Explicit Motivic Chabauty-Kim Method

D. Corwin

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Talk Overview

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Part I: Introduction: Effective Faltings and Chabauty's Method

Background: Faltings' Theorem

Question

Given a polynomial $f(x, y) \in \mathbb{Z}[x, y]$, can we find $(x, y) \in \mathbb{Q}^2$ such that $f(x, y) = 0$? And if so, how many?

Such a polynomial defines a (smooth projective) algebraic curve X - which has a geometric genus g

Theorem (Faltings '83)

If $g \geq 2$, the set $X(\mathbb{Q})$ and therefore the set of rational solutions to $f(x, y) = 0$ is finite.

In addition:

- If $g = 0$, there are always 0 or infinitely many solutions
- If $g = 1$, there is an elliptic curve associated with X, and its rational points form a finitely generated abelian group

Effective Faltings

Problem (Effective Faltings)

When $X(\mathbb{Q})$ is finite, find it.

- Computers can quickly find a list of solutions that is "likely" complete
- If we have a list of solutions, how do we prove our list is complete?
- Chabauty's method and Kim's "non-abelian" version produce p-adic analytic functions that vanish on the set of all solutions.
- Can sometimes (more detail later) use to provably find set of solutions via p-adic Newton's method

Refined Problem (Chabauty-Kim Theory)

Find p-adic analytic (Coleman) functions on $X(\mathbb{Q}_p)$ that vanish on $X(\mathbb{Q})$ using (non-abelian) Chabauty.

Classical Chabauty's Method

- Mordell conjectured in 1922 that $X(\mathbb{Q})$ is finite if X has genus $g \geq 2$
- First proof in some cases by Chabauty in 1940's using the following method:
- Embed[i] X into a abelian variety J and consider the diagram:

$$\begin{array}{ccccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) & & \\ \downarrow & & \downarrow & \searrow & \\ J(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}_p) & \xrightarrow{\log} & \mathrm{Lie}(J_{\mathbb{Q}_p}) \end{array}$$

- Mordell-Weil: $J(\mathbb{Q})$ is finitely generated; set $r := \mathrm{rank}_{\mathbb{Z}} J(\mathbb{Q})$
- Chabauty: If $r < g$, then $X(\mathbb{Q}_p) \cap J(\mathbb{Q}) \subseteq J(\mathbb{Q}_p)$ is finite.
- Idea: $\dim \mathrm{Lie}(J_{\mathbb{Q}_p}) = \dim J(\mathbb{Q}_p) = g$, $\dim X(\mathbb{Q}_p) = 1$ and $\dim \overline{J(\mathbb{Q})} = r < g \Rightarrow \dim(X(\mathbb{Q}_p) \cap J(\mathbb{Q})) = 0 \Rightarrow$ it's discrete $\Rightarrow$ it's finite (because $J(\mathbb{Q}_p)$ compact)

[i] Pointed embedding: $\exists\, b \in X(\mathbb{Q})$ mapping to $O \in J$

Chabauty-Coleman

- When $r < g$, there is a nonzero $\omega \in \Omega^1(J) = \mathrm{Lie}(J_{\mathbb{Q}_p})^\vee$ vanishing on $\log(J(\mathbb{Q}))$
- $\Rightarrow f := (\omega \circ \log)\big|_{X(\mathbb{Q}_p)}$ is a nonzero p-adic analytic function on $X(\mathbb{Q}_p)$ that vanishes on $X(\mathbb{Q})$ (by commutativity)
- Coleman: f has special properties (now "Coleman function"), so its zero set
$$Z(\{f\})$$
is discrete in $X(\mathbb{Q}_p) \Rightarrow$ finite
- More generally, let $\mathcal{I}_{CK,1}$ denote the ideal generated by all functions on $X(\mathbb{Q}_p)$ obtained in this way
- We get
$$X(\mathbb{Q}) \subseteq Z(\mathcal{I}_{CK,1}) \subseteq X(\mathbb{Q}_p)$$
and if $r < g$, then $\#Z(\mathcal{I}_{CK,1}) < \infty$.
- We can compute as described on the next slide:

