Obstructions to Rational and Integral Points

by

David Alexander Corwin

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Abstract

In this thesis, I study two examples of obstructions to rational and integral points on varieties.

The first concerns the S-unit equation, which asks for solutions to $x + y = 1$ with x and y both S-units, or units in $Z = \mathbb{Z}[1/S]$. This is equivalent to finding the set of Z-points of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We follow work of Dan-Cohen–Wewers and Brown in applying a motivic version of the non-abelian Chabauty's method of Minhyong Kim to find polynomials in p-adic polylogarithms that vanish on this set of integral points. More specifically, we extend the computations already done by Dan-Cohen–Wewers to the integer ring $\mathbb{Z}[1/3]$, and we provide some significant simplifications to a previous algorithm of Dan-Cohen, especially in the case of $\mathbb{Z}[1/S]$. One of the reasons for doing this is to verify cases of Kim's conjecture, which states that these p-adic functions precisely cut out the set of integral points. This is joint work with Ishai Dan-Cohen.

The second is about obstructions to the local-global principle. The étale Brauer-Manin obstruction of Skorobogatov can be used to explain the failure of the local-global principle for many algebraic varieties. In 2010, Poonen gave the first example of failure of the local-global principle that cannot be explained by the étale Brauer-Manin obstruction. Further obstructions such as the étale homotopy obstruction and the descent obstruction are unfortunately equivalent to the étale Brauer-Manin obstruction. However, Poonen's construction was not accompanied by a definition of a new, finer obstruction. Here, we present a possible definition for such an obstruction by applying the Brauer-Manin obstruction to each piece of every stratification of the variety. We prove that this obstruction is necessary and sufficient, over imaginary quadratic fields and totally real fields unconditionally, and over all number fields conditionally on the section conjecture. This is part of a joint project with Tomer Schlank.

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Chapter 1

Combined Introduction

A common question in number theory and algebraic geometry is to ask whether a system of polynomial equations has a solution in a ring R, usually a field or a ring of S-integers. This can be rephrased as asking whether the set $X(R)$ of R-points of a certain scheme X over R is nonempty. To answer questions like these, arithmetic geometers make two important observations:

- It is easier to understand the set of R-points when R is a completion (or collection of completions).
- There are topological invariants of schemes that contain arithmetic information, and we can understand points by their effect on these invariants.

In line with these two philosophies, one can map $X(R)$ both to a "local" version $X(R)_{\mathrm{loc}}$ and a "cohomological" or "homotopical" version $X(R)_{\mathrm{hom}}$, respectively. In this general framework, one obtains a diagram

$$\begin{array}{ccc} X(R) & \hookrightarrow & X(R)_{\mathrm{loc}} \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa_{\mathrm{loc}}} \\ X(R)_{\mathrm{hom}} & \xrightarrow[\mathrm{loc}]{} & X(R)_{\mathrm{hom,loc}} \end{array} .$$

One then considers the set $\kappa_{\mathrm{loc}}^{-1}(\mathrm{Im}(\mathrm{loc}))$ in $X(R)_{\mathrm{loc}}$ and uses it to obtain information about $X(R)$.

This thesis considers two examples of this framework.

- The first is that of Chabauty's method and its non-abelian variant due to Minhyong Kim. In this case, the local version is the set of points at a single completion of R, and the cohomological version is the non-abelian Galois (or motivic) cohomology of the unipotent fundamental group of X.

- The second is that of obstructions to the local-global principle such as the Brauer-Manin obstruction (resp., étale homotopy obstruction). In this case, the local version is the set of points with values in the adeles of R, and the cohomological version is the Brauer group of R (resp., the set of homotopy fixed points $X(hR)$).

The first example forms Part I, is joint work with Ishai Dan-Cohen, and will soon appear as a preprint. The second is Part II, is joint work with Tomer Schlank, and largely overlaps with the preprint [CS17].

Part I

Motivic Chabauty-Kim Theory for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Chapter 2

Introduction

2.1 Motivic Non-Abelian Chabauty's Method

2.1.1 Classical Chabauty's Method

The **Chabauty-Skolem method** proves that certain equations have finitely many integral solutions, and, by innovations of Coleman, allows one to find all solutions. The idea is to embed the scheme X defined by the equations (which, geometrically, should be a smooth irreducible curve of negative Euler characteristic) over an open integer scheme $Z = \mathcal{O}_K[1/S]$ (for K a number field and S a finite set of finite places of K) into a scheme J with more structure, specifically that of a semiabelian variety (to do this, one takes a generalized Jacobian of X). One then choose a closed point $\mathfrak{p}$ of Z and looks at the intersection of the p-adic closure of $J(Z)$ with $X(Z_{\mathfrak{p}})$ (with $Z_{\mathfrak{p}}$ the completed local ring of Z at $\mathfrak{p}$) inside the p-adic analytic space $J(Z_{\mathfrak{p}})$ (where $\mathfrak{p}$ is a prime of good reduction for X lying above p). The method of Coleman then sometimes allows one to find explicit p-adic analytic functions on $X(Z_{\mathfrak{p}})$ that vanish on $X(Z)$ and have finitely many zeroes (i.e., Coleman functions).

2.1.2 Minhyong Kim's Non-abelian Chabauty

This method often does not work, because $J(Z)$ can be too large. Philosophically, the reason why $J(Z)$ can be large while $X(Z)$ remains finite is because the geometric fundamental group of J is abelian (as opposed to hyperbolic curves like X, whose fundamental groups are non-abelian, even center-free)[1]. The groundbreaking work of Minhyong Kim ([Kim05]) gets around this fact. First, it reinterprets the Chabauty-Skolem method by replacing $J(Z)$ (or its p-adic closure) by a corresponding p-adic Selmer group. By the work of Bloch-Kato ([BK90]), this can be expressed intrinsically in terms of the p-adic Tate module of J, which is the abelianization of the pro-p étale fundamental group of $X_{\overline{K}}$. Then, one replaces this abelianization by a mildly non-abelian version, namely, the nth quotient of the descending central series of the pro-p étale fundamental group of $X_{\overline{K}}$ for a natural number n.

Thus the p-adic Selmer group is replaced by the **Selmer variety** $\mathrm{Sel}(X/Z)_n$ of [Kim09]. In particular, for a place $\mathfrak{p}$ of K over p, Kim constructs a commutative diagram ([Kim05],[Kim09])

$$\begin{array}{ccc} X(Z) & \hookrightarrow & X(Z_{\mathfrak{p}}) \\ \downarrow \kappa & & \downarrow \kappa_{\mathfrak{p}} \\ \mathrm{Sel}(X/Z)_n & \xrightarrow[\mathrm{loc}_n]{} & \mathrm{Sel}(X/Z_{\mathfrak{p}})_n \end{array}$$

The goal, which was realized when $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ in [Kim05], is to prove that the morphism of schemes loc_n is non-dominant for sufficiently large n. This means that there is a nonzero ideal $\mathcal{I}_n^Z$ of functions on $\mathrm{Sel}(X/Z_{\mathfrak{p}})_n$ that vanish on the image of loc_n. These functions pull back via $\kappa_{\mathfrak{p}}$ to Coleman functions on $X(Z_{\mathfrak{p}})$, which, by the commutativity of this diagram, vanish on $X(Z)$. We let $X(Z_{\mathfrak{p}})_n$ be the set of common zeroes of all pullbacks of elements of $\mathcal{I}_n^Z$, and we note that this sequence of sets is decreasing in n and contains $X(Z)$. The map κ_p can be expressed in terms of p-adic iterated integrals, which makes the method more concrete.

The original work of Kim ([Kim05]) dealt with the case of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. That work

[1] This is not to say that all varieties with abelian fundamental group have infinitely many integral points, but that in the context of the distinction between a curve and its Jacobian, this principle applies.

shows that $\mathcal{I}_n^Z$ is nonzero for sufficiently large n and hence that $X(Z_\mathfrak{p})_n$ is finite for such n, thereby reproving a 1929 theorem of Siegel that $X(Z)$ is finite. Part I of this thesis is part of the effort to develop algorithms for computing $X(Z_\mathfrak{p})_n$ and compute examples for specific n and Z, in hopes of giving evidence for the conjecture of Kim ([BDCKW]) that $X(Z) = X(Z_\mathfrak{p})_n$ for sufficiently large n.

2.1.3 Motivic Chabauty-Kim Theory

In order to compute this set concretely (i.e., in terms that can be applied to computing elements of $\mathcal{I}_n^Z$), it is actually better to upgrade this Galois cohomology to a motivic version of the Selmer variety, as developed in [Had11] and [DCW16]. This motivic Selmer variety is just the group cohomology of the **mixed Tate Galois group** $\pi_1^{\mathbf{MT}}(Z)$ of Z, as developed in [DG05], with coefficients in (a quotient depending on n of) the unipotent fundamental group $\pi_1^{\mathrm{un}}(X)$ of X. This group (as well as all of the quotients we take) has the structure of a mixed Tate motive, i.e., an action of $\pi_1^{\mathbf{MT}}(Z)$. We in fact work only with a certain quotient $\pi_1^{\mathrm{PL}}(X)$ of $\pi_1^{\mathrm{un}}(Z)$, known as the *polylogarithmic quotient*, as well as its finite-dimensional quotients $\pi_1^{\mathrm{PL}}(X)_{\geq n}$. In this case, we refer to the Chabauty-Kim ideal and loci by $\mathcal{I}_{\mathrm{PL},n}^Z$ and $X(Z_\mathfrak{p})_{\mathrm{PL},n}$, respectively. The reason why this can be made explicit is that $\pi_1^{\mathbf{MT}}(Z)$ can be described as an abstract (pro-algebraic) group over $\mathbb{Q}$, which ultimately results from Borel's computation of the algebraic K-theory of integer rings. More specifically, $\pi_1^{\mathbf{MT}}(Z) = \pi_1^{\mathrm{un}}(Z) \rtimes \mathbb{G}_m$, where $\pi_1^{\mathrm{un}}(Z)$ is isomorphic to a free pro-unipotent group on the graded vector space $\left(\bigoplus_{n=1}^{\infty} K_{2n-1}(Z)_\mathbb{Q}\right)^\vee$. This is in contrast with classical Galois groups of number fields, which cannot be easily described as abstract profinite groups.

2.1.4 Motivic Periods

While we have an abstract description of the group $\pi_1^{\mathbf{MT}}(Z)$ ultimately coming from algebraic K-theory and Voevodsky's work on motives, we need to understand this group in more concrete terms. More specifically, we need to compute the image of the map loc_n in the diagram, which is a p-adic regulator map. This leads to **motivic polylogarithms** of the form $\mathrm{Li}^{\mathfrak{u}}_n(z)$ for $n \in \mathbb{Z}_{\geq 1}$ and $z \in \mathbb{Q}$, as well as **motivic logarithms** $\log^{\mathfrak{u}}(z)$ and **motivic zeta values** $\zeta^{\mathfrak{u}}(n)$ for $n \in \mathbb{Z}_{\geq 1}$ (Section 3.2.2). These are all elements of the coordinate ring $\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$ of $\pi_1^{\mathrm{un}}(Z)$ whose p-adic realizations are the Coleman p-adic polylogarithms $\mathrm{Li}^p_n(z)$. There is an explicit formula for the (reduced) coproduct of these elements in the Hopf algebra $\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$, due to Goncharov ([Gon05a]):

$$\Delta' \mathrm{Li}^{\mathfrak{u}}_n(z) \coloneqq \Delta \mathrm{Li}^{\mathfrak{u}}_n(z) - 1 \otimes \mathrm{Li}^{\mathfrak{u}}_n(z) - \mathrm{Li}^{\mathfrak{u}}_n(z) \otimes 1 = \sum_{i=1}^{n-1} \mathrm{Li}^{\mathfrak{u}}_{n-i}(z) \otimes \frac{(\log^{\mathfrak{u}}(z))^i}{i!}.$$

Our current work, building on ideas in [DCW16] and [DC15], shows how this can be used to algorithmically compute bases of $\mathcal{O}(\pi_1^{\mathrm{un}}(Z))$ in low degrees.

2.2 This Project

Before this work, the only cases that had been computed were those for which $Z = \mathbb{Z}, \mathbb{Z}[1/2]$, both in weights $n = 2, 4$ (in [DCW16]). In addition, an algorithm had been proposed, and this algorithm was shown to compute the set of points assuming various conjectures ([DC15]).

In this work, we make the algorithm more efficient and explicit and use it to find an element of $\mathcal{I}^Z_{\mathrm{PL},4}$ for $Z = \mathbb{Z}[1/3]$. More specifically, we have:

Theorem 2.2.1 (Theorem 3.3.6)**.** *The element*

$$\zeta^{\mathfrak{u}}(3)\log^{\mathfrak{u}}(3)\,\mathrm{Li}^{\mathfrak{u}}_4 - \left(-\frac{12}{w_2(9)}\,\mathrm{Li}^{\mathfrak{u}}_4(3) + (2w_2(9))^{-1}\,\mathrm{Li}^{\mathfrak{u}}_4(9)\right)\log^{\mathfrak{u}}\mathrm{Li}^{\mathfrak{u}}_3$$
$$-\frac{(\log^{\mathfrak{u}})^3\,\mathrm{Li}^{\mathfrak{u}}_1}{24}\left(\zeta^{\mathfrak{u}}(3)\log^{\mathfrak{u}}(3) - 4\left(-\frac{12}{w_2(9)}\,\mathrm{Li}^{\mathfrak{u}}_4(3) + (2w_2(9))^{-1}\,\mathrm{Li}^{\mathfrak{u}}_4(9)\right)\right)$$

of $\mathcal{O}(\Pi_{\mathrm{PL},4} \times \pi_1^{\mathrm{un}}(Z))$ (defined in Section 3.1.6) is in $\mathcal{I}^Z_{\mathrm{PL},4}$ for $Z = \operatorname{Spec}\mathbb{Z}[1/3]$, where $w_2(9)$ is a number p-adically close to $-\dfrac{26}{3}$ for $p = 5, 7$.

By computing values of this function at elements of $X(Z_{\mathfrak{p}})_{\mathrm{PL},2}$ already found in [BDCKW], we get:

Theorem 2.2.2 (Theorem 3.3.7)**.** *For* $Z = \operatorname{Spec} \mathbb{Z}[1/3]$*, we have* $X(Z_{\mathfrak{p}})_{\mathrm{PL},4} \subseteq \{-1\}$ *for* $p = 5, 7$.

Part of the reason for computing information about these Chabauty-Kim loci is that they help verify cases of the following conjecture of Kim:

Conjecture 2.2.3 (Conjecture 3.1 of [BDCKW])**.** $X(Z) = X(Z_{\mathfrak{p}})_n$ *for sufficiently large* n*.*

In order to do computations, it is easier to work only with polylogarithms, i.e., only with the polylogarithmic quotient. Luckily, not only is $\mathcal{I}_n^Z$ nonzero for sufficiently large n, but $\mathcal{I}_{\mathrm{PL},n}^Z$ is as well, and hence $X(Z_{\mathfrak{p}})_{\mathrm{PL},n}$ is finite for such n. To make the method truly computable, one would like an analogue of Conjecture 2.2.3 for the polylogarithmic quotient, which leads one to ask the question:

Question 2.2.4 (Question 3.1.24). Does $X(Z) = X(Z_{\mathfrak{p}})_{\mathrm{PL},n}$ for sufficiently large n?

A positive answer would be a strengthening of Conjecture 2.2.3. However, we prove:

Theorem 2.2.5 (Theorem 3.4.2)**.** *For any prime* ℓ *and positive integer* n*, we have*

$$-1 \in X(Z_{\mathfrak{p}})_{\mathrm{PL},n},$$

where $Z = \operatorname{Spec} \mathbb{Z}[1/\ell]$.

In particular, for ℓ *odd, Question 2.2.4 has a negative answer.*

In particular, it does not make sense to make a version of Kim's conjecture with $X(Z_{\mathfrak{p}})_{\mathrm{PL},n}$ in place of $X(Z_{\mathfrak{p}})_n$. However, there is an action of S_3 on the scheme X, and we may use it to propose a strengthening of Kim's conjecture. We write

$$X(Z_{\mathfrak{p}})_{\mathrm{PL},n}^{S_3} \coloneqq \bigcap_{\sigma \in S_3} \sigma(X(Z_{\mathfrak{p}})_{\mathrm{PL},n}).$$

Our strengthened Kim's conjecture is the following:

Conjecture 2.2.6 (Conjecture 3.1.25)**.** $X(Z) = X(Z_{\mathfrak{p}})_{\mathrm{PL},n}^{S_3}$ *for sufficiently large* n*.*

As explained in Section 3.1.7, Conjecture 2.2.6 implies Conjecture 2.2.3.

We then use our computation from Theorem 2.2.2 to verify Conjecture 2.2.6 for $Z = \mathbb{Z}[1/3]$ and $p = 5, 7$:

Theorem 2.2.7 (Theorem 3.4.4)**.** *For $Z = \operatorname{Spec} \mathbb{Z}[1/3]$ and $p = 5, 7$, Conjecture 2.2.6 (and hence Conjecture 3.1.23) holds (with $n = 4$).*

2.3 Notation

For a scheme Y, we let $\mathcal{O}(Y)$ denote its coordinate ring. If R is a ring, we let $Y \otimes R$ or Y_R denote the product (or 'base-change') $Y \times \operatorname{Spec} R$. If Y and $\operatorname{Spec} R$ are over an implicit base scheme S (often $\operatorname{Spec} \mathbb{Q}$), we take the product over S. Similarly, if M is a linear object (such as a module, an algebra, a Lie algebra, or a Hopf algebra), then M_R denotes $M \otimes R$ (again, with the tensor product taken over an implicit base ring).

If $f \colon Y \to Z$ is a morphism of schemes, we denote by $f^{\#} \colon \mathcal{O}(Z) \to \mathcal{O}(Y)$ the corresponding homomorphism of rings. Similarly, if $\alpha \in Y(R)$, we have a homomorphism $\alpha^{\#} \colon \mathcal{O}(Y) \to R$.

Chapter 3

Main Result

3.1 Technical Preliminaries

This work builds on the work of [DCW16], [DC15], and [Bro17]. We recall some of the important objects in the theory.

3.1.1 Generalities on Graded Pro-Unipotent Groups

Conventions for Graded Vector Spaces

Our definition of graded vector space is the following:

Definition 3.1.1. A graded vector space is a collection of vector spaces V_i indexed by $i \in \mathbb{Z}$.

Definition 3.1.2. A graded vector space is positive (respectively negative, strictly positive, strictly negative) if $V_i = 0$ for $i < 0$ (respectively, for $i > 0$, for $i \leq 0$, for $i \geq 0$).

In general, we will only consider graded vector spaces satisfying one of these four conditions.

Furthermore, *unless otherwise stated, we will only consider graded vector spaces* $\{V_i\}$ *such that each* V_i *is finite-dimensional,* which ensures that the double dual is the identity. One must then be careful when taking tensor constructions, as follows. Specifically, we only consider the tensor product between two graded vector spaces if they are either both positive or both negative, and we consider the tensor algebra only of a strictly positive or strictly negative graded vector space.

We now make a technical remark. For a collection

$$V = \{V_i\}_{i\in\mathbb{Z}}$$

of (finite-dimensional) vector spaces indexed by the integers, we may either take the direct sum

$$V^{\bigoplus} := \bigoplus_i V_i$$

or the direct product

$$V^{\prod} := \prod_i V_i$$

as our notion of a graded vector space. In general, we use the former for coordinate rings and Lie coalgebras and the latter for universal enveloping algebras and Lie algebras. As all of our pro-unipotent groups will be negatively graded, we use the $\bigoplus$ notion for positively graded vector spaces and the $\prod$ notion for negatively graded vector spaces.

When considering the $\prod$ notion, we take completed tensor product instead of tensor product (and similarly for tensor algebras and universal enveloping algebras), and a coproduct is a complete coproduct, i.e., a homomorphism $V \to V\widehat{\otimes}V$. In addition, a set of homogeneous elements is considered a basis if it generates V in each degree, and a similar remark applies to bases of algebras and Lie algebras. When taking the dual of a negatively graded vector space, we take the graded dual and then view the resulting (positively) graded vector space via the $\bigoplus$ notion. In particular, the double dual is always the original vector space. Nonetheless, we have the following relation between graded and ordinary (non-graded) duals:

$$(V^{\bigoplus})^\vee = (V^\vee)^{\prod}.$$

Graded Pro-unipotent Groups

Let U be a pro-unipotent group over $\mathbb{Q}$.

Then U is a group scheme over $\mathbb{Q}$, so its coordinate ring $\mathcal{O}(U)$ is a Hopf algebra over $\mathbb{Q}$. We recall that if $\mathcal{O}(U)$ is a Hopf algebra over $\mathbb{Q}$, then it is equipped with a product $\mathcal{O}(U)\otimes\mathcal{O}(U) \to \mathcal{O}(U)$, a coproduct $\Delta\colon \mathcal{O}(U) \to \mathcal{O}(U) \otimes \mathcal{O}(U)$, and a counit $\epsilon\colon \mathcal{O}(U) \to \mathbb{Q}$. The kernel $I(U)$ of ϵ is known as the *augmentation ideal*. We also write $\Delta'(x) \coloneqq \Delta(x) - x \otimes 1 - 1 \otimes x$ for the *reduced coproduct*. Finally, we say that an element is *primitive* if it is in the kernel of Δ'.

