A QUICK PROOF THAT THE CHARACTER TABLE OF A FINITE GROUP IS A SQUARE

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I've attended more than one class on representation theory of finite groups, and the proof that the number of irreducible representations is at least the number of conjugacy classes always seems unnecessarily complicated. One takes an arbitrary class function and then proves that if it is orthonormal to all of the characters, then it is trivial. I was happy to find the following proof while playing around. Although it might be equivalent in some sense to the usual proof, I find it useful to write this up, and I think that teachers of representation theory might benefit from it.

Lemma (Schur's Lemma)**.** *Let V, W be irreducible representations of a finite group G over an algebraically closed field K of characteristic 0. Then every homorphism $V \to W$ of representations is either 0 or an isomorphism, and every homomorphism from V to itself is multiplication by a constant.*

Theorem. *The number of irreducible representations of G is at least the number of conjugacy classes of G.*

Proof. We now use the group algebra $K[G]$ and view it as a representation of G. We recall that a representation of G on a vector space is the same as an algebra homomorphism of $K[G]$ into the endomorphisms of that vector space.

Let $C[G] \subseteq K[G]$ be the subalgebra of $K[G]$ generated by elements of the form $\sum_{g \in C} g$ for a conjugacy class C in G. Then all elements of $C[G]$ commute with the rest of $K[G]$, so they act as homomorphisms of G-representations. We have a K-algebra homormorphism

$$C[G] \to \bigoplus_{V \text{irreducible}} K$$

whose component corresponding to V sends an element of $C[G]$ to its action on V (which is a scalar by Schur's Lemma). Splitting $K[G]$ into a direct sum of irreducible representations of G, we see that the kernel of this homomorphism is trivial, and the desired inequality follows. □

As usual, one proves the other direction by showing that the characters of irreducible representations are orthonormal by examining the representation $\mathrm{Hom}(V, W)$ and uses this to conclude.