Chabauty-Coleman: Effectivity

- Coleman shows

$$f(P) := (\omega \circ \log)(P) = \int_b^P \omega$$

for an explicit notion of p-adic integration on X

- One computes such ω by finding an explicit basis for $J(\mathbb{Q})$
- Newton's method lets us find the size[ii] of $Z(\mathcal{I}_{CK,1})$ and approximate it p-adically
- If we find $\#Z(\mathcal{I}_{CK,1})$ elements of $X(\mathbb{Q})$, then we know the list is complete!
- Two problems with getting Effective Faltings from this:
  1. Sometimes $r \geq g$
  2. Even if $r < g$, might have $\#X(\mathbb{Q}) < \#Z(\mathcal{I}_{CK,1})$

[ii] Technically, the degree as a p-adic analytic space

Non-Abelian Chabauty

- Kim defines a series

$$\mathcal{I}_{CK,1} \subseteq \mathcal{I}_{CK,2} \subseteq \mathcal{I}_{CK,3} \subseteq \cdots$$

  of sets of p-adic analytic functions on $X(\mathbb{Q}_p)$ with

$$X(\mathbb{Q}) \subseteq \cdots \subseteq Z(\mathcal{I}_{CK,2}) \subseteq Z(\mathcal{I}_{CK,1}) \subseteq X(\mathbb{Q}_p)$$

- If ranks of certain "generalized Selmer groups" (depending on n) are not too large, then $\mathcal{I}_{CK,n} \neq \{0\}$ (and hence $\#Z(\mathcal{I}_{CK,n}) < \infty$)
- The Quadratic Chabauty method of Balakrishnan et al computes $\mathcal{I}_{CK,2}$ in many cases by relating the functions to p-adic heights.
- Approach of C–Dan-Cohen–Wewers theoretically applies to $\mathcal{I}_{CK,n}$ for all n
- By considering $\mathcal{I}_{CK,n}$ for sufficiently large n, one hopes to solve the "Two Problems":

Effectivity and Kim's Conjecture

- Conjectures on Galois cohomology imply that $\mathcal{I}_{CK,n} \neq \{0\}$ for $n >> 0 \Rightarrow$ This solves ❶
- For Kim's method to solve ❷, we need:

Conjecture (Kim et al., 2014)

For sufficiently large n, the set[a] $Z(\mathcal{I}_{CK,n})$ of common zeroes of functions in $\mathcal{I}_{CK,n}$ is precisely $X(\mathbb{Q})$.

[a] In fact, one really wants the (not necessarily reduced) analytic subspace.

This conjecture reduces effective Faltings to the problem of computing $\mathcal{I}_{CK,n}$

- In the next Part, we review some results of Dan-Cohen–Wewers, Dan-Cohen, C–Dan-Cohen, and C on $\mathcal{I}_{CK,n}$ for $n = 2, 3, 4$.
- If time, we will discuss the actual method in Parts III and IV

Results of C–Dan-Cohen–Wewers

Part II: Results of C–Dan-Cohen–Wewers

Siegel-Faltings

- C–Dan-Cohen–Wewers work with Kim's method in a simpler kind of Diophantine problem: Effective Siegel's Theorem
- Siegel's Theorem is best explained as part of a general statement that includes Faltings' Theorem:

Theorem: Siegel-Faltings

If X is a smooth *hyperbolic* curve over $R = \mathcal{O}_K[1/S]$ for a number field K and finite set of places S, then $X(R)$ is finite.