We say that a Hopf algebra A is a *graded Hopf algebra* if the multiplication $A \otimes A \to A$, the coproduct $A \to A \otimes A$, unit $\mathbb{Q} \to A$, and counit $A \to \mathbb{Q}$, are morphisms of graded vector spaces, where $A \otimes A$ has the standard grading on a tensor product, and $\mathbb{Q}$ is in degree zero.

Definition 3.1.3. By a *grading* on U, we mean a positive grading on $\mathcal{O}(U)$ as a $\mathbb{Q}$-vector space such that the degree zero part of $\mathcal{O}(U)$ is one-dimensional over $\mathbb{Q}$.

Definition 3.1.4. If A is a Hopf algebra graded in the sense of Definition 3.1.3, we let Δ_n and Δ'_n denote the restrictions of Δ and Δ', respectively, to A_n, the nth graded piece.

Furthermore, Δ_n and Δ'_n map A_n into the nth graded piece of $A \otimes A$, which is

$$\bigoplus_{i+j=n} A_i \otimes A_j.$$

Definition 3.1.5. For $i + j = n$ and $i, j \geq 0$, we let $\Delta_{i,j}$ and $\Delta'_{i,j}$ denote the projections of Δ_n and Δ'_n, respectively, to $A_i \otimes A_j$. One may check via the axioms defining a Hopf algebra that $\Delta'_{i,j} = 0$ when either of i or j is zero.

The reduced coproduct Δ' induces a (graded) Lie coalgebra structure on $I(U)/I(U)^2$, and the dual Lie algebra $(I(U)/I(U)^2)^\vee$ is the Lie algebra $\mathfrak{n}$ of U. It is a strictly negatively graded pro-nilpotent Lie algebra. We let $\mathcal{U}U = \mathcal{U}\mathfrak{n}$ denote the dual Hopf algebra of $\mathcal{O}(U)$, which is the (completed) universal enveloping algebra of $\mathfrak{n}$. The composition

$$\ker(\Delta') \hookrightarrow \mathcal{U}U = \mathcal{O}(U)^\vee \twoheadrightarrow I(U)^\vee$$

induces an isomorphism between the set of primitive elements of $\mathcal{U}U$ and the Lie algebra $\mathfrak{n} = (I(U)/I(U)^2)^\vee \subseteq I(U)^\vee$.

Furthermore, for a $\mathbb{Q}$-algebra R, we may identify $U(R)$ with the group of grouplike elements in $(\mathcal{U}U)_R$, i.e., x such that

$$\Delta x = x \otimes x.$$

Evaluation of an an element of $\mathcal{O}(U)$ on an element of $U(R)$ is given by evaluation on the corresponding grouplike element of $(\mathcal{U}U)_R$.

The functor sending U to $\mathfrak{n}$ is known to be an equivalence of categories between graded pro-unipotent groups and strictly negatively graded pro-nilpotent Lie algebras. For each positive integer n, the set of elements $\mathfrak{n}_{<-n}$ is a Lie ideal, and we denote by $\mathfrak{n}_{\geq -n}$ the quotient $\mathfrak{n}/\mathfrak{n}_{<-n}$. We denote the corresponding quotient pro-unipotent group by

$$U_{\geq -n},$$

and it is a unipotent algebraic group. In fact, U is the inverse limit

$$\varprojlim_n U_{\geq -n}.$$

Example 3.1.6. If $\mathfrak{n}$ is a one-dimensional Lie algebra generated by an element x, then $\mathfrak{n}$ is nilpotent. We have $\mathcal{U}\mathfrak{n} = \mathbb{Q}[[x]]$, and $\mathcal{O}(U) = \mathbb{Q}[f_x]$, with both x and f_x primitive. Then U is the group $\mathbb{G}_a$, and the set of grouplike elements of $\mathcal{U}\mathfrak{n}$ is the set of elements of the form $\exp(rx)$ for $r \in \mathbb{Q}$. In particular, this demonstrates the usefulness of taking completed universal enveloping algebras.

Free Pro-unipotent Groups

Definition 3.1.7. If V is a strictly negative graded vector space, we may form the free pro-nilpotent Lie algebra on V as follows. We take the graded tensor algebra TV on V and put the unique coproduct on it such that all elements of V are primitive. The subspace of primitive elements of TV forms a graded pro-nilpotent Lie algebra, denoted $\mathfrak{n}(V)$, with corresponding pro-unipotent group $U(V)$. Then $\mathfrak{n}(V)$, $U(V)$ are known as the *free pro-nilpotent Lie algebra* and *free pro-unipotent group*, respectively, on the graded vector space V.

The construction $V \mapsto \mathfrak{n}(V)$ is left adjoint to the forgetful functor from graded pro-nilpotent Lie algebras to graded vector spaces.

Finally, if I is an index set with a degree function $d\colon I \to \mathbb{Z}_{<0}$ with finite fibers, then the *free pro-unipotent group on the set* I is just the free pro-unipotent group on the free graded vector space on the set I. In particular, the free pro-unipotent group on a graded vector space is isomorphic to the free pro-unipotent group on a (graded) basis of that vector space. The Lie algebra is the pro-nilpotent completion[1] of the free Lie algebra on the set I and as such is generated by the elements of I.

The graded dual Hopf algebra of $TV = \mathcal{U}\mathfrak{n}(V)$ is the coordinate ring $\mathcal{O}(U(V))$. Let $\{x_i\}$ be a graded basis of V, so that words w in the $\{x_i\}$ form a basis of $\mathcal{U}\mathfrak{n}(V)$, and let $\{f_w\}_w$ denote the basis of $\mathcal{O}(U(V))$ dual to $\{w\}$. Then $\mathcal{O}(U(V))$ is isomorphic to the *free shuffle algebra* on the graded vector space $V^\vee$. Its coproduct is known as the *deconcatenation coproduct* and is given by

$$\Delta f_w \coloneqq \sum_{w_1 w_2 = w} f_{w_1} \otimes f_{w_2},$$

and the (commutative) product Ш on $\mathcal{O}(U(V))$, known as the *shuffle product*, is given by:

$$f_{w_1} Ш f_{w_2} \coloneqq \sum_{\sigma \in Ш(\ell(w_1), \ell(w_2))} \sigma(f_{w_1 w_2}),$$

where ℓ denotes the length of a word, $Ш(\ell(w_1), \ell(w_2)) \subseteq S_{\ell(w_1)+\ell(w_2)}$ denotes the group of shuffle permutations of type $(\ell(w_1), \ell(w_2))$, and $w_1 w_2$ denotes concatenation of words.

Remark 3.1.8. It follows from the definition of the deconcatenation coproduct that a word consisting of a single letter is a primitive element of the free shuffle algebra.

[1]In fact, if one takes a free Lie algebra in the graded sense and views it via the $\prod$ notion, then it is already pro-nilpotent.

Conventions for Products

Let α and β be two paths in a space X. Then in the literature, the sybmol $\alpha\beta$ can have two different meanings. It can either denote:

(i) The path given by going along α and then β

(ii) The path given by going along β and then α

The first is known as the 'lexical' order and the second as the 'functional' order. We will use the lexical order, in contrast to the convention of [DCW16] (but consistent with [Bro17]). However, we would like to take a moment to explain how these two conventions help clarify differing conventions in the literature for iterated integrals, multiple zeta values, polylogarithms, and more. We will refer to these differing conventions as the lexical and functional conventions, respectively.

In fact, either convention necessitates a particular convention for iterated integrals. In general, one wants a "coproduct" formula to hold for iterated integrals (c.f. Section 2.1 of [Bro12b], 5.1(ii) of [Hai94], or Proposition 5 of [Hai05]), by which we mean

$$\int_{\alpha\beta} \omega_1 \cdots \omega_n = \sum_{i=0}^{n} \int_{\alpha} \omega_1 \cdots \omega_i \int_{\beta} \omega_{i+1} \cdots \omega_n \tag{3.1}$$

In order for the coproduct to take this nice form (3.1), our convention for path composition determines our convention for iterated integrals. More specifically, those that use the lexical order for path composition use the formula

$$I(\gamma(0); \omega_1, \cdots, \omega_n; \gamma(1)) \coloneqq \int_{\gamma} \omega_1 \cdots \omega_n = \int_{0 \le t_1 \le \cdots \le t_n \le 1} f_1(t_1) \cdots f_n(t_n) dt_1 \cdots dt_n,$$

where $\gamma^*(\omega_i) = f_i(t)dt$, and those that use the function order for path composition use the formula

$$I(\gamma(0);\omega_1,\cdots,\omega_n;\gamma(1)) \coloneqq \int_\gamma \omega_1\cdots\omega_n = \int_{0\le t_n\le\cdots\le t_1\le 1} f_1(t_1)\cdots f_n(t_n)dt_1\cdots dt_n.$$

Given that these conventions are opposite, the corresponding conventions for the iterated integral expression for polylogarithms must be opposite. More precisely, the iterated integral defining a multiple polylogarithm must always begin with $\frac{dz}{1-z}$ in the lexical convention, while it must always end with $\frac{dz}{1-z}$ in the functional convention. More precisely, let us define

$$\mathrm{Li}_{s_1,\cdots,s_r}(z) \coloneqq \sum_{0<k_1<\cdots<k_r} \frac{z^{k_r}}{k_1^{s_1}\cdots k_r^{s_r}}. \tag{3.2}$$

Set $e^0 = \frac{dz}{z}$ and $e^1 = \frac{dz}{1-z}$. Then using the lexical convention for iterated integration, we have:

$$\mathrm{Li}_{s_1,\cdots,s_r}(z) = I(0; e^1, \underbrace{e^0,\cdots,e^0}_{s_1-1}, e^1, \underbrace{e^0,\cdots,e^0}_{s_2-1},\cdots, e^1, \underbrace{e^0,\cdots,e^0}_{s_r-1}; z).$$

In fact, the definition itself of $\mathrm{Li}_{s_1,\cdots,s_r}(z)$ depends on the convention. More specifically, if we were to use the functional convention for iterated integrals in tandem with (3.2), we would get:

$$\mathrm{Li}_{s_1,\cdots,s_r}(z) = I(0; \underbrace{e^0,\cdots,e^0}_{s_r-1}, e^1, \underbrace{e^0,\cdots,e^0}_{s_{r-1}-1},\cdots, e^1, \underbrace{e^0,\cdots,e^0}_{s_1-1}, e^1; z).$$

This is precisely the formula that appears in [BL11] (1.4). However, most authors prefer the s_i's to appear in the iterated integral in the same order as they do in the argument of the function. Therefore, almost all articles that use the functional convention for iterated integration will write:

$$\mathrm{Li}_{s_1,\cdots,s_r}(z) \coloneqq \sum_{k_1>\cdots>k_r>0} \frac{z^{k_r}}{k_1^{s_1}\cdots k_r^{s_r}}.$$

As a result, one then writes:

$$\mathrm{Li}_{s_1,\cdots,s_r}(z) = I(0; \underbrace{e^0,\cdots,e^0}_{s_1-1}, e^1, \underbrace{e^0,\cdots,e^0}_{s_2-1}, \cdots, e^1, \underbrace{e^0,\cdots,e^0}_{s_r-1}, e^1; z).$$

Thus, the convention one uses for path composition determines the convention one uses for $\mathrm{Li}_{s_1,\cdots,s_r}(z)$ (except in [BL11]). Similarly, the two conventions for multiple zeta values follow this paradigm. Specifically, those who use the lexical convention write

$$\zeta(s_1,\cdots,s_r) = \sum_{k_1>\cdots>k_r>0} \frac{1}{k_1^{s_1}\cdots k_r^{s_r}},$$

and those who use the functional convention write

$$\zeta(s_1,\cdots,s_r) = \sum_{k_1>\cdots>k_r>0} \frac{1}{k_1^{s_1}\cdots k_r^{s_r}}.$$

Note, however, that Li_n always denotes the same function (both as a multi-valued complex analytic function, a Coleman function, and an abstract function on the de Rham fundamental group), no matter which convention one uses. In fact, this brings us back to the two conventions for path composition. The fact that some write $\mathrm{Li}_n^{\mathfrak{u}}$ (c.f. Section 3.2.1) as $e^1 \underbrace{e^0\cdots e^0}_{n-1}$ and others write it as $\underbrace{e^0\cdots e^0}_{n-1} e^1$, yet both denote the exact same regular function on the unipotent de Rham fundamental group of $\mathbb{P}^1 \setminus \{0,1,\infty\}$, is due to the differing conventions for path composition.

As we use the lexical convention, one will find the coproduct formula

$$\Delta' \mathrm{Li}_n^{\mathfrak{u}} = \sum_{i=1}^{n-1} \mathrm{Li}_{n-i}^{\mathfrak{u}} \otimes \frac{(\log^{\mathfrak{u}})^{\text{Ш}i}}{i!} \tag{3.3}$$

in this thesis (3.2.1). With the other convention, one must write $\Delta' \mathrm{Li}_n^{\mathfrak{u}} = \sum_{i=1}^{n-1} \frac{(\log^{\mathfrak{u}})^{\text{Ш}i}}{i!} \otimes \mathrm{Li}_{n-i}^{\mathfrak{u}}$.

More subtly, these conventions affect the convention one uses for the motivic coproduct.

More specifically, if we use the lexical convention, we want to also be able to write

$$\Delta' \operatorname{Li}^{\mathfrak{u}}_n(z) = \sum_{i=1}^{n-1} \operatorname{Li}^{\mathfrak{u}}_{n-i}(z) \otimes \frac{(\log^{\mathfrak{u}}(z))^{\text{Ш}i}}{i!} \tag{3.4}$$

rather than its opposite. This formula is correct as long as we use the lexical order for composition in $\pi_1^{\mathrm{un}}(Z)$ (which, in particular, comes out in how one writes the Goncharov coproduct of [Gon05a]; Goncharov uses the lexical order himself). Theoretically, one could use one convention for composition in $\pi_1^{\mathrm{un}}(X)$ and another for composition in $\pi_1^{\mathrm{un}}(Z)$, but that would cause formulas (3.3) and (3.4) not to appear identical.

One important implication of the difference in formulas for the motivic coproduct is:

Remark 3.1.9. Our $f_{\sigma\tau}$ is actually the $\phi_{1.3}$ of 7.6.1 of [DCW16], even though the notation would suggest it is $\phi_{3.1}$.

In terms of authors and sources, one may find the lexical convention and/or the other conventions that go along with it in [Bro14] (Section 1.2), [Hai94] (Section 5),[Hai05] (Definition 2), [Bro17] (9.1), [Hai87] (Definition 1.1), [Bro12b] (Definition 2.1), [Gon05a], [Che77] (1.1.1), [Fur04] (0.1), [Gon05b] (3.4.3), [Gon] (Formula (4) and Definition 1.2), [Gon95] (Section 12), [Zag93], [Zag94], [Ter02], and [Rab96]. One may also find conventions for multiple zeta values consistent with this convention in other articles by Francis Brown, e.g., [Bro12a], [Bro], and [Bro13].

The sources [DCW16] (1.16), [DG05] (5.16.1),[Car02] (Formula (79), [Del10] (Formula 0.1 and 5.1A), [Del13], [Hof97], [Rac02], [Sou10], [DC15] (2.2.4), and [Bro04] use the functional conventions.

3.1.2 The Various Fundamental Groups

The Mixed Tate Fundamental Group of Z

Definition 3.1.10. An *open integer scheme* is an open subscheme of $\operatorname{Spec} \mathcal{O}_K$, where K is a number field and $\mathcal{O}_K$ its ring of integers.

Let $Z \subseteq \operatorname{Spec} \mathcal{O}_K$ be an open integer scheme, $\mathbf{MT}(Z, \mathbb{Q})$ its Tannakian category of mixed

Tate motives with $\mathbb{Q}$-coefficients, which exists by [DG05][2]. This is a category with realization functors

$$\mathrm{real}^\sigma\colon\ \mathbf{MT}(Z,\mathbb{Q}) \to \mathrm{MHS}_{\mathbb{Q}}$$

to mixed Hodge structures with $\mathbb{Q}$-coefficients for each embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ and

$$\mathrm{real}^\ell\colon\ \mathbf{MT}(Z,\mathbb{Q}) \to \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)$$

to ℓ-adic representations of G_K for each prime ℓ. The image of each realization functor consists of mixed Tate objects, i.e., objects with a composition series consisting of tensor powers of the image of the Tate object $\mathbb{Q}(1) \coloneqq h_2(\mathbb{P}^1;\mathbb{Q})$.

Definition 3.1.11. A continuous $\mathbb{Q}_\ell$-representation of G_K for a number field K is said to have *good reduction* at a non-archimedean place v of K if either $v \nmid \ell$, and the representation is unramified at v, or if $v \mid \ell$, and the representation is crystalline at v.

The ℓ-adic realizations of an object of $\mathbf{MT}(Z,\mathbb{Q})$ form a compatible system of $\mathbb{Q}_\ell$-Galois representations with good reduction at closed points of Z (in particular, crystalline at primes dividing ℓ). If (X,D) is a pair of a scheme and codimension 1 subscheme, both smooth and proper over Z and rationally connected, then the relative cohomology $h^*(X,D;\mathbb{Q})$ is an object of this category such that

$$\mathrm{real}^\sigma(h^*(X,D;\mathbb{Q})) = H^*_{\mathrm{Betti}}(X^{\mathrm{an}}_\sigma(\mathbb{C}), D^{\mathrm{an}}_\sigma(\mathbb{C});\mathbb{Q}),$$

$$\mathrm{real}^\ell(h^*(X,D;\mathbb{Q})) = H^*_{\text{ét}}(X_{\overline{K}}, D_{\overline{K}};\mathbb{Q}_\ell),$$

with their associated mixed Hodge structure and continuous G_K-action, respectively.

The only simple objects of $\mathbf{MT}(Z,\mathbb{Q})$ are the objects $\mathbb{Q}(n) \coloneqq \mathbb{Q}(1)^{\otimes n}$ for $n \in \mathbb{Z}$, each

[2] It is constructed by putting a t-structure on a certain subcategory of Voevodsky's triangulated category $DM_{gm}(K)$ of [Voe00], taking the heart of that t-structure, and finally taking the subcategory of objects with good reduction at closed points of Z. The category $DM_{gm}(K)$ is defined by taking a certain localization of the category of complexes of smooth varieties over a field with correspondences as morphisms, then taking its pseudo-abelian envelope, and finally inverting the Tate object $\mathbb{Q}(1)$. However, we will only need the properties of $\mathbf{MT}(Z,\mathbb{Q})$, not its construction.

object has a finite composition series, and the extensions are determined by the fact that

$$\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(Z)_{\mathbb{Q}}$$

$$\mathrm{Ext}^i = 0 \qquad \forall\, i \geq 2.$$

The groups $K_{2n-1}(Z)_{\mathbb{Q}}$ are known by the work of Borel ([Bor74]). For $n = 1$, we have $K_1(Z) = \mathcal{O}(Z)^\times$, and for $n \geq 2$, $K_{2n-1}(Z)_{\mathbb{Q}} = K_{2n-1}(K)_{\mathbb{Q}}$ has dimension r_2 for n even and $r_1 + r_2$ for n odd, where r_1 and r_2 are the numbers of real and complex places of K, respectively.

Definition 3.1.12. Let M be an object of $\mathbf{MT}(Z, \mathbb{Q})$. Then M has an increasing filtration $W_i M$ known as the *weight filtration.* The quotient $W_i M / W_{i-1} M$ is trivial when i is odd and is isomorphic to a direct sum of copies of $\mathbb{Q}(-i/2)$ when i is even. We let

$$\mathrm{Can}(M) \coloneqq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{MT}(Z,\mathbb{Q})}(\mathbb{Q}(-i), W_{2i} M / W_{2i-1} M),$$

and we call this the *canonical fiber functor.*

Then the category $\mathbf{MT}(Z, \mathbb{Q})$ is a neutral $\mathbb{Q}$-linear Tannakian category with fiber functor Can, and we let $\pi_1^{\mathbf{MT}}(Z)$ denote its Tannakian fundamental group, which is therefore a pro-algebraic group over $\mathbb{Q}$.

The subcategory of simple objects of $\mathbf{MT}(Z, \mathbb{Q})$ consists of direct sums of tensor powers of $\mathbb{Q}(1)$ and is therefore equivalent as a Tannakian category to the category of representations of $\mathbb{G}_m$. This inclusion induces a quotient map $\pi_1^{\mathbf{MT}}(Z) \twoheadrightarrow \mathbb{G}_m$, and we let $\pi_1^{\mathrm{un}}(Z)$ denote the kernel of this quotient. The functor sending an object M of $\mathbf{MT}(Z, \mathbb{Q})$ to the direct sum $\bigoplus_{i \in \mathbb{Z}} W_i M / W_{i-1} M$ gives a splitting of this inclusion of categories, which implies that the quotient map $\pi_1^{\mathbf{MT}}(Z) \twoheadrightarrow \mathbb{G}_m$ splits.

This implies that $\mathbf{MT}(Z, \mathbb{Q})$ has fundamental group

$$\pi_1^{\mathbf{MT}}(Z) = \pi_1^{\mathrm{un}}(Z) \rtimes \mathbb{G}_m,$$

where $\pi_1^{\text{un}}(Z)$ the maximal pro-unipotent subgroup of $\pi_1^{\mathbf{MT}}(Z)$. The action of $\mathbb{G}_m$ on $\pi_1^{\text{un}}(Z)$ by conjugation gives an action of $\mathbb{G}_m$, or equivalently, a grading, on the Hopf algebra of $\pi_1^{\text{un}}(Z)$. This associated graded Hopf algebra is denoted

$$\bigoplus_{i=0}^{\infty} A(Z)_i = A(Z) := \mathcal{O}(\pi_1^{\text{un}}(Z)),$$

where $A(Z)_i$ denotes the nth graded piece. We refer to the degree on $A(Z)$ as the *half-weight*, as it is half the ordinary motivic weight.

