal{F}(U_i)\right) \xrightarrow{g} \mathcal{F}(U)$$

is an isomorphism.

Definition 15.3. In the notation of Definition 15.2, if g is assumed only to be injective, we call C a *separated precosheaf*.

For any scheme X, we let X_{Zar} denote the (big or small) Zariski site of X.

Definition 15.4. Let R be a k-algebra and X a scheme over k. There is a precosheaf $\mathcal{F}_R$ on X_{Zar} associating to every $U \in \operatorname{ob}(X_{\mathrm{Zar}})$ the set $U(R)$.

Lemma 15.5. *The precosheaf $\mathcal{F}_R$ is always separated. If R is a local ring (e.g. a field), then $\mathcal{F}_R$ is a cosheaf.*

Proof. Since the lemma is about any scheme X, it suffices to consider Zariski open covers $\{U_i\}_i$ of the whole scheme X.

Consider any two elements of the domain of g whose images in $\mathcal{F}_R(X)$ are equal. Let $x_1, x_2 \in \coprod_i \mathcal{F}_R(U_i)$ be representatives for these. Suppose $x_1 \in \mathcal{F}_R(U_i)$ and $x_2 \in \mathcal{F}_R(U_j)$. The universal property of fiber products of schemes then gives us a map $x_1 \times x_2\colon \operatorname{Spec} R \to U_i \times_X U_j$. Thus x_1 and x_2 each come from an element of $\coprod_{i,j} \mathcal{F}(U_i \times_U U_j)$, so they correspond to the same element of the domain of g. It follows that g is injective, so $\mathcal{F}_R$ is separated.

Now suppose that R is a local ring. Let $x : \operatorname{Spec} R \to X$. Then the closed point of $\operatorname{Spec} R$ maps to some physical point of X, which must be contained in U_i for some i. Thus the preimage of U_i under x contains the closed point of $\operatorname{Spec} R$, so it is all of $\operatorname{Spec} R$. It follows that x factors as $\operatorname{Spec} R \to U_i \to X$, i.e., comes from $\mathcal{F}_R(U_i)$. As x was arbitrary, the map g from Definition 15.2 is surjective. As g is surjective and injective, it is bijective, so $\mathcal{F}_R$ is a cosheaf. □

Example 15.6. Let $X = \mathbb{P}^1_{\mathbb{Q}}$. If S contains at least two elements, then $\mathcal{F}_{\mathbb{A}_{k,S}}$ is not a cosheaf. Indeed, let $v_1, v_2 \in S$. Let $\alpha \in X(\mathbb{A}_{k,S})$ have coordinate 0 in the v_1-component and ∞ in the v_2-component. Then α is a $\mathbb{A}_{k,S}$-point of neither $X \setminus \{0\}$ nor $X \setminus \{\infty\}$, so the cosheaf condition is violated.

Definition 15.7. Let (ω, S, k) be an obstruction datum and X/k a scheme. By Proposition 3.8, there is a precosheaf $\underline{X(\mathbb{A}_{k,S})^{\omega}}$ associating to $U \in \mathrm{ob}(X_{\mathrm{Zar}})$ the set

$$U(\mathbb{A}_{k,S})^{\omega}.$$

Furthermore, in the terminology of Section 3.1, we denote by $\underline{X(\mathbb{A}_{k,S})^{\mathscr{C},\omega}}$ the precosheaf associating to U the set

$$U(\mathbb{A}_{k,S})^{\mathscr{C},\omega}.$$

Definition 15.8. By Theorem 2.1(a) of [Pra16], the inclusion functor from cosheaves on C to precosheaves on C has a right adjoint, known as *cosheafification.*

Proposition 15.9. *Suppose that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a separated precosheaf (e.g., suppose the answer to Question 3.2 is yes for (ω, S, k)) or that $\underline{X(\mathbb{A}_{k,S})^{\omega}}$ is a separated precosheaf. Suppose X is a variety over k or $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a cosheaf. Then the cosheafification of $\underline{X(\mathbb{A}_{k,S})^{\omega}}$ is $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$.*

Proof. We first must show that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a cosheaf. Let $U \in \mathrm{ob}(X_{\mathrm{Zar}})$, and let $\{U_i\}_{i\in I}$ be a finite open cover of U. We need to show that the associated g from Definition 15.2 is surjective and injective. We start by showing that it is surjective.