Definition: Hyperbolic Curve

An algebraic curve X over a subring R of $\mathbb{C}$ is *hyperbolic* if the manifold $X(\mathbb{C})$ has negative Euler characteristic
Equivalently: iff its fundamental group is non-abelian

- Kim's method applies equally to Effective Siegel: find $X(R)$ given X, R when X is an open subset of $\mathbb{P}^1$ or an elliptic curve

Siegel Case 1: The Unit Equation

Observation

The following sets are naturally in bijection for a ring R

- $x, w \in R^\times$ such that $x + w = 1$ (the "Unit Equation")
- $x, y \in R$ such that $x(x-1)y = 1$ (via $y = -\frac{1}{xw}$)
- $X(R)$ with $X = \mathbb{P}^1 \setminus \{0, 1, \infty\} = \operatorname{Spec} \mathbb{Z}\left[x, \frac{1}{x(x-1)}\right]$

- For $R = \mathcal{O}_K[1/S]$, finite by Faltings-Siegel (originally proven by Siegel around 1929 using Diophantine approximation)
- Reproven by Minhyong Kim in 2004 for $K = \mathbb{Q}$ (aka $R = \mathbb{Z}[1/N]$) $\Rightarrow$ first test case of Kim's method!
- The p-adic analytic functions in this case involve p-adic polylogarithms for some $p \nmid N$; we now review these

(p-adic) Polylogarithms

- We first recall the definition of complex polylogarithms

Definition

The k-logarithm is $\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}$

- These functions satisfy the recursive differential equation

$$\frac{d}{dz}\mathrm{Li}_k(z) = \frac{1}{z}\mathrm{Li}_{k-1}(z),$$

with $\mathrm{Li}_1(z) = -\log(1-z) = \int \frac{dz}{1-z}$ and $\mathrm{Li}_k(0) = 0$ for all k.

- p-adic polylogarithms $\mathrm{Li}_k^p(z)$ are defined as p-adic analytic functions satisfying the same differential equations and initial conditions
- As the p-adics are totally disconnected, one must use Coleman's theory to ensure the differential equations have a unique solution

Recent Explicit Results

X is affine $\Rightarrow$ set $\mathcal{I}^R_{CK,n}$ depends on $R = \mathbb{Z}[1/N]$ (and vanishes on $X(R)$)

Theorem (Dan-Cohen–Wewers, 2013)

- For $R = \mathbb{Z}[1/\ell]$ and all $p \neq \ell$, the following Coleman function is in $\mathcal{I}^R_{CK,2}$:

$$2\mathrm{Li}_2^p(z) - \log^p(z)\mathrm{Li}_1^p(z)$$

- For $R = \mathbb{Z}[1/2]$ and $p \neq 2$, the following Coleman function is in $\mathcal{I}^R_{CK,4}$:

$$24\log^p(2)\zeta^p(3)\mathrm{Li}_4^p(z) + \frac{8}{7}\left(\log^p(2)^4 + 24\mathrm{Li}_4^p(\frac{1}{2})\right)\log^p(z)\mathrm{Li}_3^p(z)$$
$$+\left(\frac{4}{21}\log^p(2)^4 + \frac{32}{7}\mathrm{Li}_4^p(\frac{1}{2}) + \log^p(2)\zeta^p(3)\right)\log^p(z)^3\log^p(1-z)$$

Recent Results, cont.

- In 2015, Dan-Cohen posted a preprint[iii] showing that this could be made into an algorithm, whose halting is conditional on refinements of conjectures due to Kim and Goncharov.

Theorem (C–Dan-Cohen, 2017)

For $R = \mathbb{Z}[1/3]$ and $p \neq 2, 3$, the following Coleman function is in $\mathcal{I}^R_{CK,4}$:

$$\zeta^p(3)\log^p(3)\mathrm{Li}_4^p(z) - \left(\frac{18}{13}\mathrm{Li}_4^p(3) - \frac{3}{52}\mathrm{Li}_4^p(9)\right)\log^p(z)\mathrm{Li}_3^p(z)$$
$$-\frac{(\log^p(z))^3\mathrm{Li}_1^p(z)}{24}\left(\zeta^p(3)\log^p(3) - 4\left(\frac{18}{13}\mathrm{Li}_4^p(3) - \frac{3}{52}\mathrm{Li}_4^p(9)\right)\right),$$

13 in the denominator is related to $\mathcal{L}_3(\xi_2) = \frac{13}{6}\zeta(3)$ on p.6 of *Classical and Elliptic Polylogarithms and Special Values of L-Series* by Zagier and Gangl

[iii] Now published in Algebra and Number Theory

Recent results, cont.