In fact, the description of the Ext groups gives us the following information. It gives a canonical embedding of graded vector spaces

$$\bigoplus_{n=1}^{\infty} K_{2n-1}(Z)_{\mathbb{Q}} \hookrightarrow A(Z),$$

with $K_{2n-1}(Z)_{\mathbb{Q}}$ in degree n, and whose image is the set of primitive elements of $A(Z)$. Equivalently, this gives a canonical isomorphism

$$\pi_1^{\text{un}}(Z)^{\text{ab}} \cong \left(\bigoplus_{n=1}^{\infty} K_{2n-1}(Z)_{\mathbb{Q}} \right)^{\vee}.$$

In fact, this canonical isomorphism extends to an isomorphism between $\pi_1^{\text{un}}(Z)$ and the free pro-unipotent group (Definition 3.1.7) on the graded vector space $\left(\bigoplus_{n=1}^{\infty} K_{2n-1}(Z)_{\mathbb{Q}} \right)^{\vee}$, with $(K_{2n-1}(Z))^{\vee}$ in degree $-n$, but this extension is not canonical. This last fact tells us that there is a non-canonical isomorphism between $A(Z)$ and the free shuffle algebra on the graded vector space $\bigoplus_{n=1}^{\infty} K_{2n-1}(Z)_{\mathbb{Q}}$, with $K_{2n-1}(Z)$ in degree n. This non-canonicity is the key to a later consideration; see Remark 3.3.2.

Furthermore, in Proposition 3.3.3, we will show that $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n))$ is isomorphic to the space of degree n primitive elements of $A(Z)$.

For $Z' \subseteq Z$ an open subscheme, we have an inclusion $\mathbf{MT}(Z, \mathbb{Q}) \subseteq \mathbf{MT}(Z', \mathbb{Q})$, which gives rise to a quotient map $\pi_1^{\mathbf{MT}}(Z') \twoheadrightarrow \pi_1^{\mathbf{MT}}(Z)$ that is an isomorphism on $\mathbb{G}_m$ and hence to an inclusion

$$A(Z) \subseteq A(Z')$$

of graded Hopf algebras. There is also a graded Hopf algebra $A(\operatorname{Spec} K)$, which is the union of $A(Z)$ for $Z \subseteq \operatorname{Spec} \mathcal{O}_K$, and we may view all such $A(Z)$ as lying inside $A(\operatorname{Spec} K)$. If $\mathfrak{p}$ is a point of $\operatorname{Spec} \mathcal{O}_K \setminus Z$ and $\alpha \in A(Z)$, we say α is *unramified at* $\mathfrak{p}$ if

$$\alpha \in A(Z \cup \{\mathfrak{p}\}) \subseteq A(Z).$$

The de Rham Unipotent Fundamental Group of X

For the rest of Part I, we let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{Z}$.

Definition 3.1.13. We let

$$\pi_1^{\mathrm{un}}(X)$$

denote the unipotent de Rham fundamental group of $X_{\mathbb{Q}}$. It is the fundamental group of the Tannakian category of algebraic vector bundles with connection on $X_{\mathbb{Q}}$.

This pro-unipotent group over $\mathbb{Q}$ is a free pro-unipotent group the graded vector space consisting of $H_1^{\mathrm{dR}}(X_{\mathbb{Q}})$ in degree -1 (and zero in other degrees). Dually, its coordinate ring is generated by words in holomorphic differential forms on X, which are integrands of interated integrals.

By the construction in Section 3 of [DG05], $\pi_1^{\mathrm{un}}(X)$ is in $\mathbf{MT}(Z, \mathbb{Q})$ (in the sense that its coordinate ring and Lie algebra are each an Ind-object and pro-object, respectively, of $\mathbf{MT}(Z, \mathbb{Q})$). It therefore carries an action of $\pi_1^{\mathbf{MT}}(Z)$, whose restriction to $\mathbb{G}_m$ induces the grading. While the group $\pi_1^{\mathrm{un}}(X)$ itself is independent of basepoint, the action of $\pi_1^{\mathbf{MT}}(Z)$ is not, and we always use the tangential basepoint $\vec{1}_0$, i.e., the element $1 \in T_0(\mathbb{P}^1_{\mathbb{Q}}) = \mathbb{Q}$ at $0 \in \mathbb{P}^1(\mathbb{Q})$.

Remark 3.1.14. By an argument analogous to the proof of Corollary 2.9 of [BDCKW], one may check that all of these constructions (in particular, the Chabauty-Kim locus, c.f., Section 3.1.5) are the same if we replace $\vec{1}_0$ by any other Z-integral basepoint of X.

Remark 3.1.15. We will often consider $\pi_1^{\mathbf{MT}}(Z)$-equivariant quotients $\pi_1^{\mathrm{un}}(X) \twoheadrightarrow \Pi$, especially when Π is finite-dimensional as a scheme over $\mathbb{Q}$ (hence an algebraic group). *Unless otherwise stated, it is always understood that* Π *is such a quotient.*

A standard example is $\Pi_n \coloneqq \pi_1^{\mathrm{un}}(X)_{\geq -n}$. However, in most of our calculations, we will be concerned with quotients that factor through $\pi_1^{\mathrm{PL}}(X)$, a specific quotient to be introduced in 3.1.6.

We write $\mathfrak{n}(Z)$, $\mathfrak{n}(X)$, and $\mathfrak{n}^{\mathrm{PL}}(X)$ for the corresponding Lie algebras.

3.1.3 Cohomology and Cocycles

We let $H^1(\pi_1^{\mathbf{MT}}(Z), \pi_1^{\mathrm{un}}(X))$ denote the (pointed) set of $\pi_1^{\mathbf{MT}}(Z)$-equivariant torsor schemes under $\pi_1^{\mathrm{un}}(X)$.

For $b \in X(Z)$, there is the torsor of paths ${}_bP_{\vec{1}_0}$ constructed in Section 3 of [DG05] (as [DG05] uses the functional convention for path composition; this is in fact the torsor of paths from $\vec{1}_0$ to b). It is a $\pi_1^{\mathbf{MT}}(Z)$-equivariant torsor under ${}_{\vec{1}_0}P_{\vec{1}_0} = \pi_1^{\mathrm{un}}(X)$.

We therefore have the Kummer map

$$X(Z) \xrightarrow{\kappa} H^1(\pi_1^{\mathbf{MT}}(Z), \pi_1^{\mathrm{un}}(X)),$$

and by composition with the map induced by $\pi_1^{\mathrm{un}}(X) \twoheadrightarrow \Pi$,

$$X(Z) \xrightarrow{\kappa} H^1(\pi_1^{\mathbf{MT}}(Z), \Pi).$$

Definition 3.1.16. For any $\mathbb{Q}$-algebra R, we define

$$Z_{\Pi}^{1,\mathbb{G}_m}(R) := Z^1(\pi_1^{\mathrm{un}}(Z)_R, \Pi_R)^{\mathbb{G}_m},$$

which is the set of morphisms of schemes from $\pi_1^{\mathrm{un}}(Z)_R$ to Π_R over R, equivariant with respect to the $\mathbb{G}_m$-action, and satisfying the cocycle condition on the R'-points for any R-algebra R'.

Proposition 6.4 of [Bro17] ensures that this is representable by a scheme (also see Corollary

3.2.10). We thus write

$$Z_\Pi^{1,\mathbb{G}_m} = Z_\Pi^{1,\mathbb{G}_m}(Z) := Z^1(\pi_1^{\mathrm{un}}(Z), \Pi)^{\mathbb{G}_m}$$

for the scheme of $\mathbb{G}_m$-equivariant cocycles. If Π is finite-dimensional, then this is in fact a finite-dimensional variety over $\mathbb{Q}$.

By Proposition 5.2.1 of [DCW16], we have

$$H^1(\pi_1^{\mathbf{MT}}(Z), \Pi) = Z_\Pi^{1,\mathbb{G}_m}(\mathbb{Q}).$$

We have a *universal cocycle evaluation map*

$$\mathcal{C}_\Pi \colon Z_\Pi^{1,\mathbb{G}_m} \times \pi_1^{\mathrm{un}}(Z) \to \Pi \times \pi_1^{\mathrm{un}}(Z).$$

It is defined on the functor of points as follows. For a $\mathbb{Q}$-algebra R and an element $(c, \gamma) \in (Z_\Pi^{1,\mathbb{G}_m})_R(R) \times \pi_1^{\mathrm{un}}(Z)_R(R) = (Z_\Pi^{1,\mathbb{G}_m} \times \pi_1^{\mathrm{un}}(Z))(R)$, we have $c(\gamma) \in \Pi_R(R) = \Pi(R)$. We define $\mathcal{C}_\Pi(c, \gamma)$ to be the pair $(c(\gamma), \gamma)$.

In fact, the morphism $\mathcal{C}_\Pi$ lies over the identity morphism on $\pi_1^{\mathrm{un}}(Z)$, so letting $\mathcal{K}$ denote the function field of $\pi_1^{\mathrm{un}}(Z)$, we also have the base change from $\pi_1^{\mathrm{un}}(Z)$ to $\operatorname{Spec} \mathcal{K}$

$$\mathcal{C}_\Pi^{\mathcal{K}} \colon (Z_\Pi^{1,\mathbb{G}_m})_{\mathcal{K}} \to \Pi_{\mathcal{K}}.$$

Remark 3.1.17. If Π is finite dimensional, it will turn out that this is a morphism of affine spaces over the field $\mathcal{K}$, and our "geometric step" (see 3.3.2) consists in computing its scheme-theoretic image.

Definition 3.1.18. We let $\mathcal{I}_\Pi^Z$ denote the ideal of functions in the coordinate ring of $\Pi_{\mathcal{K}}$ vanishing on the image of $\mathcal{C}_\Pi^{\mathcal{K}}$. This is known as the *Chabauty-Kim ideal* (associated to Π). For $\Pi = \Pi_n$, we denote it by $\mathcal{I}_n^Z$.

3.1.4 $\mathfrak{p}$-adic Realization and Kim's Cutter

Let $\mathfrak{p}$ be a closed point of Z. For simplicity, we suppose that $Z_\mathfrak{p} \cong \operatorname{Spec} \mathbb{Z}_p$.

If $\omega \in \mathcal{O}(\pi_1^{\text{un}}(X))$, and a, b are $\mathbb{Z}_p$-basepoints of X (rational or tangential), then we can extract an element of $\mathbb{Q}_p$ known as the Coleman iterated integral $\int_a^b \omega$. The Coleman iterated integral is originally due to Coleman ([Col82]) and was reformulated by Besser [Bes02] into the form that is used in [DCW16]. Fixing $a = \vec{1}_0$ and letting b vary over $X(Z_\mathfrak{p})$, one may define local Kummer map

$$X(Z_\mathfrak{p}) \xrightarrow{\kappa_\mathfrak{p}} \pi_1^{\text{un}}(X)(\mathbb{Q}_p)$$

by sending a regular function ω on $\pi_1^{\text{un}}(X)$ to the Coleman function $\kappa_\mathfrak{p}^\sharp(\omega) \colon X(Z_\mathfrak{p}) \to \mathbb{Q}_p$ defined by $b \mapsto \int_a^b \omega$.

Its composition with $\pi_1^{\text{un}}(X)(\mathbb{Q}_p) \to \Pi(\mathbb{Q}_p)$ is also denoted by $\kappa_\mathfrak{p}$. This map is Coleman-analytic, meaning that regular functions on $\pi_1^{\text{un}}(X)$ pull back to Coleman functions on $X(Z_\mathfrak{p})$. These are locally analytic functions, and a nonzero such function has finitely many zeroes.

Remark 3.1.19. The local Kummer map in this form was originally referred to in [Kim05] as the *p-adic unipotent Albanese map*. It is the same as the map α of 1.3 of [DCW16]. The latter is defined by sending b to the torsor of paths from a to b (with its structure as a filtered ϕ-module)

$\mathfrak{p}$-adic Period Map

In addition, there is a morphism $\operatorname{Spec} \mathbb{Q}_p \to \pi_1^{\text{un}}(Z)$, or equivalently, a $\mathbb{Q}$-algebra homomorphism

$$\operatorname{per}_\mathfrak{p} \colon A(Z) \to \mathbb{Q}_p,$$

known as the $\mathfrak{p}$*-adic period map.* While elements of $A(Z)$ are represented by formal (motivic) iterated integrals, this homomorphism takes the value of the iterated integral in the sense of Coleman integration.

The following may be regarded as a p-adic analogue of a small piece of the Kontsevich-Zagier period conjecture ([KZ01]). It has been in folklore for some time and appears in the literature for $K/\mathbb{Q}$ abelian as Conjecture 4 of [Yam10].

Conjecture 3.1.20 (p-adic Period Conjecture)**.** *For any open integer scheme Z, the period map* $\mathrm{per}_{\mathfrak{p}}\colon A(Z) \to \mathbb{Q}_p$ *is injective.*

Kim's Cutter

Viewing $\mathcal{C}_\Pi$ as a morphism of schemes over $\pi_1^{\mathrm{un}}(Z)$ and base-changing along the $\mathfrak{p}$-adic period map $\operatorname{Spec}\mathbb{Q}_p \to \pi_1^{\mathrm{un}}(Z)$, we get a morphism $(Z_\Pi^{1,\mathbb{G}_m})_{\mathbb{Q}_p} \to \Pi_{\mathbb{Q}_p}$. The induced map on $\mathbb{Q}_p$-points is denoted by loc_Π. Denoting the composition $X(Z) \xrightarrow{\kappa} H^1(\pi_1^{\mathbf{MT}}(Z), \Pi) = Z_\Pi^{1,\mathbb{G}_m}(\mathbb{Q}) \subseteq Z_\Pi^{1,\mathbb{G}_m}(\mathbb{Q}_p)$ by κ as well, this fits into a diagram:

$$\begin{array}{ccc} X(Z) & \hookrightarrow & X(Z_{\mathfrak{p}}) \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa_{\mathfrak{p}}} \\ Z_\Pi^{1,\mathbb{G}_m}(\mathbb{Q}_p) & \xrightarrow[\mathrm{loc}_\Pi]{} & \Pi(\mathbb{Q}_p) \end{array},$$

which we call *Kim's Cutter*.

This diagram is commutative (c.f. [DCW16], 4.9). In Section 3.2.4, we will describe κ and $\kappa_{\mathfrak{p}}$ explicitly in terms of coordinates.

3.1.5 Chabauty-Kim Locus

Let $f \in \mathcal{O}(\Pi \times \pi_1^{\mathrm{un}}(Z))$. Now $\mathrm{per}_{\mathfrak{p}}$ induces a morphism $\Pi_{\mathbb{Q}_p} \to \Pi \times \pi_1^{\mathrm{un}}(Z)$, and f pulls back via this morphism to an element of the coordinate ring of $\Pi_{\mathbb{Q}_p}$, hence a function $\Pi(\mathbb{Q}_p) \xrightarrow{f} \mathbb{Q}_p$. The composite $f \circ \kappa_{\mathfrak{p}}$ with $X(Z_{\mathfrak{p}}) \xrightarrow{\kappa_{\mathfrak{p}}} \Pi(\mathbb{Q}_p)$ is a Coleman function on $X(Z_{\mathfrak{p}})$ and is denoted $f \restriction_{X(Z_{\mathfrak{p}})}$.

We then define

$$X(Z_{\mathfrak{p}})_\Pi = X(Z_{\mathfrak{p}})_\Pi^Z \coloneqq \{z \in X(Z_{\mathfrak{p}})\colon f \restriction_{X(Z_{\mathfrak{p}})} (z) = 0 \quad \forall\, f \in \mathcal{O}(\Pi \times \pi_1^{\mathrm{un}}(Z)) \cap \mathcal{I}_\Pi^Z\}.$$

While this set depends on Z, we write $X(Z_{\mathfrak{p}})_\Pi$ when there is no confusion.

Any $f \in \mathcal{O}(\Pi \times \pi_1^{\mathrm{un}}(Z)) \cap \mathcal{I}_\Pi^Z$ vanishes on the image of $\mathcal{C}_\Pi\colon Z_\Pi^{1,\mathbb{G}_m} \times \pi_1^{\mathrm{un}}(Z) \to \Pi \times \pi_1^{\mathrm{un}}(Z)$, hence also on the image of loc_Π. By the commutativity of Kim's Cutter, $f\restriction_{X(Z_\mathfrak{p})}$ vanishes on $X(Z)$, hence

$$X(Z) \subset X(Z_\mathfrak{p})_\Pi.$$

We note that if Π' dominates Π (i.e., we have $\pi_1^{\mathrm{un}}(X) \twoheadrightarrow \Pi' \xrightarrow{p} \Pi$), then $p^\#(\mathcal{I}_\Pi^Z) \subseteq \mathcal{I}_{\Pi'}^Z$, hence $X(Z_\mathfrak{p})_{\Pi'} \subseteq X(Z_\mathfrak{p})_\Pi$.

For $\Pi = \Pi_n$, we denote $X(Z_\mathfrak{p})_\Pi$ by $X(Z_\mathfrak{p})_n$. The importance of Kim's method lies in the fact that $X(Z_\mathfrak{p})_n$ is known to be finite for sufficiently large n when K is totally real by [Kim05] and [Kim12].

Remark 3.1.21. The locus defined in [DCW16] differs slightly from our own in that it uses a version of $\mathcal{I}_\Pi^Z$ defined as the ideal of functions vanishing on the image of the base change of $\mathcal{C}_\Pi$ along $\mathrm{per}_\mathfrak{p}$, which could in principle be larger than our own. However, as explained in 4.2.6 of [DC15], Conjecture 3.1.20 implies that these two are the same, which is why we see no harm in doing it this way. Furthermore, our version of Kim's conjecture is a priori stronger than the original version, so our theorems apply to his conjecture either way.

3.1.6 The Polylogarithmic Quotient

We let N denote the kernel of the homomorphism $\pi_1^{\mathrm{un}}(X) \to \pi_1^{\mathrm{un}}(\mathbb{G}_m)$ induced by the inclusion $X \hookrightarrow \mathbb{G}_m$, where $\pi^1\mathrm{un}(\mathbb{G}_m)$ refers to the unipotent de Rham fundamental group of $\mathbb{G}_{m\mathbb{Q}}$. Then, following [Del89], we define the *polylogarithmic quotient*

$$\pi_1^{\mathrm{PL}}(X) \coloneqq \pi_1^{\mathrm{un}}(X)/[N,N].$$

The group $\pi_1^{\mathbf{MT}}(Z)$ acts on $\pi_1^{\mathrm{un}}(\mathbb{G}_m)$ as well as $\pi_1^{\mathrm{un}}(X)$, and because $\pi_1^{\mathrm{un}}(X) \to \pi_1^{\mathrm{un}}(\mathbb{G}_m)$ is induced by a map of schemes over $\mathbb{Z}$, it is $\pi_1^{\mathbf{MT}}(Z)$-equivariant. Therefore, N and hence $[N,N]$ are $\pi_1^{\mathbf{MT}}(Z)$-stable, so $\pi_1^{\mathrm{PL}}(X)$ has a structure of a motive, i.e., an action of $\pi_1^{\mathbf{MT}}(Z)$.

As a motive, it has the structure

$$\pi_1^{\mathrm{PL}}(X) = \mathbb{Q}(1) \ltimes \prod_{i=1}^{\infty} \mathbb{Q}(i),$$

so in particular the action of $\pi_1^{\mathbf{MT}}(Z)$ factors through $\mathbb{G}_m$.

Remark 3.1.22. In Section 3.2.1, we will describe $\pi_1^{\mathrm{PL}}(X)$ more explicitly in coordinates.

In the specific case $\Pi = \Pi_{\mathrm{PL},n} \coloneqq \pi_1^{\mathrm{PL}}(X)_{\geq -n}$ for a positive integer n, we write $Z^{1,\mathbb{G}_m}_{\mathrm{PL},n}$, $\mathcal{C}_{\mathrm{PL},n}$, $\mathrm{loc}_{\mathrm{PL},n}$, $\mathcal{I}^Z_{\mathrm{PL},n}$, and $X(Z_{\mathfrak{p}})_{\mathrm{PL},n}$ to denote $Z^{1,\mathbb{G}_m}_{\Pi}$, $\mathcal{C}_{\Pi}$, loc_{Π}, $\mathcal{I}^Z_{\Pi}$, and $X(Z_{\mathfrak{p}})_{\Pi}$, respectively.

Since $\pi_1^{\mathrm{un}}(Z)$ acts trivially on $\pi_1^{\mathrm{PL}}(X)$, a $\mathbb{G}_m$-equivariant cocycle

$$c\colon \pi_1^{\mathrm{un}}(Z) \to \pi_1^{\mathrm{PL}}(X)$$

is just a $\mathbb{G}_m$-equivariant homomorphism.