We therefore need to show that any $\alpha \in U(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ is in $U_i(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ for some i. If we are not already assuming this fact, we are assuming that X is a variety. As k is countable, there are countably many Zariski open subsets of X and therefore countably many finite open covers of U. We enumerate all finite open covers of U as $\mathcal{W}_k = \{W_{jk}\}_j$ for $k \in \mathbb{Z}_{>0}$. For each positive integer k, we let $\mathcal{T}_k = \{T_{jk}\}_j$ be the common refinement of $\mathcal{W}_l$ for $1 \le l \le k$, so that $\mathcal{T}_{k+1}$ is always a refinement of $\mathcal{T}_k$, and the system is still cofinal. Finally, we let $V_{ijk} = U_i \cap T_{jk}$, and we let $\mathcal{V}_k$ denote the open cover $\{V_{ijk}\}_{ij}$ (where k is fixed).

As a result, the system $\mathcal{V}_k$ of open covers satisfies the following properties:

- The system is cofinal as k ranges over the positive integers
- The covering $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$
- For each i and k, the collection $\{V_{ijk}\}_j$ is a covering of U_i

Furthermore, for fixed i, the open covers $\{V_{ijk}\}_j$ form a cofinal system of open covers of U_i as k varies, and each such covering is a refinement of the previous one.

For every $k \in \mathbb{Z}_{>0}$, let $I_k \subseteq I$ be the set of $i \in I$ for which there exists j such that $\alpha \in V_{ijk}(\mathbb{A}_{k,S})$. Because $\alpha \in U(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$, the set I_k is nonempty for all k. Furthermore,

$I_{k+1} \subseteq I_k$, as $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$. As I is finite, the set $\bigcap_k I_k$ is nonempty. Choose i in this intersection.

We conclude that $\alpha \in U_i(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$, since the open covers $\{V_{ijk}\}_j$ form a cofinal system of open covers of U_i. It follows that g is surjective.

To show that g is injective, we only need consider the case where we assume that $\underline{X(\mathbb{A}_{k,S})^\omega}$ is a separated precosheaf. Let $\alpha_i \in U_i(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ and $\alpha_j \in U_j(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ map to the same element of $U(\mathbb{A}_{k,S})$.

****** Everything here is not so sure:

Probably not true: By separatedness for $\underline{X(\mathbb{A}_{k,S})^\omega}$, they must come from some $\alpha \in (U_i \cap U_j)(\mathbb{A}_{k,S})^\omega$.

It suffices to show that $\alpha \in (U_i \cap U_j)(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$.

For this, let $\{V_i\}$ be an open cover of $U_i \cap U_j$.

******

Now let $\mathcal{G}$ be a cosheaf. It suffices to show that every precosheaf map $f\colon \mathcal{G} \to \underline{X(\mathbb{A}_{k,S})^\omega}$ comes from a unique map $\mathcal{G} \to \underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$. Uniqueness follows from the injectivity of $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}} \to \underline{X(\mathbb{A}_{k,S})^\omega}$. It is simply necessary to show that the image of any such f lands in $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$.

Let $U \in \mathrm{ob}(X_{\mathrm{Zar}})$, and let $s \in \mathcal{G}(U)$. We want to show that $f(s) \in U(\mathbb{A}_{k,S})^\omega$ actually lies in the subset $U(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$. For this, let $\{U_i\}$ be a covering of U. We want to show that $f(s) \in U_i(\mathbb{A}_{k,S})^\omega$ for some i. By the cosheaf property, there exists i such that s is the corestriction of some element $s_i \in \mathcal{G}(U_i)$. But then $f(s_i)$ is the element of $U_i(\mathbb{A}_{k,S})^\omega$ we seek, so we are done.

Under the assumption about Question 3.2, Lemma 15.5 implies that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a cosheaf. $\square$

Our main results can therefore be thought of as a description of the cosheafification of $\mathcal{F}_{(\mathrm{f-ab},f,k)}$ or $\mathcal{F}_{(\mathrm{f-cov},f,k)}$.

Furthermore, the interest of Question 14.1 is based on the fact that $\mathcal{F}_{\mathbb{A}_{k,S}}$ is not a cosheaf. In fact, by Proposition 15.9 and Lemma 15.5, this question is precisely asking to describe the cosheafification of this precosheaf.

***** Get rid of the last sentence in the paragraph above if I can't include the hypothesis that $\underline{X(\mathbb{A}_{k,S})^\omega}$ is a separated precosheaf

Finally, in this language, this cosheafification contains $\mathcal{F}_k$ precisely because $\mathcal{F}_k$ maps to $\mathcal{F}_{\mathbb{A}_{k,S}}$ and is a cosheaf.

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