- Dan-Cohen–Wewers and C–Dan-Cohen have used these functions to verify[iv] a version of Kim's Conjecture for integral points in special cases:

Conjecture (Kim et al., 2014)

The space of common zeroes of elements of $\mathcal{I}^R_{CK,n}$ is precisely $X(R)$ for sufficiently large n.

- Another article of C–Dan-Cohen presents an improved algorithm
- Kim's conjecture and some standard conjectures about mixed motives imply the algorithm halts
- If the algorithm halts, then it provably gives the correct answer

[iv] In C–Dan-Cohen, we actually showed that one needs multiple polylogarithms (not just polylogarithms) to make the conjecture work, yet all our functions above involve only polylogarithms. However, one may bring multiple polylogarithms into the picture using the S_3-action on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Siegel Case 2: Punctured Elliptic Curve

Let $X = E' = E \setminus \{O\}$ for some elliptic curve $E/\mathbb{Q}$ (given by $y^2 = x^3 + ax + b$) for which $4a^3 + 27b^2$ is a unit in $R = \mathcal{O}_K[1/S]$

Theorem (Siegel)

$|E'(R)| < \infty$; i.e., $y^2 = x^3 + ax + b$ has finitely many solutions for $x, y \in R$

Also proven by Siegel; re-proven when E is CM by Kim.

- (Multiple) polylogarithms are iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\} \Rightarrow$ need iterated integrals on E'
- For a sequence $\omega_1, \cdots, \omega_k$ of differential forms on X and basepoint $b \in X$, we define (p-adic) iterated integrals recursively by

$$d \int_b \omega_1 \cdots \omega_k = \omega_k \int_b \omega_1 \cdots \omega_{k-1}$$

as Coleman functions on X that vanish at b.

Polylogs as Iterated Integrals

Exercise: Check that

$$\mathrm{Li}_k^p = \int_0 \frac{dz}{1-z} \underbrace{\frac{dz}{z} \cdots \frac{dz}{z}}_{k-1}$$

(This is a simple comparison of Slide 14 with Slide 18!)

Chabauty-Kim for a Punctured Elliptic Curve

- Set $\eta_0 := \frac{dx}{y}$ and $\eta_1 = \frac{xdx}{y}$
- Set $J_1 := \int \eta_0$, $J_2 := \int \eta_0\eta_1$, $J_3 := \int \eta_0\eta_1\eta_0$, and $J_4 := \int \eta_0\eta_1\eta_1 + 2\int \eta_1$

Theorem (C, 2021)

If $\ell \neq p$ are primes and E is an elliptic curve over $\mathbb{Q}$ of p-Selmer rank 1 with good ordinary reduction at p and $\mathcal{L}_p(E, 2) \neq 0$, then there is a function of the form

$$c_1 J_4 + c_2 J_3 + c_3 J_1 J_2 + c_4 J_1^3 + c_5 J_1$$

in $\mathcal{I}_{CK,3}^{\mathbb{Z}[1/\ell]}$, with $c_i \in \mathbb{Q}_p$.