If Π is a quotient of $\Pi_{\mathrm{PL},n}$, then any $\mathbb{G}_m$-equivariant homomorphism $\pi_1^u n(Z) \to \Pi$ must be zero on $\pi_1^{\mathrm{un}}(Z)_{<-n}$ by the $\mathbb{G}_m$-equivariance, so $Z^{1,\mathbb{G}_m}_{\Pi}$ is the same as

$$Z^1(\pi_1^{\mathrm{un}}(Z)_{\geq -n}, \Pi)^{\mathbb{G}_m}.$$

It follows that we can in fact view $\mathcal{C}_{\mathrm{PL},n}$ as a morphism

$$Z^{1,\mathbb{G}_m}_{\mathrm{PL},n} \times \pi_1^{\mathrm{un}}(Z)_{\geq -n} \to \Pi_{\mathrm{PL},n} \times \pi_1^{\mathrm{un}}(Z)_{\geq -n}$$

lying over the identity on $\pi_1^{\mathrm{un}}(Z)_{\geq -n}$. Letting $\mathcal{K}_n$ denote the function field of $\pi_1^{\mathrm{un}}(Z)_{\geq -n}$, it induces a map

$$\mathcal{C}^{\mathcal{K}_n}_{\mathrm{PL},n}\colon (Z^{1,\mathbb{G}_m}_{\mathrm{PL},n})_{\mathcal{K}_n} \to (\Pi_{\mathrm{PL},n})_{\mathcal{K}_n}$$

of finite-dimensional affine spaces over the field $\mathcal{K}_n$, and we view $\mathcal{I}^Z_{\mathrm{PL},n}$ as an ideal in $\mathcal{O}((\Pi_{\mathrm{PL},n})_{\mathcal{K}_n})$.

3.1.7 Kim's Conjecture

We recall Conjecture 2.2.3 from the introduction:

Conjecture 3.1.23. *$X(Z) = X(Z_{\mathfrak{p}})_n$ for sufficiently large n.*

In fact, dimension counts show that $\mathcal{I}^Z_{\mathrm{PL},n}$ is nonzero and hence $X(Z_{\mathfrak{p}})_{\mathrm{PL},n}$ is finite for sufficiently large n. One might wonder the following:

Question 3.1.24. Does $X(Z) = X(Z_{\mathfrak{p}})_{\mathrm{PL},n}$ for sufficiently large n?

This is a strengthening of Conjecture 3.1.23. However, as we will show in Section 3.4, this question has a negative answer as stated (at least for $Z = \operatorname{Spec} \mathbb{Z}[1/\ell]$ and odd primes ℓ) and needs to be modified by an S_3-symmetrization.

More specifically, there is an action of S_3 on the scheme X. This induces an action of S_3 on $\pi_1^{\mathrm{un}}(X)$. For a quotient Π and $\sigma \in S_3$, we may define $\sigma(\Pi)$ by

$$\ker(\pi_1^{\mathbf{MT}}(Z) \twoheadrightarrow \sigma(\Pi)) = \sigma(\ker(\pi_1^{\mathbf{MT}}(Z) \twoheadrightarrow \Pi)).$$

It follows by independence of basepoint (Remark 3.1.14) and functoriality of all the constructions that $\sigma(X(Z_{\mathfrak{p}})_{\Pi}) = X(Z_{\mathfrak{p}})_{\sigma(\Pi)}$. We then write

$$X(Z_{\mathfrak{p}})_{\mathrm{PL},n}^{S_3} \coloneqq \bigcap_{\sigma \in S_3} \sigma(X(Z_{\mathfrak{p}})_{\mathrm{PL},n}) = \bigcap_{\sigma \in S_3} X(Z_{\mathfrak{p}})_{\sigma(\Pi_{\mathrm{PL},n})}.$$

Our symmetrized conjecture is that

Conjecture 3.1.25. *$X(Z) = X(Z_{\mathfrak{p}})_{\mathrm{PL},n}^{S_3}$ for sufficiently large n.*

We note that Π_n dominates $\sigma(\Pi_{\mathrm{PL},n})$ for all $\sigma \in S_3$; hence $X(Z_{\mathfrak{p}})_n \subseteq \sigma(X(Z_{\mathfrak{p}})_{\mathrm{PL},n})$, and so

$$X(Z_{\mathfrak{p}})_{\Pi_n} \subseteq X(Z_{\mathfrak{p}})_{\mathrm{PL},n}^{S_3},$$

so Conjecture 3.1.25 implies Conjecture 3.1.23. In Theorem 3.4.4, we use our computations to verify Conjecture 3.1.25 for $Z = \operatorname{Spec} \mathbb{Z}[1/3]$ and $p = 5, 7$.

3.2 Coordinates

3.2.1 Coordinates on the Fundamental Group

Let $\{e_0, e_1\}$ be a basis of $H_1^{\mathrm{dR}}(X_{\mathbb{Q}})$ dual to the basis $\left\{\frac{dz}{z}, \frac{dz}{1-z}\right\}$ of $H^1_{\mathrm{dR}}(X_{\mathbb{Q}})$. The algebra $\mathcal{U}\pi_1^{\mathrm{un}}(X)$ is the free (completed) non-commutative algebra on the generators e_0 and e_1, with coproduct given by declaring that e_0 and e_1 are primitive and grading given by putting both in degree -1. We refer to the words

$$e_0, e_1, e_1e_0, e_1e_0e_0, \ldots$$

as the *polylogarithmic words*. We let the elements

$$\log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \mathrm{Li}_2^{\mathfrak{u}}, \cdots \in \mathcal{O}(\pi_1^{\mathrm{un}}(X))$$

be the duals of these words with respect to the standard basis of $\mathcal{U}\pi_1^{\mathrm{un}}(X)$. More generally, for a word w in e_0, e_1, we let $\mathrm{Li}_w^{\mathfrak{u}}$ denote the dual basis element of $\mathcal{O}(\pi_1^{\mathrm{un}}(X))$.

Proposition 3.2.1. *We have*

$$\Delta' \log^{\mathfrak{u}} = 0$$

$$\Delta' \mathrm{Li}_n^{\mathfrak{u}} = \sum_{i=1}^{n-1} \mathrm{Li}_{n-i}^{\mathfrak{u}} \otimes \frac{(\log^{\mathfrak{u}})^{ш i}}{i!}.$$

Proof. The first equation follows from Remark 3.1.8.

By the discussion following Definition 3.1.7, we have the formula

$$\Delta' \mathrm{Li}_n^{\mathfrak{u}} = \Delta' \mathrm{Li}_{e_1e_0\cdots e_0}^{\mathfrak{u}} = \mathrm{Li}_{e_1}^{\mathfrak{u}} \otimes \mathrm{Li}_{e_0\cdots e_0}^{\mathfrak{u}} + \mathrm{Li}_{e_1e_0}^{\mathfrak{u}} \otimes \mathrm{Li}_{e_0\cdots e_0}^{\mathfrak{u}} + \cdots + \mathrm{Li}_{e_1e_0\cdots e_0}^{\mathfrak{u}} \otimes \mathrm{Li}_{e_0}^{\mathfrak{u}}.$$

By the definition of the shuffle product, we have the formula

$$(\log^{\mathfrak{u}})^{ш i} = (\mathrm{Li}_{e_0}^{\mathfrak{u}})^{ш i} = i!\, \mathrm{Li}_{(e_0)^i}^{\mathfrak{u}}.$$

The previous two formulas together with the definition of $\mathrm{Li}_n^{\mathfrak{u}}$ then imply the proposition.

□

Letting $\mathcal{P}^{\mathrm{PL}}$ be the subalgebra generated by $\log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \mathrm{Li}_2^{\mathfrak{u}}, \ldots$, the formula above implies that $\mathcal{P}^{\mathrm{PL}}$ is a Hopf subalgebra. It therefore corresponds to a quotient group of $\pi_1^{\mathrm{un}}(X)$. It follows from Proposition 7.1.3 of [DCW16] that this quotient group is the group $\pi_1^{\mathrm{PL}}(X)$.

Furthermore, we have

$$\mathcal{O}(\Pi_{\mathrm{PL},n}) = \mathbb{Q}[\log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \ldots, \mathrm{Li}_n^{\mathfrak{u}}]$$

as a Hopf subalgebra of $\mathcal{P}^{\mathrm{PL}} = \mathcal{O}(\pi_1^{\mathrm{PL}}(X))$.

Given a cocycle $c \in Z^{1,\mathbb{G}_m}_{\mathrm{PL}}$, we write $c^\sharp \colon \mathcal{P}^{\mathrm{PL}} \to A(Z)$ for the associated homomorphism of $\mathbb{Q}$-algebras, and we write

$$\log^{\mathfrak{u}}(c) \coloneqq c^\sharp \log^{\mathfrak{u}},$$

$$\mathrm{Li}_n^{\mathfrak{u}}(c) \coloneqq c^\sharp \,\mathrm{Li}_n^{\mathfrak{u}}.$$

Corollary 3.2.2. *For $c \in Z^{1,\mathbb{G}_m}_{\mathrm{PL}}(\mathbb{Q})$, we have*

$$\Delta' \log^{\mathfrak{u}}(c) = 0$$

$$\Delta' \,\mathrm{Li}_n^{\mathfrak{u}}(c) = \sum_{i=1}^{n-1} \mathrm{Li}_{n-i}^{\mathfrak{u}}(c) \otimes \frac{(\log^{\mathfrak{u}}(c))^{ш i}}{i!}.$$

Proof. The cocycle condition reduces to the homomorphism conditions (as we are restricting ourselves to the polylogarithmic quotient), which means, dually, that $c^\sharp$ respects the coproduct. The corollary then follows immediately from Proposition 3.2.1. □

3.2.2 Generating $A(Z)$

We take a moment to discuss coordinates for the Galois group $\pi_1^{\mathrm{un}}(Z)$. On an abstract level, this is a free unipotent group, and its structure is governed by the theory of 3.1.1. However, we need to be able to write elements of $A(Z)$ in a way that allows us to compute their p-adic periods. This essentially means writing them as explicit combinations of special values of polylogarithms and zeta functions.

We introduce the notation

$$\log^{\mathfrak{u}}(z) \coloneqq \log^{\mathfrak{u}}(\kappa(z)) \in A(Z)_1$$

$$\mathrm{Li}_n^{\mathfrak{u}}(z) \coloneqq \mathrm{Li}_n^{\mathfrak{u}}(\kappa(z)) \in A(Z)_n$$

for $z \in X(Z)$. It is the same as the motivic period $\mathrm{Li}^{\mathfrak{u}}_n(z)$ mentioned in (9.1) of [Bro17] and 2.2.4 of [DC15], which justifies the notation $\mathrm{Li}^{\mathfrak{u}}_n(c)$ in the previous section.

Because $\log^{\mathfrak{u}}$ is pulled back from $\mathbb{G}_m$, we in fact have $\log^{\mathfrak{u}}(z) \in A(Z)$ whenever $z \in \mathbb{G}_m(Z)$.

We also note:

Fact 3.2.3. For $z, w \in X(Z)$, we have

$$\log^{\mathfrak{u}}(zw) = \log^{\mathfrak{u}}(z) + \log^{\mathfrak{u}}(w)$$

and

$$\mathrm{Li}^{\mathfrak{u}}_1(z) = -\log^{\mathfrak{u}}(1-z).$$

Letting c_1 denote the cocycle corresponding to the class of the path torsor ${}_{\vec{-1}_1}P_{\vec{1}_0}$ in $H^1(\pi_1^{\mathbf{MT}}(Z), \pi_1^{\mathrm{un}}(X))$, we write

$$\zeta^{\mathfrak{u}}(n) \coloneqq \mathrm{Li}^{\mathfrak{u}}_n(c_1) \in A(Z)_n.$$

It is not clear a priori that the special values $\mathrm{Li}^{\mathfrak{u}}_n(z)$ for $z \in X(Z)$ span the space $A(Z)$ (whether we leave Z fixed or take a union over various Z, such as all Z with a fixed function field K or all Z whose function field is cyclotomic). However, there is the following conjecture of Goncharov:

Conjecture 3.2.4 ([Gon], Conjecture 7.4)**.** *The ring $A(\mathbb{Q})$ is spanned by elements of the form $\mathrm{Li}^{\mathfrak{u}}_w(z)$ for z a rational point or rational tangential basepoint of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and w a word in e_0, e_1.*

Similar to [DC15], we state a version of this conjecture with control on ramification.

Definition 3.2.5. We say that an integral scheme Z with function field $\mathbb{Q}$ is *saturated* if there exists $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ such that $Z \cong \operatorname{Spec} \mathbb{Z}[1/S]$ for $S = \{p \text{ prime}; p \leq N\}$.

Conjecture 3.2.6. *If Z is saturated, then the ring $A(Z)$ is spanned by elements of the form $\mathrm{Li}^{\mathfrak{u}}_w(z)$ for z a Z-integral point or tangential basepoint of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.*

This is a strengthening of Conjecture 2.2.6 of [DC15] and is in fact the motivation behind that conjecture. It is known in the cases $N = 1$ ([Bro12a]) and $N = 2$ ([Del10]), and the case $N = \infty$ is Conjecture 3.2.4.

In Section 3.3.3, we will demonstrate this in practice by writing elements of $A(\mathbb{Z}[1/3])$ in terms of values of polylogarithms at elements of $\mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathbb{Z}[1/6])$.

Remark 3.2.7. In doing so, we will find that all of these may be written in terms of single (rather than multiple) polylogarithms. This is in line with a conjecture of Goncharov about the depth filtration, as we now describe. We let

$$\mathfrak{n}^G(Z) := \mathfrak{n}(Z)/[\mathfrak{n}(Z)_{<-1}, [\mathfrak{n}(Z)_{<-1}, \mathfrak{n}(Z)_{<-1}]],$$

$\pi_1^G(Z)$ the corresponding quotient group, and $A^G(Z)$ its coordinate ring. It follows by the definition of $\pi_1^{\mathrm{PL}}(X)$ that

$$Z^{1,\mathbb{G}_m}_{\mathrm{PL},n} = Z^1(\pi_1^G(Z)_{\geq n}, \Pi_{\mathrm{PL},n})^{\mathbb{G}_m}.$$

In particular, each $\mathrm{Li}^{\mathfrak{u}}_n(z)$ is in $A^G(Z)$, and Goncharov's conjecture (Conjecture 7.6 of [Gon]) says that $A^G(Z)$ is spanned by elements of the form $\mathrm{Li}^{\mathfrak{u}}_n(z)$, at least when $Z = \operatorname{Spec} \mathbb{Q}$. We might expect this to hold whenever Z is saturated. Whenever K is totally real, $\mathfrak{n}^G(Z)$ agrees with $\mathfrak{n}(Z)$ in degrees ≥ -4; hence $A^G(Z)_{\leq 4} = A(Z)_{\leq 4}$ should be spanned by elements of the form $\mathrm{Li}^{\mathfrak{u}}_n(z)$.

3.2.3 Coordinates on the Space of Cocycles

Fix an arbitrary family $\Sigma = \{\sigma_{n,i}\}$ of homogeneous free generators for $\mathfrak{n}(Z)$ with $\sigma_{n,i}$ in half-weight $-n$, and for each word w of half-weight $-n$ in the above generators, let f_w denote the associated element of $A_n = A(Z)_n$.

The following proposition was communicated to the authors in an unpublished letter by Francis Brown. It provides equivalent information to the geometric algorithm in [DC15], but it allows one to avoid computing the logarithm in a unipotent group, making the algorithm much more practical.

Definition 3.2.8. For an arbitrary cocycle $c \in Z^{1,\mathbb{G}_m}_{\mathrm{PL}}(R)$ for a $\mathbb{Q}$-algebra R, a polylogarithmic word λ of half-weight $-n$, and a word w in the elements of Σ of half-weight $-n$, let

$$\phi^w_\lambda(c) \in \mathbb{Q}$$

denote the associated matrix entry of $c^\sharp$, so that in the notation above, we have

$$\mathrm{Li}^{\mathfrak{u}}_\lambda(c) = \sum_w \phi^w_\lambda(c) f_w.$$

Proposition 3.2.9. *Let $c \in Z^{1,\mathbb{G}_m}_{\mathrm{PL}}(R)$ for a $\mathbb{Q}$-algebra R. For $0 \le r < n$, $\tau_1, \dots, \tau_r \in \Sigma_{-1}$, and $\sigma \in \Sigma_{r-n}$, we have*

$$\phi^{\sigma\tau_1\cdots\tau_r}_{\underbrace{e_1 e_0 \cdots e_0}_{n}}(c) = \phi^{\tau_1}_{e_0}(c) \cdots \phi^{\tau_r}_{e_0}(c)\phi^{\sigma}_{\underbrace{e_1 e_0 \cdots e_0}_{n-r}}(c),$$

and all other matrix entries $\phi^w_\lambda(c)$ vanish.

Proof. This amounts to a straightforward verification, but we nevertheless give the details. We begin with a formal calculation, in which Σ_{-1} may be an arbitrary finite set, and $\{a^\tau\}_{\tau\in\Sigma_{-1}}$ a family of commuting coefficients. In this abstract setting, we claim that

$$\left(\sum_{\tau\in\Sigma_{-1}} a^\tau f_\tau\right)^{ш n} = n! \sum_{\tau_1,\dots,\tau_n\in\Sigma_{-1}} a^{\tau_1} \cdots a^{\tau_n} f_{\tau_1\cdots\tau_n}.$$

Indeed, the left side of the equation

$$
\begin{aligned}
&= \sum_{\tau_1,\ldots,\tau_n} (a^{\tau_1} f_{\tau_1}) ш \cdots ш (a^{\tau_n} f_{\tau_n}) \\
&= \sum_{\tau_1,\ldots,\tau_n} a^{\tau_1} \cdots a^{\tau_n} \left(\sum_{\substack{\text{permutations } p \\ \text{of } \tau_1,\ldots,\tau_n}} f_{\tau_1^p \cdots \tau_n^p} \right) \\
&= \sum_p \underbrace{\sum_{\tau_1,\ldots,\tau_n} a^{\tau_1} \cdots a^{\tau_n} f_{\tau_1^p \cdots \tau_n^p}}_{\text{independent of } p},
\end{aligned}
$$

which equals the right side of the equation.

Returning to our concrete situation, we apply the reduced coproduct Δ' to both sides of

$$
\sum_{|w|=-(n+1)} \phi_{n+1}^{w}(c) f_w = \mathrm{Li}_{n+1}^{\mathfrak{u}}(c),
$$

and compute:

$$
\begin{aligned}
\sum_{w',w''} \phi_{n+1}^{w'w''}(c) f_{w'} \otimes f_{w''} &= \sum_{i=1}^{n} \mathrm{Li}_{n+1-i}^{\mathfrak{u}}(c) \otimes \frac{(\log^{\mathfrak{u}}(c))^{ш i}}{i!} && (3.5) \\
&= \sum_{i=1}^{n} \left(\mathrm{Li}_{n+1-i}^{\mathfrak{u}}(c) \otimes \frac{(\sum_{\tau \in \Sigma_{-1}} \phi_{e_0}^{\tau}(c) f_{\tau})^{ш i}}{i!} \right) && (3.6) \\
&= \sum_{i=1}^{n} \sum_{\tau_1,\ldots,\tau_i \in \Sigma_{-1}} \sum_{|w|=n+1-i} \phi_{e_0}^{\tau_1}(c) \cdots \phi_{e_0}^{\tau_i}(c) \phi_{n+1-i}^{w}(c) f_w \otimes f_{\tau_1 \cdots \tau_i}. && (3.7)
\end{aligned}
$$

Taking the coefficient of $f_v \otimes f_\tau$, with $\tau \in \Sigma_{-1}$, and v an arbitrary word of length $n \geq 1$, we obtain

$$
\phi_{n+1}^{v\tau}(c) = \phi_{e_0}^{\tau}(c) \cdot \phi_n^{v}(c),
$$

while taking the coefficient of $f_v \otimes f_\sigma$, with $\sigma \in \Sigma_{-i<-1}$, and v an arbitrary word of length

$n+1-i \geq 1$, we obtain

$$\phi^{v\sigma}_{n+1}(c) = 0. \qquad \square$$

We have a morphism

$$\mathcal{C}_{\mathrm{PL}} \colon Z^{1,\mathbb{G}_m}_{\mathrm{PL}} \times \pi_1^{\mathrm{un}}(Z) \to \pi_1^{\mathrm{PL}}(X)$$

given by

$$(c, \gamma) \mapsto c(\gamma).$$

We refer to $\mathcal{C}_{\mathrm{PL}}$ as the *universal polylogarithmic cocycle*, and it is just the first component of the universal cocycle evaluation morphism $\mathcal{C}_{\Pi}$ for $\Pi = \pi_1^{\mathrm{PL}}(X)$. We define a morphism

$$\Psi : Z^{1,\mathbb{G}_m}_{\mathrm{PL}} \to \operatorname{Spec} \mathbb{Q}\left[\{\Phi^{\rho}_{\lambda}\}_{wt(\rho)=wt(\lambda)}\right],$$

where ρ ranges over Σ and λ ranges over the set of polylogarithmic words, by

$$c \mapsto (\phi^{\rho}_{\lambda}(c))_{\rho,\lambda}.$$

We define a homomorphism of rings

$$\theta^{\sharp} \colon \mathbb{Q}[\log^{\mathfrak{u}}, \mathrm{Li}^{\mathfrak{u}}_1, \mathrm{Li}^{\mathfrak{u}}_2, \dots] \to A(Z)[\{\Phi^{\rho}_{\lambda}\}_{\rho,\lambda}]$$

by

$$\log^{\mathfrak{u}} \mapsto \sum_{\tau \in \Sigma_{-1}} f_{\tau} \Phi^{\tau}_{e_0}$$

and

$$\mathrm{Li}^{\mathfrak{u}}_n \mapsto \sum_{\substack{\tau_1,\dots,\tau_r \in \Sigma_{-1} \\ \sigma \in \Sigma_{-s} \\ r+s=n \\ 1 \leq s \leq n}} f_{\sigma\tau_1\cdots\tau_r} \Phi^{\tau_1}_{e_0} \cdots \Phi^{\tau_r}_{e_0} \Phi^{\sigma}_{\underbrace{e_1 e_0 \cdots e_0}_{s}}.$$

Corollary 3.2.10. *The morphism Ψ is an isomorphism. In particular, $Z^{1,\mathbb{G}_m}_{\mathrm{PL}}$ is canonically* *an affine space endowed with coordinates. Moreover, the triangle*

$$\begin{array}{ccc} Z_{\mathrm{PL}}^{1,\mathbb{G}_m} \times \pi_1^{\mathrm{un}}(Z) & \xrightarrow[\sim]{\Psi \times id} & (\operatorname{Spec} \mathbb{Q}[\{\Phi_\lambda^\rho\}_{\rho,\lambda}]) \times \pi_1^{\mathrm{un}}(Z) \\ & \searrow^{\mathcal{C}_{\mathrm{PL}}} \quad \swarrow_{\theta} & \\ & \pi_1^{\mathrm{PL}}(X) & \end{array}$$

commutes.

Proof. The injectivity of Ψ, as well as the commutativity of the diagram, follow directly from Proposition 3.2.9. The surjectivity of Ψ amounts to the statement that given any family a_λ^ρ of elements of a $\mathbb{Q}$-algebra R, the R-algebra homomorphism

$$c^\sharp : R[\log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \mathrm{Li}_2^{\mathfrak{u}}, \dots] \to R \otimes A(Z)$$

given by

$$\log^{\mathfrak{u}} \mapsto \sum_{\tau \in \Sigma_{-1}} a_{e_0}^\tau f_\tau$$

and

$$\mathrm{Li}_n^{\mathfrak{u}} \mapsto \sum_{\substack{\tau_1,\dots,\tau_r \in \Sigma_{-1} \\ \sigma \in \Sigma_{r-n} \\ 0 \le r \le n-1}} a_{e_0}^{\tau_1} \cdots a_{e_0}^{\tau_r} a^\sigma_{\underbrace{e_1 e_0 \cdots e_0}_{n-r}} f_{\sigma\tau_1\cdots\tau_r}$$

is compatible with the coproduct. For $\log^{\mathfrak{u}}$, this is because both sides are primitive. For $\mathrm{Li}_n^{\mathfrak{u}}$, this follows by the computation of (3.5)–(3.7) and the fact that the terms for which $r = 0$ are primitive.

□

Variant in Bounded Weight

Let n be a positive integer. Then there is a natural isomorphism $Z^{1,\mathbb{G}_m}_{\mathrm{PL},n} \xrightarrow{\Psi_n} (\operatorname{Spec}\mathbb{Q}[\{\Phi^\rho_\lambda\}_{\rho,|\lambda|\le n}])$ such that the square

$$\begin{array}{ccc} Z^{1,\mathbb{G}_m}_{\mathrm{PL}} & \xrightarrow[\sim]{\Psi} & \operatorname{Spec}\mathbb{Q}[\{\Phi^\rho_\lambda\}_{\rho,\lambda}] \\ \downarrow & & \downarrow \\ Z^{1,\mathbb{G}_m}_{\mathrm{PL},n} & \xrightarrow[\sim]{\Psi_n} & \operatorname{Spec}\mathbb{Q}[\{\Phi^\rho_\lambda\}_{\rho,|\lambda|\le n}] \end{array}$$

commutes, where the vertical arrows are the natural projections.

3.2.4 Kummer and Period Maps in Coordinates

Given $z \in X(Z)$, we recall that, by definition,

$$\log^{\mathfrak{u}}(z) = \log^{\mathfrak{u}}(\kappa(z)),$$

$$\mathrm{Li}^{\mathfrak{u}}_n(z) = \mathrm{Li}^{\mathfrak{u}}_n(\kappa(z)).$$

We describe $\kappa_{\mathfrak{p}}$ in these coordinates. More precisely, for $z \in X(Z_{\mathfrak{p}})$, the value $\kappa_{\mathfrak{p}}$ is the element of $\Pi_{\mathrm{PL},n}(\mathbb{Q}_p)$ sending $\log^{\mathfrak{u}}$ to $\log^p(z)$ and $\mathrm{Li}^{\mathfrak{u}}_n$ to $\mathrm{Li}^p_n(z)$.

Finally, we describe $\mathrm{per}_{\mathfrak{p}}$ in these coordinates. More precisely, we have

$$\mathrm{per}_{\mathfrak{p}}(\log^{\mathfrak{u}}(z)) = \log^p(z),$$

$$\mathrm{per}_{\mathfrak{p}}(\mathrm{Li}^{\mathfrak{u}}_n(z)) = \mathrm{Li}^p_n(z),$$

which in particular expresses the commutativity of Kim's cutter.

3.3 Computations for $Z = \operatorname{Spec} \mathbb{Z}[1/S]$

3.3.1 Abstract Coordinates for $Z = \operatorname{Spec} \mathbb{Z}[1/S]$

We let ℓ denote a prime number. We want to fix free generators $(\tau_\ell)_{\ell \in S}$ and $(\sigma_{2n+1})_{n \geq 1}$ for $\mathfrak{n}(Z)$, with τ_ℓ in degree -1 and σ_{2n+1} in degree $-2n-1$. We first recall some notation from Section 3.2 of [DC15].

Letting $A_n = A(Z)_n$ denote the degree n part of $A(Z)$, and $A_{>0} = \bigoplus_{n=1}^{\infty} A_n$, we let $E_n = E_n(Z)$ denote the kernel of $\Delta' \restriction_{A_n}$ and $D_n = D_n(Z)$ the subspace of decomposable elements of degree n, i.e., the degree n elements in the image of

$$A_{>0} \otimes A_{>0} \xrightarrow{\text{mult}} A.$$

We let $P_n = P_n(Z)$ denote a vector subspace of A_n complementary to E_n and D_n. The symbols $\mathcal{E}_n$, $\mathcal{D}_n$, and $\mathcal{P}_n$ refer to bases of E_n, D_n, and P_n, respectively.

Proposition 3.3.1. *One may choose free generators $(\tau_\ell)_{\ell \in S}$ and $(\sigma_{2n+1})_{n \geq 1}$ for $\mathfrak{n}(Z)$, with τ_ℓ in degree -1 and σ_{2n+1} in degree $-2n-1$, such that $f_{\tau_\ell} = \log^{\mathfrak{u}}(\ell)$ and $f_{\sigma_{2n+1}} = \zeta^{\mathfrak{u}}(2n+1)$.*

Furthermore a choice of P_{2n+1} uniquely determines σ_{2n+1}.

Proof. A computation using Corollary 3.2.2 shows that $\log^{\mathfrak{u}}(\ell)$ and $\zeta^{\mathfrak{u}}(2n+1)$ are primitive elements of the Hopf algebra $A(Z)$ (or, in the terminology of [DC15], they lie in the space E_n of extensions). In fact, by our knowledge of the rational algebraic K-theory of Z, we know that E_n is one-dimensional when n is odd and zero-dimensional otherwise. Therefore, the elements $\log^{\mathfrak{u}}(\ell)$ and $\zeta^{\mathfrak{u}}(2n+1)$ must span the spaces E_n, and we take them as our $\mathcal{E}_n$.

By Proposition 3.2.3 of [DC15], for a choice of $\mathcal{E}_n$, $\mathcal{D}_n$, and $\mathcal{P}_n$, we get a set of generators for the Lie algebra, which are dual to the elements of $\mathcal{E}_n$. In fact, the part of the condition of being dual that depends on $\mathcal{D}_n$ and $\mathcal{P}_n$ is that the element of the Lie algebra pairs to zero with all of $\mathcal{D}_n \cup \mathcal{P}_n$, so it in fact depends only on P_n and D_n. The latter is uniquely determined, so a choice of P_n determines such a choice of generators. ∎

Remark 3.3.2. The choice of $\log^{\mathfrak{u}}(\ell)$ and $\zeta^{\mathfrak{u}}(2n+1)$ corresponds to choosing generators for the rational algebraic K-groups of Z. The arbitrariness in choosing P_n then corresponds precisely to the non-canonicity discussed toward the end of Section 3.1.2.

Give such a set of generators, we get an abstract basis for $A(Z)$. For each word w of half-weight $-n$ in the above generators, we have an element $w \in \mathcal{U}\mathfrak{n}(Z)$, and these form a basis of $\mathcal{U}\mathfrak{n}(Z)$. We let $(f_w)_w$ denote the dual basis for $A(Z)$. With the choices above, we have

$$f_{\tau_\ell} = \log^{\mathfrak{u}}(\ell)$$

$$f_{\sigma_{2n+1}} = \zeta^{\mathfrak{u}}(2n+1).$$

In order to find bases of P_n and verify relations between different $\mathrm{Li}_n^{\mathfrak{u}}(z)$'s, we need to apply the reduced coproduct to reduce the computation in degree n to the computation in degrees $m < n$. For this, we need the exact sequence

Proposition 3.3.3. *The sequence*

$$0 \to E_n \to A_n \xrightarrow{\Delta'} \bigoplus_{\substack{i+j=n \\ i,j\geq 1}} A_i \otimes A_j \xrightarrow{\Delta'\otimes\mathrm{id}-\mathrm{id}\otimes\Delta'} \bigoplus_{\substack{i+j+k=n \\ i,j,k\geq 1}} A_i \otimes A_j \otimes A_k$$

is exact, and $E_n = \mathrm{Ext}^1_{\mathbf{MT}(Z,\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(Z)_{\mathbb{Q}}$.

Proof. We have the reduced cobar complex

$$A_{>0} \xrightarrow{\Delta'} A_{>0} \otimes A_{>0} \xrightarrow{\Delta'\otimes\mathrm{id}-\mathrm{id}\otimes\Delta'} A_{>0} \otimes A_{>0} \otimes A_{>0} \xrightarrow{\Delta'\otimes\mathrm{id}\otimes\mathrm{id}-\mathrm{id}\otimes\Delta'\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{id}\otimes\Delta'} \cdots,$$

which is a complex of graded vector spaces. The proof of Lemma 3.8 of [Gon] shows that the kth cohomology of this complex is the graded vector space $\bigoplus_n \mathrm{Ext}^k_{\mathrm{GrComod}(A)}(\mathbb{Q}(0), \mathbb{Q}(n))$ (graded by n). The result then follows because the category of graded comodules over A is the same as the category of representations of $\pi_1^{\mathbf{MT}}(Z)$, or equivalently, the category $\mathbf{MT}(Z, \mathbb{Q})$, and we know that $\mathrm{Ext}^1_{\mathbf{MT}(Z,\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(Z)_{\mathbb{Q}}$.

from our knowledge of the Ext-groups in the category $\mathbf{MT}(Z, \mathbb{Q})$.

□

We recall Definitions 3.1.4 and 3.1.5. In those definitions, we let Δ'_n denote the restriction of Δ' to A_n. For $i+j=n$, we let $\Delta'_{i,j}$ denote the component of Δ'_n landing in $A_i \otimes A_j$.

The following corollary will be useful for computations in half-weight 4:

Corollary 3.3.4. *If K is totally real, then* $\ker(\Delta'_{1,2}) = \ker(\Delta'_3) = E_3$.

Proof. Let $\alpha \in A_3$. Since Δ'_1 is zero (by Remark 3.1.8), we have

$$\begin{aligned}(\Delta'_2 \otimes \mathrm{id} - \mathrm{id} \otimes \Delta'_2)(\Delta'_3(\alpha)) &= (\Delta'_2 \otimes \mathrm{id})(\Delta'_{2,1}(\alpha)) - (\mathrm{id} \otimes \Delta'_2)(\Delta'_{1,2}(\alpha)) \\ &= 0.\end{aligned}$$

As K is totally real, it has no complex places, so we have $K_3(Z)_{\mathbb{Q}} = 0$. Then by Proposition 3.3.3, we have that $E_2 = 0$. Therefore, Δ'_2 is injective, hence $\Delta'_2 \otimes \mathrm{id}$ is injective on $A_2 \otimes A_1$. By the displayed equation and the previous sentence, it follows that if $\Delta'_{1,2}(\alpha) = 0$, then $\Delta'_{2,1}(\alpha) = 0$ as well. Therefore, $\ker(\Delta'_{1,2}) = \ker(\Delta'_3) = E_3$. □

Coordinates on the Space of Cocycles for $Z = \operatorname{Spec} \mathbb{Z}[1/\ell]$

Relative to the chosen coordinates, we name coordinates for $Z^{1,\mathbb{G}_m}_{\mathrm{PL},n}$ when $S = \{\ell\}$. Specifically, we set

$$w_0 \coloneqq \phi^{\tau_\ell}_{e_0}$$

$$w_1 \coloneqq \phi^{\tau_\ell}_{e_1}$$

$$w_k \coloneqq \phi^{\sigma_{2k+1}}_{\underbrace{e_1 e_0 \cdots e_0}_{2k+1}} \quad 2 \le k \le n$$

in the notation of Proposition 3.2.9. In fact, for $k \ge 2$, this makes sense even when $|S| > 1$, and we use it in Section 3.3.3.

If $z \in X(Z)$, we write $w_k(z)$ for $w_k(\kappa(z))$. In this notation, $w_0(z) = \operatorname{ord}_\ell z$, and $w_1(z) = -\operatorname{ord}_\ell(1-z)$, both by Fact 3.2.3.

In this case, for $n \geq k$, we have

$$\mathcal{C}^{\#}_{\mathrm{PL},n}(\log^{\mathfrak{u}}) = \mathcal{C}^{\#}_{\mathrm{PL}}(\log^{\mathfrak{u}}) = w_0 f_\tau$$

$$\mathcal{C}^{\#}_{\mathrm{PL},n}(\mathrm{Li}^{\mathfrak{u}}_k) = \mathcal{C}^{\#}_{\mathrm{PL}}(\mathrm{Li}^{\mathfrak{u}}_k) = w_1 w_0^{k-1} f_\tau^k / k! + \sum_{i=2}^{\lceil \frac{k}{2} \rceil} w_i w_0^{k-2i+1} f_{\sigma_{2i-1}\tau^{k-2i+1}}. \tag{3.8}$$

3.3.2 The Geometric Step for $\mathbb{Z}[1/\ell]$ in Half-Weight 4

Coordinates for the Galois Group

Let $Z = \mathbb{Z}[1/\ell]$. Then $\mathfrak{n}(Z)_{\geq -4}$ is three-dimensional as a vector space, generated by $\tau = \tau_\ell$, $\sigma = \sigma_3$, and $[\sigma, \tau]$. As $P_3 = 0$ when $|S| = 1$, the elements τ, σ are already well-defined. We choose f_τ, f_σ, and $f_{\sigma\tau}$ as a set of affine coordinates on $\pi_1^{\mathrm{un}}(Z)_{\geq -4}$.

Coordinates for the Selmer Variety

In this case, the only nonzero coordinates are w_0, w_1, and w_2.

The Universal Cocycle Evaluation Morphism

We now write the morphism $\mathcal{C}_{\mathrm{PL},4}$ in these coordinates. We have $\mathcal{C}^{\#}_{\mathrm{PL},4}(f_i) = f_i$ for $i = \tau, \sigma, \sigma\tau$.

Using Equation (3.8), we find that $\mathcal{C}^{\#}_{\mathrm{PL},4}(\log^{\mathfrak{u}}) = w_0 f_\tau$, $\mathcal{C}^{\#}_{\mathrm{PL},4}(\mathrm{Li}^{\mathfrak{u}}_1) = w_1 f_\tau$, and

$$\mathcal{C}^{\#}_{\mathrm{PL},4}(\mathrm{Li}^{\mathfrak{u}}_2) = w_0 w_1 f_\tau^2 / 2.$$

At this point, we already see that the function on $\Pi_{\mathrm{PL},4} \times \pi_1^{\mathrm{un}}(Z)_{\geq -4}$ given by

$$\mathrm{Li}^{\mathfrak{u}}_2 - \frac{1}{2} \log^{\mathfrak{u}} \mathrm{Li}^{\mathfrak{u}}_1$$

vanishes on the image of $\mathcal{C}_{\mathrm{PL},4}$, i.e., is in $\mathcal{I}^{Z}_{\mathrm{PL},4}$.

Again, by Equation (3.8), we have

$$\mathcal{C}^{\#}_{\mathrm{PL},4}(\mathrm{Li}^{\mathfrak{u}}_3) = w_1 w_0^2 f_\tau^3/6 + w_2 f_\sigma,$$

$$\mathcal{C}^{\#}_{\mathrm{PL},4}(\mathrm{Li}^{\mathfrak{u}}_4) = w_1 w_0^3 f_\tau^4/24 + w_0 w_2 f_{\sigma\tau}.$$

To construct a second element of $\mathcal{I}^{Z}_{\mathrm{PL},4}$, we first eliminate w_2 by considering

$$\begin{aligned}\mathcal{C}^{\#}_{\mathrm{PL},4}(f_\sigma f_\tau \mathrm{Li}^{\mathfrak{u}}_4 - f_{\sigma\tau}\log^{\mathfrak{u}}\mathrm{Li}^{\mathfrak{u}}_3) &= w_1 w_0^3 f_\sigma f_\tau^5/24 - f_{\sigma\tau} w_1 w_0^3 f_\tau^4/6 \\ &= \frac{w_1 w_0^3 f_\tau^4}{24}(f_\sigma f_\tau - 4 f_{\sigma\tau}) \\ &= \frac{\mathcal{C}^{\#}_{\mathrm{PL},4}((\log^{\mathfrak{u}})^3 \mathrm{Li}^{\mathfrak{u}}_1)}{24}(f_\sigma f_\tau - 4 f_{\sigma\tau})\end{aligned}$$

It follows that

$$\mathcal{C}^{\#}_{\mathrm{PL},4}\left(f_\sigma f_\tau \mathrm{Li}^{\mathfrak{u}}_4 - f_{\sigma\tau}\log^{\mathfrak{u}}\mathrm{Li}^{\mathfrak{u}}_3 - \frac{(\log^{\mathfrak{u}})^3\mathrm{Li}^{\mathfrak{u}}_1}{24}(f_\sigma f_\tau - 4 f_{\sigma\tau})\right) = 0$$

In other words,

Proposition 3.3.5. *The two functions*

$$\mathrm{Li}^{\mathfrak{u}}_2 - \frac{1}{2}\log^{\mathfrak{u}}\mathrm{Li}^{\mathfrak{u}}_1$$

$$f_\sigma f_\tau \mathrm{Li}^{\mathfrak{u}}_4 - f_{\sigma\tau}\log^{\mathfrak{u}}\mathrm{Li}^{\mathfrak{u}}_3 - \frac{(\log^{\mathfrak{u}})^3\mathrm{Li}^{\mathfrak{u}}_1}{24}(f_\sigma f_\tau - 4 f_{\sigma\tau})$$

on $\Pi_{\mathrm{PL},4} \times \pi_1^{\mathrm{un}}(Z)_{\geq -4}$ *are in* $\mathcal{I}^{Z}_{\mathrm{PL},4}$ *for* $Z = \operatorname{Spec}\mathbb{Z}[1/\ell]$.

3.3.3 Coordinates on the Galois Group for $Z = \mathbb{Z}[1/\ell]$

In order to evaluate the functions of Proposition 3.3.5, we need to choose a prime $p \neq \ell$ and interpret f_τ, f_σ, and $f_{\sigma\tau}$ in such a way that we can take (approximate) their p-adic periods. Essentially, this means writing them as special values of polylogarithms. As mentioned in

Proposition 3.3.1, we have chosen the first two to correspond to $\log^{\mathfrak{u}}(\ell)$ and $\zeta^{\mathfrak{u}}(3)$, respectively. It remains to understand $f_{\sigma\tau}$.

This will depend on ℓ. We start with the case $\ell = 2$, in an effort to re-derive Theorem 1.16 of [DCW16].

The Case $Z = \mathbb{Z}[1/2]$

We let $A = A(Z)$ as usual. We compute using Corollary 3.2.2 and Fact 3.2.3 that

$$\Delta'_{1,2}(\mathrm{Li}_3^{\mathfrak{u}}(1/2)) = \mathrm{Li}_1^{\mathfrak{u}}(1/2) \otimes (\log^{\mathfrak{u}}(1/2))^2/2 = -\log^{\mathfrak{u}}(1/2) \otimes (\log^{\mathfrak{u}}(2))^2/2 = f_\tau \otimes f_\tau^2/2.$$

Using the fact that Δ is a ring homomorphism, we note that $\Delta'_{1,2}(\log^{\mathfrak{u}}(2)^3) = 3\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(2)^2$. Therefore, Corollary 3.3.4 implies that

$$\mathrm{Li}_3^{\mathfrak{u}}(1/2) - \frac{(\log^{\mathfrak{u}}(2))^3}{6}$$

is in E_3. As $E_3 = K_5(Z)_{\mathbb{Q}}$ is one-dimensional, this is a rational multiple of $\zeta^{\mathfrak{u}}(3)$.

The identity in the appendix to [DCW16] says that

$$\mathrm{Li}_3^{\mathfrak{u}}(1/2) = \frac{(\log^{\mathfrak{u}}(2))^3}{6} + \frac{7}{8}\zeta^{\mathfrak{u}}(3). \tag{3.9}$$

By Definition 3.2.8, we have

$$\mathcal{C}^{\#}_{\mathrm{PL},4}(\mathrm{Li}_4^{\mathfrak{u}}) = w_1 w_0^3 f_\tau^4/24 + w_0 w_2 f_{\sigma\tau}.$$

It is easy to check using the formulas at the end of Section 3.3.1 that $w_0(\kappa(1/2)) = -1$ and $w_1(\kappa(1/2)) = 1$, and (3.9) together with Definition 3.2.8 implies that $w_2(\kappa(1/2)) = 7/8$.