In progress: computing for specific elliptic curves, projective examples

Can say more about the c_i:

Bonus Slide: What the Function *Actually* Looks Like

$$(-2f_{\pi_0}f_\tau f_{\sigma_0})J_4 + (f_{\pi_0}f_\tau f_{\sigma_1})J_3 - (f_{\sigma_1}(f_{\tau\pi_0} - f_{\pi_0\tau}) - 2f_{\sigma_0}f_{\pi_1\tau})J_1J_2$$
$$-\left(\frac{f_{\pi_0}f_\tau(f_{\sigma_1}f_{\pi_0\pi_1\pi_0} - 2f_{\sigma_0}f_{\pi_0\pi_1^2}) - (f_{\sigma_1}(f_{\tau\pi_0} - f_{\pi_0\tau}) - 2f_{\sigma_0}f_{\pi_1\tau})(f_{\pi_0}f_{\pi_0\pi_1})}{f_{\pi_0}^3}\right)J_1^3$$
$$+(4f_{\pi_1}f_\tau f_{\sigma_0})J_1$$

with

- $f_{\pi_i} = \int_b^z \eta_i$ for z a generator of $E(\mathbb{Q})$ [v]
- $f_\tau = \log^p(\ell)$ [vi]
- f_{σ_i} related to syntomic regulator from $K_2(E)^{(2)}_{\mathrm{odd}}$
- Combinations are Tannakian periods satisfying "shuffle relations"

[v] aka syntomic regulator applied to $z \in K_0(E)$

[vi] aka syntomic regulator applied to $\ell \in K_1(R)$

Explaining the p-adic L-Function

- The condition $\mathcal{L}_p(E,2) \neq 0$ is used to show that $H^1_f(G_{\mathbb{Q}}; V_p(E)^\vee) = 0$
- Here H^1_f is a Bloch-Kato Selmer group, and $V_p(E)^\vee = V_p(E)(-1)$ is the dual of the p-adic Tate module
- The implication $\mathcal{L}_p(E,2) \neq 0 \Rightarrow H^1_f(G_{\mathbb{Q}}; V_p(E)^\vee) = 0$ uses modularity of E and Kato's Iwasawa Main Conjecture
- To check $\mathcal{L}_p(E,2) \neq 0$, we may approximate it using Manin symbols:

$$\mathcal{L}_p(E,2) = \lim_{n\to\infty} \sum_{1 \leq a < p^n} \left(\frac{a}{\alpha^n} \left[\frac{a}{p^n} \right]^+ - \frac{a}{\alpha^{n+1}} \left[\frac{a}{p^{n-1}} \right]^+ \right)$$

  where α is the unit root of $T^2 - a_p(E)T + p$ and

$$[r]^+ = \frac{1}{\Omega_E} \operatorname{Re} \left(2\pi i \int_{i\infty}^{r} f(z)\, dz \right)$$

  is a Manin symbol, with f the modular form associated to E.
- Manin symbols are rational numbers and are implemented in SageMath!

Part III: How Non-Abelian Chabauty's Method Works

Motivating Non-Abelianness

- The hypothesis in Siegel-Faltings suggests a mysterious relationship between fundamental groups and rational points
- $=$ Grothendieck's anabelian philosophy:
  non-abelian fundamental group $\Rightarrow$ few rational points
- Grothendieck: there is a map

  $$X(K) \to H^1(\mathrm{Gal}(\overline{K}/K); \pi_1\widehat{(X(\mathbb{C})}))$$

  for a smooth geometrically connected variety X over a field $K \subseteq \mathbb{C}$
- The Jacobian J can have infinitely many rational points, and its fundamental group is abelian
- In fact $\pi_1(J) = \pi_1(X)^{\mathrm{ab}} := \pi_1(X)/[\pi_1(X), \pi_1(X)]$
- Anabelian philosophy suggests: when $r = \mathrm{rank}_{\mathbb{Z}} J(\mathbb{Q})$ is too large, the problem is that $\pi_1(J)$ is abelian!
- Kim's idea: replace $\pi_1(X)^{\mathrm{ab}}$ with a larger quotient of $\pi_1(X)$

Towards non-abelian Chabauty

- E.g., replace $\pi_1^{\mathrm{ab}} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ with $\pi_1(X)/[[\pi_1(X), \pi_1(X)], \pi_1(X)]$
- To replace $\pi_1(X)^{\mathrm{ab}}$ with a larger quotient of $\pi_1(X)$, we first need to rewrite all of Classical Chabauty in terms of $\pi_1(X)^{\mathrm{ab}}$! How?
- There is an embedding $J(\mathbb{Q}) \to \mathrm{Sel}(J/\mathbb{Q}, p^{\infty})$ into a *Selmer group* ("essentially" isomorphism by BSD conjecture)
- $\mathrm{Sel}(J/\mathbb{Q}, p^{\infty})$ may be defined as a Galois cohomology group

$$H^1_f(G_{\mathbb{Q}}; H_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}; \mathbb{Q}_p))$$

- $H_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}; \mathbb{Q}_p)$ is étale homology and $= \pi_1(X)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p$
- $H_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}; \mathbb{Q}_p)$ is a p-adic vector space. What is a "non-abelian vector space"?
- E.g., what kind of object is $(\pi_1(X)/[[\pi_1(X), \pi_1(X)], \pi_1(X)]) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p$?

Unipotent Groups

- The underlying set of an n-dimensional vector space over a field F is represented by affine space $\mathbb{A}^n$
- Addition of vectors makes it into a group; for $n = 1$ this is called $\mathbb{G}_a$, the "additive group scheme"
- An n-dimensional vector space corresponds to $\mathbb{G}_a^n$ (up to isomorphism)
- $\mathbb{G}_a$ naturally embeds into GL_2 as

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

  the group of *unipotent matrices*
- More generally, we may consider the group of all unipotent matrices in GL_n, which is a *unipotent group* (non-abelian for $n \geq 3$)
- Thus the generalization of a $\mathbb{Q}_p$-vector space with continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action (aka p-adic Galois representation) is:
  a unipotent algebraic group U over $\mathbb{Q}_p$ with continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $U(\mathbb{Q}_p)$

Unipotent Fundamental Group

- For each positive integer n, there is a p-adic unipotent group

$$U_n := \pi_1^{\text{ét,un}}(X_{\overline{\mathbb{Q}}})_n$$

  with $U_1 \cong H_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}; \mathbb{Q}_p)$ and
  $U_2 = (\pi_1(X)/[[\pi_1(X), \pi_1(X)], \pi_1(X)]) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p$
- There is a Zariski dense map $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}) \to U_n(\mathbb{Q}_p)$
- There is a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $U_n(\mathbb{Q}_p)$ making the above map equivariant
- The analogue of Lie $J_{\mathbb{Q}_p}$ is a quotient U_n/F^0
- Kim's diagram becomes:

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ H^1_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), U_n) & \longrightarrow & U_n/F^0 \end{array}$$

- $\mathcal{I}_{CK,n}$ defined using this diagram

Part IV: Explicit Motivic Chabauty-Kim

Toward Tannakian Formalism

- Problem: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is complicated (so is f)
- Idea: consider the category of all relevant $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations (depends on J and R)
- This category is not as hard to describe:
  - Its semisimple objects form the category of representations of the Mumford-Tate group $\mathrm{MT}(J)$ of J
  - Its Ext^1's are Galois cohomology (whose dimensions are predicted by the Bloch-Kato conjectures)
  - Ext^i (conjecturally) vanishes for $i \geq 2$
- The Tannakian formalism says it is the category of representations of a pro-algebraic group $\pi_1^J(\mathbb{Q})$ - an extension of $\mathrm{MT}(J)$ by a pro-unipotent group
- The coordinate ring of the pro-unipotent part is a Hopf algebra we call

$$A^J(R)$$

- For $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ it is graded; for $X = E'$, it has a GL_2-action; more generally, an action of $\mathrm{MT}(J)$