We therefore get

$$\mathrm{Li}_4^{\mathfrak{u}}(1/2) = -(\log^{\mathfrak{u}}(2))^4/24 - \frac{7}{8} f_{\sigma\tau}.$$

It follows that

$$f_{\sigma\tau} = -\frac{8}{7}\left(\frac{\log^{\mathfrak{u}}(2)^4}{24} + \mathrm{Li}_4^{\mathfrak{u}}(1/2)\right).$$

Choosing $P_3(\operatorname{Spec}\mathbb{Z}[1/6])$

To deal with the case $Z = \operatorname{Spec}\mathbb{Z}[1/3]$, we have to consider $Z' = \operatorname{Spec}\mathbb{Z}[1/6]$, as Z is not saturated. In this case, the Lie algebra $\mathfrak{n}(Z')$ is generated by $\tau_2, \tau = \tau_3$, and $\sigma = \sigma_3$.

In this case, $P_1(Z')$ is zero. As $P_3(Z')$ is nonzero, there is some choice in the definition of σ as an element of $\mathfrak{n}(Z')$. We therefore seek to choose a set $\mathcal{P}_3(Z')$.

While $A(Z')_1$ has basis $\{\log^{\mathfrak{u}}(2), \log^{\mathfrak{u}}(3)\}$, we must first find a basis of $A(Z')_2$. As $E_2(Z') = 0$, the map Δ_2' is injective. We may take $\mathcal{D}_2(Z') = \{\log^{\mathfrak{u}}(2)^2, \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3), \log^{\mathfrak{u}}(3)^2\}$. Then

$$\Delta'(\mathrm{Li}_2^{\mathfrak{u}}(-2)) = \mathrm{Li}_1^{\mathfrak{u}}(-2) \otimes \log^{\mathfrak{u}}(-2) = -\log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(2),$$

which is independent from

$$\{\Delta' \log^{\mathfrak{u}}(2)^2, \Delta' \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3), \Delta' \log^{\mathfrak{u}}(3)^2\}$$

in $A_1(Z') \otimes A_1(Z')$ (as seen by checking via the basis of $A_1(Z') \otimes A_1(Z')$ induced by the basis $\{\log^{\mathfrak{u}}(2), \log^{\mathfrak{u}}(3)\}$ of $A_1(Z')$). In fact, $\mathrm{Li}_2^{\mathfrak{u}}(-2) = -f_{\tau\tau_2}$, as they have the same reduced coproduct.

This gives us a basis $\{\log^{\mathfrak{u}}(2)^2, \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3), \log^{\mathfrak{u}}(3)^2, \mathrm{Li}_2^{\mathfrak{u}}(-2)\}$ of $A_2(Z')$, so by taking all degree 3 products of basis elements of $A_1(Z')$ and $A_2(Z')$, we may take

$$\mathcal{D}_3(Z') = \{\log^{\mathfrak{u}}(2)^3, \log^{\mathfrak{u}}(3)^3, \log^{\mathfrak{u}}(2)^2\log^{\mathfrak{u}}(3), \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3)^2, \log^{\mathfrak{u}}(2)\,\mathrm{Li}_2^{\mathfrak{u}}(-2), \log^{\mathfrak{u}}(3)\,\mathrm{Li}_2^{\mathfrak{u}}(-2)\}.$$

We would like to show that

$$\mathcal{B} := \{\log^{\mathfrak{u}}(2)^3, \log^{\mathfrak{u}}(3)^3, \log^{\mathfrak{u}}(2)^2\log^{\mathfrak{u}}(3), \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3)^2, \log^{\mathfrak{u}}(2)\,\mathrm{Li}_2^{\mathfrak{u}}(-2), \log^{\mathfrak{u}}(3)\,\mathrm{Li}_2^{\mathfrak{u}}(-2), \mathrm{Li}_3^{\mathfrak{u}}(-2), \mathrm{Li}_3^{\mathfrak{u}}(3)\}$$

is a basis of $A_3(Z')/E_3(Z')$, which would imply that $\{\mathrm{Li}_3^{\mathfrak{u}}(-2), \mathrm{Li}_3^{\mathfrak{u}}(3)\}$ can be taken as $\mathcal{P}_3(Z')$.

To do this, we apply $\Delta'_{1,2}$ to each element of $\mathcal{B}$ and expand in the basis

$$\{\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(2)^2, \log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3), \log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(3)^2, \log^{\mathfrak{u}}(2) \otimes \mathrm{Li}_2(-2),$$
$$\log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(2)^2, \log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3), \log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(3)^2, \log^{\mathfrak{u}}(3) \otimes \mathrm{Li}_2(-2)\}$$

of $A_1(Z') \otimes A_2(Z')$. Using Corollary 3.2.2, this produces the matrix

$\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(2)^2$	3	0	0	0	0	0	0	0
$\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3)$	0	0	2	0	0	0	0	0
$\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(3)^2$	0	0	0	1	0	0	0	$-1/2$
$\log^{\mathfrak{u}}(2) \otimes \mathrm{Li}_2(-2)$	0	0	0	0	1	0	0	0
$\log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(2)^2$	0	0	1	0	-1	0	$-1/2$	0
$\log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(2)\log^{\mathfrak{u}}(3)$	0	0	0	2	0	-1	0	0
$\log^{\mathfrak{u}}(3) \otimes \log^{\mathfrak{u}}(3)^2$	0	3	0	0	0	0	0	0
$\log^{\mathfrak{u}}(3) \otimes \mathrm{Li}_2(-2)$	0	0	0	0	0	1	0	0

,

where the columns correspond to the element of $\mathcal{B}$. This matrix has determinant 9, which by Corollary 3.3.4 and the fact that $A_3(Z')/E_3(Z')$ is eight-dimensional (by our knowledge of the shuffle algebra $A(Z')$) implies that $\mathcal{B}$ is in fact a basis of $A_3(Z')/E_3(Z')$. We choose $P_3(Z')$ to be the space generated $\mathrm{Li}^{\mathfrak{u}}_3(-2)$ and $\mathrm{Li}^{\mathfrak{u}}_3(3)$, and σ such that it pairs to 0 with this choice of $P_3(Z')$.

The Case $Z = \mathbb{Z}[1/3]$

Armed with our choice of $\sigma \in \mathfrak{n}(Z')$, we seek to write $f_{\sigma\tau} \in A(Z)$ as an explicit combination of motivic polylogarithms. With our choice of σ, we have $w_2(-2) = w_2(3) = 0$ because $\mathrm{Li}^{\mathfrak{u}}_3(-2)$ and $\mathrm{Li}^{\mathfrak{u}}_3(-3)$ are elements of $\mathcal{P}_3$.

From the matrix above, we see that

$$\Delta'_{1,2}\,\mathrm{Li}^{\mathfrak{u}}_3(3) = -\frac{\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(3)^2}{2}.$$

We also compute

$$\Delta'_{1,2}\operatorname{Li}_3^{\mathfrak{u}}(9) = \operatorname{Li}_1^{\mathfrak{u}}(9) \otimes \frac{\log^{\mathfrak{u}}(9)^2}{2} = -6\log^{\mathfrak{u}}(2) \otimes \log^{\mathfrak{u}}(3)^2,$$

so $\operatorname{Li}_3^{\mathfrak{u}}(9) = 12\operatorname{Li}_3^{\mathfrak{u}}(3)$ is in $E_3(Z')$, hence a rational multiple of $\zeta^{\mathfrak{u}}(3)$ (because $E_3(Z') = K_5(Z')_{\mathbb{Q}}$ is one-dimensional). In fact, since $w_2(3) = 0$, this rational number is the f_σ-coordinate of $\operatorname{Li}_3^{\mathfrak{u}}(9)$ in the basis $(f_w)_w$ of $A(Z')$, hence it equals $w_2(9)$ because $f_\sigma = \zeta^{\mathfrak{u}}(3)$.

Numerical computation using the code [DCC] in the 5-adic and 7-adic realizations suggests that

$$w_2(9) = \frac{\operatorname{Li}_3^{\mathfrak{u}}(9) - 12\operatorname{Li}_3^{\mathfrak{u}}(3)}{\zeta^{\mathfrak{u}}(3)} = -\frac{26}{3}.$$

We may now compute using Definition 3.2.8 that

$$\operatorname{Li}_4^{\mathfrak{u}}(3) = w_2(3)\phi_{e_0}^{\tau}(3) f_{\sigma\tau} + \phi_{e_1}^{\tau_2}(3)\phi_{e_0}^{\tau}(3)^3 f_{\tau_2\tau\tau\tau} = -f_{\tau_2\tau\tau\tau}$$

and

$$\operatorname{Li}_4^{\mathfrak{u}}(9) = w_2(9)\phi_{e_0}^{\tau}(9) f_{\sigma\tau} + \phi_{e_1}^{\tau_2}(9)\phi_{e_0}^{\tau}(9)^3 f_{\tau_2\tau\tau\tau} = 2w_2(9) f_{\sigma\tau} - 24 f_{\tau_2\tau\tau\tau}$$

This implies that

$$-\frac{12}{w_2(9)}\operatorname{Li}_4^{\mathfrak{u}}(3) + (2w_2(9))^{-1}\operatorname{Li}_4^{\mathfrak{u}}(9) = f_{\sigma\tau}.$$

We note that this corresponds precisely to $f_{\sigma\tau}$ in the image of $\mathcal{O}(\pi_1^{\mathrm{un}}(Z)) \hookrightarrow \mathcal{O}(\pi_1^{\mathrm{un}}(Z'))$.

Plugging this into the second equation of Proposition 3.3.5, we find

Theorem 3.3.6. *The element*

$$\zeta^{\mathfrak{u}}(3)\log^{\mathfrak{u}}(3)\operatorname{Li}_4^{\mathfrak{u}} - \left(-\frac{12}{w_2(9)}\operatorname{Li}_4^{\mathfrak{u}}(3) + (2w_2(9))^{-1}\operatorname{Li}_4^{\mathfrak{u}}(9)\right)\log^{\mathfrak{u}}\operatorname{Li}_3^{\mathfrak{u}}$$
$$-\frac{(\log^{\mathfrak{u}})^3\operatorname{Li}_1^{\mathfrak{u}}}{24}\left(\zeta^{\mathfrak{u}}(3)\log^{\mathfrak{u}}(3) - 4\left(-\frac{12}{w_2(9)}\operatorname{Li}_4^{\mathfrak{u}}(3) + (2w_2(9))^{-1}\operatorname{Li}_4^{\mathfrak{u}}(9)\right)\right)$$

of $\mathcal{O}(\Pi_{\mathrm{PL},4} \times \pi_1^{\mathrm{un}}(Z))$ *is in* $\mathcal{I}_{\mathrm{PL},4}^{Z}$ *for* $Z = \operatorname{Spec}\mathbb{Z}[1/3]$, *where* $w_2(9)$ *is a number* p*-adically close to* $-\frac{26}{3}$ *for* $p = 5, 7$.

For $p \neq 2, 3$, the corresponding Coleman function is

$$\zeta^p(3)\log^p(3)\mathrm{Li}_4^p(z) - \left(-\frac{12}{w_2(9)}\mathrm{Li}_4^p(3) + (2w_2(9))^{-1}\mathrm{Li}_4^p(9)\right)\log^p(z)\mathrm{Li}_3^p(z) \qquad (3.10)$$
$$-\frac{(\log^p(z))^3\mathrm{Li}_1^p(z)}{24}\left(\zeta^p(3)\log^p(3) - 4\left(-\frac{12}{w_2(9)}\mathrm{Li}_4^p(3) + (2w_2(9))^{-1}\mathrm{Li}_4^p(9)\right)\right).$$

3.3.4 The Chabauty-Kim Locus for $\mathbb{Z}[1/3]$ in Half-Weight 4

As noted in Section 8.2 of [BDCKW], the function $\mathrm{Li}_2^p(z) - \frac{1}{2}\log^p(z)\log^p(1-z)$ has the zero set $\{2, \frac{1}{2}, -1\}$ for $p = 5, 7$. By numerical evaluation of (3.10) at 2, $\frac{1}{2}$, and -1, we conclude that $2, \frac{1}{2} \notin X(Z_{\mathfrak{p}})_{\mathrm{PL},4}$. It follows that:

Theorem 3.3.7. *For $Z = \operatorname{Spec}\mathbb{Z}[1/3]$, we have $X(Z_{\mathfrak{p}})_{\mathrm{PL},4} \subseteq \{-1\}$ for $p = 5, 7$.*

3.4 Answer to Question 3.1.24 and S_3-Symmetrization

3.4.1 Answer to Question 3.1.24

We fix a set of generators as in Section 3.3.1.

We first need the following lemma:

Lemma 3.4.1. *We have $\mathrm{Li}_k^p(-1) = 0$ for $k \geq 2$ even.*

Proof. This follows from the identity $2^{-k}\mathrm{Li}_k^p(z^2) = \mathrm{Li}_k^p(z) + \mathrm{Li}_k^p(-z)$, which is Proposition 6.1 of [Col82] for $m = 1$. Indeed, setting $z = 1$ in the identity shows $\mathrm{Li}_k^p(-1) = (2^{-k} - 1)\mathrm{Li}_k^p(1)$, and since $\mathrm{Li}_k^p(1) = \zeta^p(k)$, which is 0 for k even, we have $\mathrm{Li}_k^p(-1) = 0$. □

Theorem 3.4.2. *For any prime ℓ and positive integer n, we have*

$$-1 \in X(Z_{\mathfrak{p}})_{\mathrm{PL},n},$$

where $Z = \operatorname{Spec}\mathbb{Z}[1/\ell]$.

In particular, for ℓ odd, Question 3.1.24 has a negative answer.

Proof. We use the coordinates of Section 3.3.1. We also write $\tau = \tau_\ell$. We have $f_\tau = \log^{\mathfrak{u}}(\ell)$.

To prove the theorem, we produce an element c_{-1} of $Z^{1,\mathbb{G}_m}_{\mathrm{PL},n}(\mathcal{K})$ whose image α_{-1} under $\mathcal{C}_{\mathrm{PL},n}$ lies in $\Pi_{\mathrm{PL},n}(A(Z)) \subseteq \Pi_{\mathrm{PL},n}(\mathcal{K})$ and satisfies $\mathrm{per}_{\mathfrak{p}}(\alpha_{-1}) = \kappa_{\mathfrak{p}}(-1) \in \Pi_{\mathrm{PL},n}(\mathbb{Q}_p)$. Since any element of $\mathcal{O}(\Pi_{\mathrm{PL},n} \times \pi_1^{\mathrm{un}}(Z)) \cap \mathcal{I}^Z_{\mathrm{PL},n}$ vanishes on the image of $\mathcal{C}^{\mathcal{K}}_{\mathrm{PL},n}$, it vanishes on α_{-1} and therefore on $\kappa_{\mathfrak{p}}(-1)$, which proves the theorem.

We define c_{-1} by setting

$$w_0(c_{-1}) \coloneqq 0,$$
$$w_1(c_{-1}) \coloneqq \frac{\mathrm{Li}^{\mathfrak{u}}_1(-1)}{\log^{\mathfrak{u}}(\ell)},$$
$$w_k(c_{-1}) \coloneqq \frac{\mathrm{Li}^{\mathfrak{u}}_{2k-1}(-1)}{\zeta^{\mathfrak{u}}(2k-1)} \quad k = 2, \cdots, n.$$

Setting $\alpha_{-1} \coloneqq \mathcal{C}_{\mathrm{PL},n}(c_{-1})$, we now compute $\log^{\mathfrak{u}}(\alpha_{-1})$ and $\mathrm{Li}^{\mathfrak{u}}_k(\alpha_{-1})$ for $1 \le k \le n$.

We have $\mathcal{C}^{\#}_{\mathrm{PL},n}(\log^{\mathfrak{u}}) = w_0 f_\tau$, so

$$\log^{\mathfrak{u}}(\alpha_{-1}) = w_0(c_{-1}) f_\tau = 0.$$

We have $\mathcal{C}^{\#}_{\mathrm{PL},n}(\mathrm{Li}^{\mathfrak{u}}_1) = w_1 f_\tau$, so

$$\mathrm{Li}^{\mathfrak{u}}_1(\alpha_{-1}) = w_1(c_{-1}) f_\tau = \frac{\mathrm{Li}^{\mathfrak{u}}_1(-1)}{\log^{\mathfrak{u}}(\ell)} \log^{\mathfrak{u}}(\ell) = \mathrm{Li}^{\mathfrak{u}}_1(-1).$$

For $k \geq 2$ even, we have

$$\begin{aligned} \mathcal{C}^{\#}_{\mathrm{PL},n}(\mathrm{Li}^{\mathfrak{u}}_k) &= w_1 w_0^{k-1} f_\tau^k / k! + \sum_{i=2}^{\lceil \frac{k}{2} \rceil} w_i w_0^{k-2i+1} f_{\sigma_{2i-1}\tau^{k-2i+1}} \\ &= w_0 \left(w_1 w_0^{k-1} f_\tau^k / k! + \sum_{i=2}^{\lceil \frac{k}{2} \rceil} w_i w_0^{k-2i} f_{\sigma_{2i-1}\tau^{k-2i+1}} \right), \end{aligned}$$

and since $w_0(c_{-1}) = 0$, we have $\mathrm{Li}^{\mathfrak{u}}_k(\alpha_{-1}) = 0$.

For $k \geq 3$ odd, we have

$$\begin{aligned}
\mathcal{C}^{\#}_{\mathrm{PL},n}(\mathrm{Li}^{\mathfrak{u}}_k) &= w_1 w_0^{k-1} f_\tau^k/k! + \sum_{i=2}^{\lceil \frac{k}{2} \rceil} w_i w_0^{k-2i+1} f_{\sigma_{2i-1}\tau^{k-2i+1}} \\
&= w_0 \left(w_1 w_0^{k-1} f_\tau^k/k! + \sum_{i=2}^{\lceil \frac{k}{2} \rceil - 1} w_i w_0^{k-2i} f_{\sigma_{2i-1}\tau^{k-2i+1}} \right) + w_{\frac{k+1}{2}} f_{\sigma_k},
\end{aligned}$$

so by $w_0(c_{-1}) = 0$, we have

$$\begin{aligned}
\mathrm{Li}^{\mathfrak{u}}_k(\alpha_{-1}) &= w_{\frac{k+1}{2}}(-1) f_{\sigma_k} \\
&= \frac{\mathrm{Li}^{\mathfrak{u}}_k(-1)}{\zeta^{\mathfrak{u}}(k)} \zeta^{\mathfrak{u}}(k) \\
&= \mathrm{Li}^{\mathfrak{u}}_k(-1).
\end{aligned}$$

We have thus shown that $\log^{\mathfrak{u}}(\alpha_{-1}) = 0$, $\mathrm{Li}^{\mathfrak{u}}_k(\alpha_{-1}) = 0$ for k even, and $\mathrm{Li}^{\mathfrak{u}}_k(\alpha_{-1}) = \mathrm{Li}^{\mathfrak{u}}_k(-1)$ for k odd. This shows that $\alpha_{-1} \in \Pi_{\mathrm{PL},n}(A(Z))$.

The key fact is that $\mathrm{Li}^p_k(-1) = 0$ for $k \geq 2$ even. This follows from the identity $2^{-k}\mathrm{Li}^p_k(z^2) = \mathrm{Li}^p_k(z) + \mathrm{Li}^p_k(-z)$, which is Proposition 6.1 of [Col82] for $m = 1$. Indeed, setting $z = 1$ in the identity shows $\mathrm{Li}^p_k(-1) = (2^{-k} - 1)\mathrm{Li}^p_k(1)$, and since $\mathrm{Li}^p_k(1) = \zeta^p(k)$, which is 0 for k even, we have $\mathrm{Li}^p_k(-1) = 0$.

By Lemma 3.4.1, $\mathrm{Li}^p_k(-1) = 0$ for $k \geq 2$ even. Since we also have $\log^p(-1) = 0$, we get that $\mathrm{Li}^p_k(-1) = \log^p(-1)$ for $k \geq 2$. This implies that $\mathrm{per}_{\mathfrak{p}}(\alpha_{-1}) = \kappa_{\mathfrak{p}}(-1)$, as desired. □

3.4.2 S_3-Symmetrization

We recall our strengthening of Conjecture 3.1.23. The S_3-action on X induces an $\pi_1^{\mathbf{MT}}(Z)$-equivariant action on $\pi_1^{\mathrm{un}}(X)$, and for a quotient Π and $\sigma \in S_3$, we may refer to $\sigma(\Pi)$. We then write

$$X(Z_{\mathfrak{p}})^{S_3}_{\mathrm{PL},n} := \bigcap_{\sigma \in S_3} \sigma(X(Z_{\mathfrak{p}})_{\mathrm{PL},n}) = \bigcap_{\sigma \in S_3} X(Z_{\mathfrak{p}})_{\sigma(\Pi)}.$$

Our symmetrized conjecture is that

Conjecture 3.4.3. *$X(Z) = X(Z_\mathfrak{p})^{S_3}_{\mathrm{PL},n}$ for sufficiently large n.*

Verification for $Z = \operatorname{Spec} \mathbb{Z}[1/3]$

We can use our computations in Section 3.3 to show verify a case of this conjecture:

Theorem 3.4.4. *For $Z = \operatorname{Spec} \mathbb{Z}[1/3]$ and $p = 5, 7$, Conjecture 3.4.3 (and hence Conjecture 3.1.23) holds (with $n = 4$).*

Proof. By Theorem 3.3.7, $X(Z_\mathfrak{p})_{\mathrm{PL},4} \subseteq \{-1\}$. But -1 is not fixed by the action of S_3, so

$$X(Z_\mathfrak{p})^{S_3}_{\mathrm{PL},4} \coloneqq \bigcap_{\sigma \in S_3} \sigma(X(Z_\mathfrak{p})_{\mathrm{PL},4}) = \emptyset$$

for $p = 5, 7$. In particular, Conjecture 3.4.3 and hence Conjecture 3.1.23 holds in these cases. □

Part II

Obstructions to the Local-Global Principle

Chapter 4

Introduction and Setup

4.1 Introduction

Given a variety X over a global field k, a major problem is to decide whether $X(k) = \emptyset$. By [Poo09], it suffices to consider the case that X is smooth, projective, and geometrically integral. As a first approximation one can consider the set $X(\mathbb{A}_k) \supset X(k)$, where $\mathbb{A}_k$ is the adele ring of k. It is a classical theorem of Minkowski and Hasse that if X is a quadric, then $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$. When a variety X satisfies this implication, we say that it satisfies the Hasse principle, or local-global principle. In the 1940s, Lind and Reichardt ([Lin40] [Rei42]) gave examples of genus 1 curves that do not satisfy the Hasse principle. More counterexamples to the Hasse principle were given throughout the years, until in 1971 Manin [Man71] described a general obstruction to the Hasse principle that explained many of the counterexamples to the Hasse principle that were known at the time. The obstruction (known as the Brauer-Manin obstruction) is defined by considering a certain set $X(\mathbb{A}_k)^{\mathrm{Br}}$, such that $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. If X is a counterexample to the Hasse principle, we say that it is accounted for or explained by the Brauer-Manin obstruction if $\emptyset = X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k) \neq \emptyset$.