Using the Tannakian Formalism

- The problem of understanding $H^1_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), U_n)$ becomes a matter of understanding

$$H^1(\pi_1^J(\mathbb{Q}); U_n)$$

- This reduces to a question about cocycle maps between algebraic groups!
- To understand $\pi_1^J(\mathbb{Q})$, we want to understand $A^J(R)$
- Our knowledge of the category tells us the abstract structure of $A^J(R)$ (as a Hopf algebra with action of $\mathrm{MT}(J)$)
- We also need to know the image of the map

$$H^1_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), U_n) \to U_n/F_0$$

- We thus need to relate the abstract structure to a certain "period map" $A^J(R) \to \mathbb{Q}_p$

Understanding $A(R)$ Explicitly

- We review what happens in the case of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$
- For any $z \in X(\mathbb{Z}_p)$ and $k \in \mathbb{Z}_{\geq 1}$, we have $\mathrm{Li}_k^p(z) \in \mathbb{Q}_p$
- But, when $z \in X(R)$, we can do better:
- The period map is a (conjecturally injective) ring homomorphism

$$\mathrm{per}_p \colon A(R) = A^J(R) \to \mathbb{Q}_p$$

- Each $z \in X(R)$ gives rise to an element $\mathrm{Li}_k^{\mathfrak{u}}(z) \in A(R)$ such that

$$\mathrm{Li}_k^p(z) = \mathrm{per}_p(\mathrm{Li}_k^{\mathfrak{u}}(z))$$

- We have a coproduct formula

$$\Delta \mathrm{Li}_k^{\mathfrak{u}}(z) = 1 \otimes \mathrm{Li}_k^{\mathfrak{u}}(z) + \sum_{i=0}^{k-1} \mathrm{Li}_{k-i}^{\mathfrak{u}}(z) \otimes \frac{(\log^{\mathfrak{u}}(z))^i}{i!}.$$

Our Work

- To compute $\mathcal{I}_{CK,n}^{R}$, must compute bases of $A(R)$ in low degrees, which is basically just computing relations between polylogarithms!
- E.g., in C–Dan-Cohen, we showed that a certain special element of $A(\mathbb{Z}[1/3])$ was

$$\frac{18}{13}\mathrm{Li}_4^{\mathfrak{u}}(3) - \frac{3}{52}\mathrm{Li}_4^{\mathfrak{u}}(9)$$

- For an elliptic curve, we consider $A^E(R)$, which is "graded" by the irreducible representations of GL_2
- Elements of this are related to special values of iterated integrals on E'
- A recent preprint of C describes part of $A^E(R)$ for $R = \mathbb{Z}[1/2]$ and $E = \{y^2 + xy = x^3 + x^2 - 2x\} =$ '$102a1$' (outside of range of Quadratic Chabauty)

Appendix: Polylogarithms as Motivic Periods

- How on earth do analytic functions like polylogarithms relate to $A(Z)$, the Hopf algebra of a Tannakian category of Galois representations??
- The differential equation shows that polylogarithms can be expressed via interated integration on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$
- These integrals show up in relative cohomology of powers $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^n$
- The relevant relative cohomology groups give objects the appropriate category of Galois representations
- Such an object is thus a representation of $\pi_1^J(\mathbb{Q})$
- Choosing coordinates on the cohomology allows one to write down functions on $\pi_1^J(\mathbb{Q})$

Useful References/Credits

The following are on arXiv:

- Mixed Tate Motives and the Unit Equation, Ishai Dan-Cohen and Stefan Wewers
- Mixed Tate Motives and the Unit Equation II, Ishai Dan-Cohen
- Single-Valued Motivic Periods, Francis Brown
- Motivic Periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Francis Brown
- Notes on Motivic Periods, Francis Brown
- Integral Points on Curves, the Unit Equation, and Motivic Periods, Francis Brown.
- The polylog quotient and the Goncharov quotient in computational Chabauty-Kim theory I, David Corwin and Ishai Dan-Cohen
- The polylog quotient and the Goncharov quotient in computational Chabauty-Kim theory II, David Corwin and Ishai Dan-Cohen
- Explicit Motivic Mixed Elliptic Chabauty-Kim, David Corwin

Thank You!