In 1999, Skorobogatov ([Sko99]) defined a refinement of the Brauer-Manin obstruction known as the étale Brauer-Manin obstruction and used it to produce an example of a variety X such that $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$ but $X(k) = \emptyset$. Namely, he described a set $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ for which

$X(k) \subset X(\mathbb{A}_k)^{\text{ét,Br}} \subset X(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k)$ and found a variety X such that $\emptyset = X(\mathbb{A}_k)^{\text{ét,Br}} \subset X(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$.

In his paper [Poo10], Poonen constructed the first example of a variety X such that $\emptyset = X(k) \subset X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$. However, Poonen's method of showing that $X(k) = \emptyset$ relies on the details of his specific construction and is not explained by a new finer obstruction. While it was hoped ([Pál10]) that an étale homotopy obstruction might solve the problem, it was shown ([HS13]) that this provides nothing new.

Therefore, one wonders if Poonen's counterexample can be accounted for by an additional refinement of $X(\mathbb{A}_k)^{\text{ét,Br}}$. Namely, can one give a general definition of a set

$$X(k) \subset X(\mathbb{A}_k)^{\text{new}} \subset X(\mathbb{A}_k)^{\text{ét,Br}}$$

such that Poonen's variety X satisfies $X(\mathbb{A}_k)^{\text{new}} = \emptyset$.

In this paper, we provide such a refinement and prove that it is necessary and sufficient over many number fields. The obstruction is essentially given by applying the finite abelian descent obstruction to open covers of X or decompositions of X as a disjoint union of locally closed subvarieties. We shall make this statement precise below. As an example of our results, we can prove the following:

Corollary 4.1.1 (of Theorem 4.3.3)**.** *Let $X/\mathbb{Q}$ be a variety for which $X(\mathbb{Q}) = \emptyset$. Then there is a Zariski open cover $X = \bigcup_i U_i$ such that $U_i(\mathbb{A}_k)^{\text{Br}} = \emptyset$ for all i.*

For a summary of all of the variants of this result, the reader may proceed to Section 4.3.

Chapter 5 contains the main proofs of these results, one method unconditional and one method assuming the section conjecture in anabelian geometry.

In Chapter 6, we examine the section conjecture more closely, in particular what happens to cuspidal sections in fibrations. Our results are of independent interest for the section conjecture, but they also allow us to reprove some of the results in Section 5 via a slightly different method (Corollary 6.3.16). In this chapter, we also derive an important result about the behavior of the étale-Brauer obstruction in fibrations (Theorem 6.2.10) that will be useful in Chapter 7. Finally, as our discussion touches on the concept of cuspidal sections for higher dimensional varieties, we explore this topic in more detail in Section 5.3.1.

Finally, in Chapter 7, we explicitly find a Zariski open cover of the example of [Poo10] and prove that its pieces have empty étale-Brauer set with the aid of one of our results on homotopy fixed points in fibrations (Theorem 6.2.10). In fact, we introduce a few variants of this result, depending on different hypotheses. Motivated by one of these examples, we also introduce the notion of quasi-torsors (Definition 7.2.1), which leads us to propose a new approach to computing obstructions for open subvarieties, known as the ramified étale-Brauer obstruction (Definition 7.2.2).

4.1.1 Notation and Conventions

Whenever we speak of a field k, we implicitly fix a separable closure k_s throughout, which produces a geometric point $\mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$ of k. We write X/k to mean that X is a scheme over $\mathrm{Spec}\, k$, and we denote by X^s the base change of X to k_s, following [Poo17]. We let G_k denote $\mathrm{Gal}(k_s/k)$. If X is a scheme, $\mathcal{O}(X)$ denotes $\mathcal{O}_X(X)$.

All cohomology is taken in the étale topology unless otherwise stated, and fundamental group always denotes an étale fundamental group. The same is true for higher homotopy groups, once they are defined in Definition 6.1.12. If k is a field, we write $H^*(k, -)$ for Galois cohomology of the field k.

For a global field k, we let $\mathbb{A}_k$ denote the ring of adeles of k. For a place v of k, we let k_v denote the completion of k at v, $\mathcal{O}_v$ the ring of integers in k_v, $\mathfrak{m}_v$ the maximal ideal in $\mathcal{O}_v$, π_v a generator of $\mathfrak{m}_v$, $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$, and q_v the size of $\mathbb{F}_v$. If S is a subset of the set of places of k, we let $\mathbb{A}_{k,S} = \prod'_{v \in S}(k_v, \mathcal{O}_v)$. This must not be confused with $\mathbb{A}_k^S = \prod'_{v \notin S}(k_v, \mathcal{O}_v)$. We let $\mathcal{O}_{k,S}$ denote the S-integers of k, i.e.,, in which primes in S are inverted. In general, we replace S by the letter “f” when S is the set of finite places of k. We also use $\mathbb{A}_k^f$ to denote $\mathbb{A}_{k,S}$ when S is the set of finite places, and we believe this does not lead to any confusion.

We use loc to denote the map from the version of some set over a global field to the adelic version (or the product over all places of the local version), e.g. for Galois cohomology or homotopy fixed points. We write loc_S when we use the $\mathbb{A}_{k,S}$ version, and we write loc_v to denote the map from the version over the global field k to that over k_v.

If X is a variety over a local field k_v, we let $X(k_v)_\bullet$ denote the set of connected components of $X(k_v)$ under the v-adic topology, and we similarly set $X(\mathbb{A}_{k,S})_\bullet = \prod'_{v \in S}(X(k_v)_\bullet, X(\mathcal{O}_v)_\bullet)$.

4.2 Obstructions to Rational Points

4.2.1 Generalized Obstructions

Let k be a global field.

Definition 4.2.1. Let ω be a subfunctor of the functor $X \mapsto X(\mathbb{A}_k)$ from k-varieties to sets. We write $X(\mathbb{A}_k)^\omega$ instead of $\omega(X)$. We say that ω is a *generalized obstruction (to the local-global principle)* if $X(k) \subseteq X(\mathbb{A}_k)^\omega$ for every k-variety X.

If S is a subset of the places of k, we define $X(\mathbb{A}_{k,S})^\omega$ to be the projection of $X(\mathbb{A}_k)^\omega$ from $X(\mathbb{A}_k)$ to $X(\mathbb{A}_{k,S})$.

We note the trivial but important lemmas:

Lemma 4.2.2. *Let ω and ω' be two generalized obstructions. If $X(\mathbb{A}_k)^\omega \subseteq X(\mathbb{A}_k)^{\omega'}$, then $X(\mathbb{A}_{k,S})^\omega \subseteq X(\mathbb{A}_{k,S})^{\omega'}$.*

Proof. The containment $X(\mathbb{A}_k)^\omega \subseteq X(\mathbb{A}_k)^{\omega'}$ implies that the containment remains true when we project from $X(\mathbb{A}_k)$ to $X(\mathbb{A}_{k,S})$. □

Lemma 4.2.3. *Let S, S' be two nonempty sets of places of k. Then $X(\mathbb{A}_{k,S})^\omega = \emptyset$ if and only if $X(\mathbb{A}_{k,S'})^\omega = \emptyset$.*

Proof. It suffices to suppose S' is the set of all places of k. The set $X(\mathbb{A}_{k,S})^\omega$ is the projection of $X(\mathbb{A}_{k,S'})^\omega$ from $X(\mathbb{A}_{k,S'})$ to $X(\mathbb{A}_{k,S})$, so one is empty if and only if the other is. □

Lemma 4.2.4. *The association $X \mapsto X(\mathbb{A}_{k,S})^\omega$ is a subfunctor of the covariant functor from k-varieties to sets represented by* $\mathrm{Spec}(\mathbb{A}_{k,S})$.

Proof. Let $f: X \to Y$ be a map of k-varieties. It suffices to show that f maps $X(\mathbb{A}_{k,S})^\omega$ into $Y(\mathbb{A}_{k,S})^\omega$. Let $\alpha \in X(\mathbb{A}_{k,S})^\omega$. Then α is the projection of some $\alpha' \in X(\mathbb{A}_k)^\omega$. But $f(\alpha')$ is in $Y(\mathbb{A}_k)^\omega$ and projects to $f(\alpha)$, so $f(\alpha) \in Y(\mathbb{A}_{k,S})^\omega$. □

Very Strong Approximation

Definition 4.2.5. If k is a global field, S is a nonempty set of places of k, and ω is a generalized obstruction, then we call (ω, S, k) an *obstruction datum.*

When S is the set of finite places of k, we also write (ω, f, k) in this case.

We will sometimes leave out S and write (ω, k), in which case *we understand S to be the set of all places of k.* We often write (ω, S) when k is understood.

Definition 4.2.6. Let (ω, S, k) be an obstruction datum. We say that a variety X/k *is* or *satisfies very strong approximation* (VSA) for (ω, S) if $X(k) = X(\mathbb{A}_{k,S})^\omega$.

Lemma 4.2.7. *Let $W \subseteq X$ be be a (locally closed) subvariety of X over k. If X is VSA for (ω, S, k), then so is W.*

Proof. Let $\alpha \in W(\mathbb{A}_{k,S})^\omega$. Then $\alpha \in X(\mathbb{A}_{k,S})^\omega$ by functoriality. Since X is VSA, $\alpha \in X(k)$; i.e., $\operatorname{Spec} \mathbb{A}_{k,S} \xrightarrow{\alpha} W$ factors through a map $\operatorname{Spec} k \to X$.

Let $v \in S$. Then the v-component α_v of α is a k_v-point of X coming from a k-point of X. But because $\alpha \in W(\mathbb{A}_{k,S})$, this implies that $\alpha_v \in W(k_v)$. But if it is a k-point as a point of X, then we also have $\alpha_v \in W(k)$. □

Remark 4.2.8. It is the need to prove Lemma 4.2.7 that prevents us from replacing the set of points at the real places by the set of their connected components, as the map from $W(\mathbb{A}_k)_\bullet$ to $X(\mathbb{A}_k)_\bullet$ need not be injective.

Lemma 4.2.9. *Suppose X and Y are k-varieties that are VSA for (ω, S, k). Then so is $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)(\mathbb{A}_{k,S})^\omega$. By functoriality of ω applied to the projections $X \times Y \to X, Y$, we have $x \in X(\mathbb{A}_{k,S})^\omega$ and $y \in Y(\mathbb{A}_{k,S})^\omega$. By VSA for X and Y, this implies $x \in X(k)$ and $y \in Y(k)$, so $(x, y) \in (X \times Y)(k)$. □

4.2.2 Functor Obstructions

Let k be a global field. Following the formalism in Section 8.1 of [Poo17], let F be a contravariant functor from schemes over k to sets (a.k.a. a presheaf of sets on the category of k-schemes). For X a k-scheme and $A \in F(X)$, we have the following commutative diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ \downarrow & & \downarrow \\ F(k) & \xrightarrow{\text{loc}} & \prod_v F(k_v), \end{array}$$

where the vertical arrows denote pullback of A from X to k or k_v. We define $X(\mathbb{A}_k)^A$ as the subset of $X(\mathbb{A}_k)$ whose image in the lower right object is in the image of loc. We then define the obstruction set

$$X(\mathbb{A}_k)^F = \bigcap_{A \in F(X)} X(\mathbb{A}_k)^A.$$

If $\mathcal{F}$ is a collection of such functors F, we define

$$X(\mathbb{A}_k)^{\mathcal{F}} = \bigcap_{F \in \mathcal{F}} X(\mathbb{A}_k)^F.$$

Lemma 4.2.10. *The functor $X \mapsto X(\mathbb{A}_k)^{\mathcal{F}}$ is a generalized obstruction.*

Proof. It is clear that $X(k) \subseteq X(\mathbb{A}_k)^{\mathcal{F}}$, so it suffices to verify functoriality, which is a simply diagram chase.

□

Lemma 4.2.11. *Let $\mathcal{F}' \subseteq \mathcal{F}$. Then $X(\mathbb{A}_{k,S})^{\mathcal{F}} \subseteq X(\mathbb{A}_{k,S})^{\mathcal{F}'}$ for all X.*

Proof. If $\alpha \in X(\mathbb{A}_k)^{\mathcal{F}}$, then $\alpha \in X(\mathbb{A}_k)^F$ for all $F \in \mathcal{F}$, hence for all $F \in \mathcal{F}'$. But this implies $\alpha \in X(\mathbb{A}_k)^{\mathcal{F}'}$, i.e. $X(\mathbb{A}_k)^{\mathcal{F}} \subseteq X(\mathbb{A}_k)^{\mathcal{F}'}$. The result for general S follows by Lemma 4.2.2. □

4.2.3 Descent Obstructions

Let G be a group scheme over k. We let F_G denote the contravariant functor from k-schemes to sets for which $F_G(X)$ is the set of isomorphism classes of G-torsors for the fppf topology over X. By Theorem 6.5.10(i) of [Poo17], we have $F_G(X) = H^1_{\mathrm{fppf}}(X, G)$ when G is affine. If G is also smooth, it is isomorphic to the étale cohomology, denoted $H^1(X, G)$ (c.f. Notation and Conventions).

Let $\mathcal{G}$ be a subset of the set of isomorphism classes of finite type group schemes over k. For $\mathcal{F} = \{F_G\}_{[G]\in\mathcal{G}}$, we set

$$X(\mathbb{A}_k)^{\mathcal{G}} = X(\mathbb{A}_k)^{\mathcal{F}}.$$

Usually, $\mathcal{G}$ will consist only of algebraic groups over k, i.e., finite type separated group schemes over k.

Definition 4.2.12. For $\mathcal{G}$ the set of smooth affine algebraic groups over k, we call this the *descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{descent}}$ (following [Poo17]).

Definition 4.2.13. For $\mathcal{G}$ the set of finite smooth (étale) algebraic groups over k, we call this the *finite descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{f-cov}}$ (following [Sto07],[Sko09]).

Definition 4.2.14. For $\mathcal{G}$ the set of finite solvable smooth algebraic groups over k, we call this the *finite solvable descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{f-sol}}$ (following [Sto07]).

Definition 4.2.15. For $\mathcal{G}$ the set of finite commutative smooth algebraic groups over k, we call this the *finite abelian descent obstruction* and denote it by $X(\mathbb{A}_k)^{\mathrm{f-ab}}$ (following [Sto07]).

Remark 4.2.16. By the equivalence between (i) and (i') in Theorem 2.1 of [HS12], it suffices to take only finite *constant* algebraic groups over k in Definition 4.2.13.

When we wish to refer to the obstruction and specify a specific S, we write $(\mathrm{f-ab}, S, k)$, etc.

Proposition 4.2.17. *We have* $X(\mathbb{A}_k)^{\mathrm{descent}} \subseteq X(\mathbb{A}_k)^{\mathrm{f-cov}} \subseteq X(\mathbb{A}_k)^{\mathrm{f-sol}} \subseteq X(\mathbb{A}_k)^{\mathrm{f-ab}}$.

Proof. This follows by Lemma 4.2.11 because the corresponding collections of isomorphism classes of group schemes become smaller as we progress from descent to $\mathrm{f-cov}$ to $\mathrm{f-sol}$ to $\mathrm{f-ab}$. □

Corollary 4.2.18. *For fixed k and S, VSA for* $\mathrm{f-ab}$ *implies VSA for* $\mathrm{f-sol}$*, which implies VSA for* $\mathrm{f-cov}$*, which implies VSA for* descent*.*

Proof. This follows immediately from Proposition 4.2.17 and Lemma 4.2.2. □

Alternative Description in Terms of Images of Adelic Points

Let G be a smooth affine algebraic group over k, and let $Z \in F_G(X)$ be given by, with abuse of notation, $f : Z \to X$. If $\tau \in F_G(k)$, one can define the twist $f^\tau \colon Z^\tau \to X$ as in Example 6.5.12 of [Poo17].

By Theorem 8.4.1 of [Poo17], if $x \in X(k)$, then $x \in f^\tau(Y^\tau(k))$ if and only if τ is the pullback of Z under $x\colon \operatorname{Spec} k \to X$.

It follows that

$$X(k) = \bigcup_{\tau \in F_G(k)} f^\tau(Z^\tau(k)) \subseteq \bigcup_{\tau \in F_G(k)} f^\tau(Z^\tau(\mathbb{A}_{k,S})).$$

Proposition 4.2.19. *We have*

$$\bigcup_{\tau \in F_G(k)} f^\tau(Z^\tau(\mathbb{A}_k)) = X(\mathbb{A}_k)^Z.$$

Proof. Suppose $\alpha \in X(\mathbb{A}_k)^Z$. We think of this point as a map $\alpha\colon \operatorname{Spec} \mathbb{A}_k \to X$, and we know by definition that the pullback of Z along this map is in the image of

$$H^1(k, G) \xrightarrow{\text{loc}} \prod_v H^1(k_v, G),$$

say of $\tau \in H^1(k, G)$. One can easily check that twisting commutes with pullback (from k to $\mathbb{A}_k$). Thus the proof of 8.4.1 of [Poo17] tells us that the pullback of Z^τ under this adelic point is now the trivial element of $\prod_v H^1(k_v, G)$. But this means that the fiber of Z^τ over α is a trivial torsor and thus contains an adelic point; i.e., $\alpha \in f^\tau(Z^\tau(\mathbb{A}_k))$.

Conversely, suppose $\alpha \in f^\tau(Z^\tau(\mathbb{A}_k))$ for some $\tau \in H^1(k, G)$. Then for each v, we have $\alpha_v \in f^\tau(Z^\tau(k_v))$, so Theorem 8.4.1 tells us that the pullback of Z under $\alpha_v \colon \operatorname{Spec} k_v \to X$ is $\operatorname{loc}_v(\tau)$. But this implies that the pullback of Z under α is in the image of loc; i.e., $\alpha \in X(\mathbb{A}_k)^Z$. $\square$

4.2.4 Comparison with Brauer and Homotopy Obstructions

There is the Brauer-Manin obstruction set, $X(\mathbb{A}_{k,S})^{\mathrm{Br}}$, which is the obstruction for $F(-) = H^2(-, \mathbb{G}_m)$.

There is also the étale Brauer-Manin obstruction

$$X(\mathbb{A}_k)^{\text{ét,Br}} = \bigcap_{\substack{\text{finite étale } G \\ \text{all } G-\text{torsors} f: Y \to X}} \bigcup_{\tau \in H^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathrm{Br}}).$$

By Section 4.2.3, this is contained in $X(\mathbb{A}_{k,S})^{\mathrm{f-cov}}$.

A series of obstructions is defined in [HS13] when k is a number field:

$$X(\mathbb{A}_k)^h \subseteq \cdots \subseteq X(\mathbb{A}_k)^{h,2} \subseteq X(\mathbb{A}_k)^{h,1} \subseteq X(\mathbb{A}_k)$$

and

$$X(\mathbb{A}_k)^{\mathbb{Z}h} \subseteq \cdots \subseteq X(\mathbb{A}_k)^{\mathbb{Z}h,2} \subseteq X(\mathbb{A}_k)^{\mathbb{Z}h,1} \subseteq X(\mathbb{A}_k)$$

such that $X(\mathbb{A}_k)^{h,n} \subseteq X(\mathbb{A}_k)^{\mathbb{Z}h,n}$ for all n, $X(\mathbb{A}_k)^{\mathbb{Z}h} = \bigcap_n X(\mathbb{A}_k)^{\mathbb{Z}h,n}$, and $X(\mathbb{A}_k)^h = \bigcap_n X(\mathbb{A}_k)^{h,n}$. The definition of $X(\mathbb{A}_k)^h$ will be discussed in more detail in Section 6.1.2. When discussing obstruction data, we write (h, S, k) and $(\mathbb{Z}h, S, k)$.

By Theorem 9.1 of [HS13] and Lemma 4.2.2, we have for a smooth geometrically connected variety X that:

$$\begin{aligned} X(\mathbb{A}_{k,S})^{h} &= X(\mathbb{A}_{k,S})^{\text{ét,Br}} \\ X(\mathbb{A}_{k,S})^{\mathbb{Z}h} &= X(\mathbb{A}_{k,S})^{\text{Br}} \\ X(\mathbb{A}_{k,S})^{h,1} &= X(\mathbb{A}_{k,S})^{\text{f}-\text{cov}} \\ X(\mathbb{A}_{k,S})^{\mathbb{Z}h,1} &= X(\mathbb{A}_{k,S})^{\text{f}-\text{ab}} \end{aligned}$$

In particular, we have the two inclusions:

$$X(\mathbb{A}_{k,S})^{\text{ét,Br}} \subseteq X(\mathbb{A}_{k,S})^{\text{Br}} \subseteq X(\mathbb{A}_{k,S})^{\text{f}-\text{ab}}$$

$$X(\mathbb{A}_{k,S})^{\text{ét,Br}} \subseteq X(\mathbb{A}_{k,S})^{\text{f}-\text{cov}} \subseteq X(\mathbb{A}_{k,S})^{\text{f}-\text{ab}}.$$

4.3 Statement of Our Main Results

For the rest of this section, we assume that k is a number field.

The main reason to care about these obstructions is that they help prove that a variety X/k has no rational points. The most powerful obstruction currently known is $X(\mathbb{A}_k)^{\text{ét,Br}} = X(\mathbb{A}_k)^{\text{descent}}$. But, as stated in the introduction, even in this case, there is a variety X with $\emptyset = X(k) \subseteq X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$, as found in [Poo10]. In the method of proof that $X(k) = \emptyset$, it is clear that the étale Brauer-Manin obstruction and its avatars still appear, but they are applied separately to different *pieces* of X. From this point of view, it's natural to ask the following question:

Question 4.3.1. Let X be a k-variety with $X(k) = \emptyset$. Does there exist a finite open cover or stratification of X for which each constituent part has empty étale-Brauer set? More strongly, is the same true for any of the other obstruction sets from Section 4.2.3?

If true, this proves that $X(k) = \emptyset$, because if each constituent part has no rational points, then X does not.

One could ask for the stronger statement that each constituent part satisfies VSA:

Question 4.3.2. Let X be a k-variety and (ω, S, k) an obstruction datum. Does there exist a finite open cover or stratification of X for which each constituent part satisfies VSA for (ω, S, k)?

In this paper, we obtain the following result.

Theorem 4.3.3 (Corollary 5.2.4 and Corollary 5.2.9)**.** *The answer to Question 4.3.2 is yes for* $(\mathrm{f}-\mathrm{ab}, f, k)$ *when* k *is a imaginary quadratic or totally real number field.*

This is actually a very strong result when it applies, because $\mathrm{f}-\mathrm{ab}$ is weaker than the Brauer-Manin, finite descent, or étale Brauer-Manin obstruction.

To see whether we expect this to hold for all number fields, we prove the following conditional result:

Theorem 4.3.4 (Corollary 5.3.27)**.** *Assuming Grothendieck's section conjecture (5.3.25), the answer to Question 4.3.2 is yes for* $(\mathrm{f}-\mathrm{cov}, S, k)$*, where* S *is a nonempty set of finite places.*

This leaves only the question for $(\mathrm{f}-\mathrm{ab}, f, k)$ and arbitrary number fields k. Here, we can only make a conjecture:

Conjecture 4.3.5 (Conjecture 5.4.2)**.** *For any number field* k*, the answer to Question 4.3.2 is true for* $(\mathrm{f}-\mathrm{ab}, f, k)$

The primary method of proof consists in reducing the problem to proving VSA for an open subset of $\mathbb{P}^1$, a condition we call $A(\omega, f, k)$. Theorem 5.1.3 then tells us in general that this answers Question 4.3.2 (and therefore Question 4.3.1) affirmatively. This is described in Chapter 5.

In Chapter 6, we examine the section conjecture more closely, in particular what happens to cuspidal sections in fibrations. Our results are of independent interest for the section conjecture, but they also allow us to reprove Theorem 4.3.4 in Corollary 6.3.16. While this might seem logically unnecessary if the theorem is already proven, the second proof provides a possibly different open cover than that of the first proof.

In Chapter 7, we review the original example of [Poo10] in terms of our results. We also introduce the notion of quasi-torsors (Definition 7.2.1), which leads us to propose the following conjecture:

Conjecture 4.3.6 (Conjecture 7.2.4)**.** *For any number field k and variety X/k with $X(k) = \emptyset$, does there exist a stratification*[1] $X = \coprod_i X_i$ *and for each i, a quasi-torsor Y_i over the closure $\overline{X_i}$ of X_i such that Y_i restricts to a torsor over X_i and $Y_i^\sigma(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ for all twists Y_i^σ of Y_i.*

4.3.1 A New Obstruction

Let $\mathscr{X} = \{X_i\}$ be a finite collection of locally closed subvarieties of X whose set-theoretic union is X (e.g., a stratification or open cover) and (ω, S, k) an obstruction datum. We define the following two sets, where unions take place in $X(\mathbb{A}_{k,S})$:

$$X(\mathbb{A}_{k,S})^{\mathscr{X},\omega} = \bigcup_i X_i(\mathbb{A}_{k,S})^\omega$$

Functoriality then gives us the inclusion:

$$X(k) \subseteq X(\mathbb{A}_{k,S})^{\mathscr{X},\omega} \subseteq X(\mathbb{A}_{k,S})^\omega.$$

Finally, we must consider the case where $\mathscr{C}$ is a *collection of finite collections of locally closed subvarieties*. For example, $\mathscr{C}$ might be the collection of all open covers of X, denoted by $\mathcal{OPEN}$, or the collection of all finite stratifications, denoted by $\mathcal{STRAT}$.

We then define

$$X(\mathbb{A}_{k,S})^{\mathscr{C},\omega} = \bigcap_{\mathscr{X}\in\mathscr{C}} X(\mathbb{A}_{k,S})^{\mathscr{X},\omega}.$$

In fact, $\mathcal{OPEN}$ and $\mathcal{STRAT}$ are really rules that assign such a $\mathscr{C}$ to each X/k.

We can then rephrase Theorem 5.1.3:

[1] We are defining a finite stratification of X as a finite partially-ordered index set I along with a locally closed subset $S_i \subseteq X$ for every $i \in I$ such that X is the disjoint union of all S_i, and the closure of any given S_i in X is the union $\bigcup_{j \leq i} S_j$.

Theorem 4.3.7. *If $A(\omega, S, k)$ holds, then for any X/k, there is a finite Zariski open cover $\mathscr{X}$ such that*

$$X(k) = X(\mathbb{A}_{k,S})^{\mathscr{X},\omega}.$$

As a result,

$$X(k) = X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}.$$

Just for completeness, we note a couple of basic properties of these types of obstructions.

Proposition 4.3.8. *Let (ω, S, k) be an obstruction datum and $f : X \to Y$ a map of k-schemes. Then f maps $X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ into $Y(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ and $X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$ into $Y(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$, respectively.*

Proof. Let $\alpha \in X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$, and let $\{Y_i\}_i$ be a finite open cover of Y. Then $\{f^{-1}(Y_i)\}_i$ is a finite open cover of X, so $\alpha \in f^{-1}(Y_i)(\mathbb{A}_{k,S})^{\omega}$. Then functoriality tells us that $f(\alpha) \in Y_i(\mathbb{A}_{k,S})^{\omega}$. As $\{Y_i\}_i$ was arbitrary, $f(\alpha) \in Y(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$.

The other part goes nearly word-for-word. Let $\alpha \in X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$, and let $\{Y_i\}_i$ be a finite stratification of Y. Then $\{f^{-1}(Y_i)\}_i$ is a finite stratification of X, so $\alpha \in f^{-1}(Y_i)(\mathbb{A}_{k,S})^{\omega}$. Then functoriality tells us that $f(\alpha) \in Y_i(\mathbb{A}_{k,S})^{\omega}$. As $\{Y_i\}_i$ was arbitrary, $f(\alpha) \in Y(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$. □

The following proposition explains why answering Question 4.3.2 positively for open covers rather than stratifications is more powerful:

Proposition 4.3.9. *For any (ω, S, k), we have*

$$X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega} \subseteq X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$$

Proof. Suppose $\alpha \in X(\mathbb{A}_{k,S})^{\mathcal{STRAT},\omega}$. We must show that for every finite open cover $X = \bigcup_{i=1}^{n} X_i$, we have $\alpha \in X_i(\mathbb{A}_{k,S})^{\omega}$ for some i.

We build a stratification out of this open cover as follows. For $i = 1, \cdots, n$, we let

$$S_i = X_i \setminus \bigcup_{j<i} X_j.$$

Then $\{S_i\}$ forms a stratification, with $\overline{S_i} = \bigcup_{j \geq i} S_j$ for all i. Thus $\alpha \in S_i(\mathbb{A}_{k,S})^\omega$ for some i. But $S_i \subseteq X_i$, so functoriality tells us that $\alpha \in X_i(\mathbb{A}_{k,S})^\omega$. $\square$

If one wishes to define $X(\mathbb{A}_k)^{\text{new}}$ as in the introduction, the natural candidates are $X(\mathbb{A}_k)^{\mathcal{OPEN},\text{f-ab}}$ and $X(\mathbb{A}_k)^{\mathcal{STRAT},\text{f-ab}}$. The real power in our results, however, is that one need only take *a single open cover*.

Remark 4.3.10. It is interesting to consider the constructions in this subsection applied when ω is such that $X(\mathbb{A}_k)^\omega = X(\mathbb{A}_k)$ for all X. This is discussed in Appendix A.

Remark 4.3.11. This refinement of obstructions via open covers is related to the notion of cosheafification. This connection is discussed in Appendix B.

4.3.2 Relationship to the Birational Section Conjecture

The reader might think that the ideas in this paper can be used to prove the birational section conjecture. The idea is this: let X be a proper smooth hyperbolic curve over $\mathbb{Q}$. If there is a birational Galois section (over $\mathbb{Q}$), then the p-adic birational section conjecture ([Koe05]) shows that there is a $\mathbb{Q}_p$-point for every place p. This provides an adelic point of X whose associated birational section comes from a global section. In other words, it is in the finite descent set of every open subset of our curve, and by the results here, it is rational.

The flaw in this argument is that an adelic point on X might not be an adelic point of *any proper open subvariety*. In the language of Appendix B, this is the fundamental reason why $\mathcal{F}_{\mathbb{A}_k^f}$ is not a cosheaf. Therefore, one would have to use the fact that its birational section is comes from a birational section over a global field in a deeper way to make such an argument work. All of this is in fact implicit in [Sti15] and Theorems 3.2, 3.3, 4.2, and 4.3 of [HS12].

Chapter 5

Main Result via Embeddings

5.1 General Setup

We set up the basic formalism for the various unconditional and conditional results proven via embeddings.

Definition/Theorem 5.1.1. *Let (ω, S, k) be an obstruction datum. The following statements are equivalent:*

(i) There is a nonempty open k-subscheme of $\mathbb{P}^1_k$ that is VSA for (ω, S, k).

(ii) There is a nonempty open k-subscheme of $\mathbb{A}^1_k$ that is VSA for (ω, S, k).

(iii) For each positive integer n, there is a nonempty open k-subscheme of $\mathbb{A}^n_k$ that is VSA for (ω, S, k).

(iv) For each positive integer n, there is a nonempty open k-subscheme of $\mathbb{P}^n_k$ that is VSA for (ω, S, k).

If this is the case, we say that the property $A(\omega, S, k)$ is true.

Proof. If (i) holds, let this open subscheme be U. Then $V := U \cap \mathbb{A}^1_k$ is nonempty because any nonempty open is dense, and V is then VSA by Lemma 4.2.7, so (ii) holds.

If (ii) holds, then V^n is a nonempty open subscheme of $\mathbb{A}_k^n$ for all n, which is VSA by Lemma 4.2.9, so (iii) holds.

If (iii) holds, then the VSA nonempty open subscheme of $\mathbb{A}_k^n$ is also a VSA nonempty open subscheme of $\mathbb{P}_k^n$, so (iv) holds.

Finally, (iv) implies (i) by setting $n = 1$. □

We note that $\mathrm{PGL}_2(k)$ acts on $\mathbb{P}_k^1$ by k-automorphisms, and hence $(\mathrm{PGL}_2(k))^n$ acts on $(\mathbb{P}_k^1)^n$ in the same way.

Lemma 5.1.2. *Let k be an infinite field and x a closed point of $(\mathbb{P}_k^1)^n$. Then the orbit of x under $(\mathrm{PGL}_2(k))^n$ is Zariski dense in $(\mathbb{P}_k^1)^n$.*

Proof. We choose an algebraic closure $\overline{k}$ of k and a point $\overline{x}$ of $(\mathbb{P}^1(\overline{k}))^n$ representing x. We equivalently wish to show that for any open $U \subseteq (\mathbb{P}_k^1)^n$, the orbit of $\overline{x}$ under $(\mathrm{PGL}_2(k))^n$ intersects $U(\overline{k})$.

We first prove this for $n = 1$. The points $(\overline{x} + a)_{a \in k}$ of $\mathbb{P}^1$ are distinct. As there are infinitely many of them, and $\mathbb{P}^1$ is one-dimensional, they are Zariski dense. But these points are all in the orbit of $\overline{x}$, so this orbit is Zariski dense.

Let $\overline{x} = (x_1, \cdots, x_n)$ with $x_i \in \mathbb{P}_k^1(\overline{k})$, and let S_i be the $\mathrm{PGL}_2(k)$-orbit of x_i. The previous paragraph implies that each S_i is Zariski-dense in $\mathbb{P}^1(\overline{k})$, so $\prod_{i=1}^{n} S_i$ is dense in $(\mathbb{P}^1(\overline{k}))^n$.

A less detailed and less general version of this argument is given in Lemma 6.3 of [SS16]. □

Theorem 5.1.3. *Suppose that $A(\omega, S, k)$ holds. Let X be a k-variety. Then there exists a finite affine open cover $X = \bigcup_i V_i$ such that V_i is VSA for (ω, S, k) for every i.*

Proof. Let x be any closed point. By the definition of a scheme, there is an affine open neighborhood of X containing x, so we can assume that X is affine.

We now embed X into $\mathbb{A}^n$ for some sufficiently large n. As $\mathbb{A}^1$ is an open subscheme of $\mathbb{P}^1$, we have an open inclusion $\mathbb{A}^n = (\mathbb{A}^1)^n \hookrightarrow (\mathbb{P}^1)^n$, so we get an embedding $\phi : X \hookrightarrow (\mathbb{P}^1)^n$.

Assuming $A(\omega, S, k)$, there is an open subset $U \subseteq (\mathbb{P}^1)^n$ that is VSA for (ω, S, k). By Lemma 5.1.2, there is a k-automorphism g of $(\mathbb{P}^1)^n$ sending $\phi(x)$ into U. Then U contains

$g(\phi(x))$. But then $U \cap g(\phi(X))$ is an open subscheme of X containing x and is a locally closed subscheme of U. It is therefore VSA for (ω, S, k) by Lemma 4.2.7. Choose an affine open neighborhood V_x of x in $U \cap g(\phi(X))$, and V_x is again VSA.

We do this for every x, and we obtain an affine open cover $X = \bigcup_{x \in X} V_x$ such that V_x is VSA for (ω, S, k) for every x. As varieties are quasi-compact, this has a finite subcover, which proves the theorem. □

Corollary 5.1.4. *Let X be a k-variety with $X(k) = \emptyset$, and suppose $A(\omega, S, k)$. Then there exists a finite open cover of X such that each constituent open set has empty obstruction set for (ω, S, k). The same is true for any other nonempty set S' of places of k.*

Proof. Let $\bigcup_i V_i$ be an open cover as in Theorem 5.1.3. As $X(k) = \emptyset$, we have $V_i(k) = \emptyset$. By VSA, this implies $V_i(\mathbb{A}_{k,S})^\omega = \emptyset$.

By Lemma 4.2.3, we also have $V_i(\mathbb{A}_{k,S'})^\omega = \emptyset$ for every i. □

5.2 Embeddings of Varieties into Tori

For the rest of Section 5.2, we assume that k is a number field.

Lemma 5.2.1. *Let T be an algebraic torus over k, let S be a nonempty set of places of k, and let $\alpha \in T(\mathbb{A}_{k,S})$. The first statement implies the second, and if S consists only of finite places of k, the second implies the third:*

(i) $\alpha \in T(\mathbb{A}_{k,S})^{\mathrm{f-ab}}$.

(ii) For every $n \in \mathbb{Z}_{\geq 0}$, there exists $a_n \in T(k)$ and $\beta_n \in T(\mathbb{A}_{k,S})$ for which $\alpha = a_n(\beta_n)^n$.

(iii) $\alpha \in \overline{T(k)}$, with the closure taken in $T(\mathbb{A}_{k,S})$.

Proof. (i) $\implies$ (ii): For every n, there is a standard torsor $T \xrightarrow{[n]} T$ under the n-torsion scheme $T[n]$ of T given by the nth power map $T \to T$. The Kummer map is the map sending $x \in T(k)$ to the pullback of $T \xrightarrow{[n]} T$ under x, the result of which is a torsor under $T[n]$ over k. In this case, it is given by the boundary map $T(k) \to H^1(k, T[n])$ in Galois cohomology coming from the short exact sequence $0 \to T[n] \to T \to T \to 0$ of étale sheaves. In particular, the image of the Kummer map is canonically $T(k)/nT(k)$. The same holds with k_v in place of k, and all maps respect the inclusions $k \to k_v$. If $\alpha \in T(\mathbb{A}_{k,S})^{\text{f-ab}}$, it must be in the image of

$$T(k)/nT(k) \to \prod_{v \in S} T(k_v)/nT(k_v),$$

which amounts to saying that $\alpha = a_n(\beta_n)^n$ for $\beta_n \in \prod_{v \in S} T(k_v)$. As $a_n, \alpha \in T(\mathbb{A}_{k,S})$, so is β_n.

(ii) $\implies$ (iii): Let K be any open subgroup of $T(\mathbb{A}_{k,S})$. By Theorem 5.1 of [Bor63] (also c.f. [Con06])[1], $T(\mathbb{A}_{k,S})/T(k)K$ is finite, say of order h. We know that $\alpha = a_h(\beta_h)^h$. But $(\beta_h)^h \in T(k)K$, and therefore so is α. In other words, the coset αK contains an element of $T(k)$. The set of all such K and their cosets form a basis for the topology of $T(\mathbb{A}_{k,S})$ because S only contains finite places. It follows that every open neighborhood of α contains an element of $T(k)$, or $\alpha \in \overline{T(k)}$.

□

Lemma 5.2.2. *Let T be an algebraic torus over k and K an open subgroup of $T(\mathbb{A}_{k,S})$ such that $T(k) \cap K$ is finite. Then $T(k)$ is closed in $T(\mathbb{A}_k^f)$. For example, this happens for $S = f$ if T has a model $\mathcal{T}$ over $\mathcal{O}_k$ with $\mathcal{T}(\mathcal{O}_k)$ finite.*

Proof. As $T(k)$ is a subgroup, each coset of K also has finite intersection with $T(k)$. The topological space $T(\mathbb{A}_{k,S})$ is the disjoint union of the cosets of K. This space is T_0, so finite sets are closed. Thus $T(k)$ is closed in each coset of K. This implies that it is closed in all of $T(\mathbb{A}_{k,S})$.

By the definition of the adelic topology, the subgroup

$$\mathcal{T}(\widehat{\mathcal{O}_k}) := \prod_v \mathcal{T}(\mathcal{O}_v) \subseteq T(\mathbb{A}_k^f)$$

is open. We can then conclude because $\mathcal{T}(\mathcal{O}_k) = \mathcal{T}(k) \cap \mathcal{T}(\widehat{\mathcal{O}_k})$. □

[1] In fact, [Bor63] only covers the case that $\mathbb{A}_k = \mathbb{A}_{k,S}$, but the result holds in general by considering the continuous projection $\mathbb{A}_k \to \mathbb{A}_{k,S}$.

5.2.1 The Result When k is Imaginary Quadratic

In this subsection, we prove $A(\mathrm{f}-\mathrm{ab}, f, k)$ when k has finitely many units.

Remark 5.2.3. By Dirichlet's unit theorem, k has finitely many units if and only if k is $\mathbb{Q}$ or imaginary quadratic.

Corollary 5.2.4. *For k as above, $\mathbb{G}_m$ is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$. In particular, $A(\mathrm{f}-\mathrm{ab}, f, k)$ holds.*

Proof. That k has finitely many units means that $\mathbb{G}_m(\mathcal{O}_k)$ is finite. By Lemma 5.2.2, this implies that $\mathbb{G}_m(k)$ is closed in $\mathbb{G}_m(\mathbb{A}_k^f)$. By Lemma 5.2.1, this implies that $\mathbb{G}_m(k) = \mathbb{G}_m(\mathbb{A}_k^f)^{\mathrm{f}-\mathrm{ab}}$; i.e., $\mathbb{G}_m$ is VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$. $\square$

5.2.2 The Result When k is Totally Real

We now prove $A(\mathrm{f}-\mathrm{ab}, f, F)$ when k is totally real. The material in this subsection is inspired by [Sti15].

In fact, we can choose the subscheme of $\mathbb{P}_k^1$ as in Definition/Theorem 5.1.1(i) to be the complement in $\mathbb{P}_k^1$ of the vanishing scheme of any quadratic polynomial over k with totally negative discriminant.

Definition 5.2.5. For k a totally real number field and E/k a totally imaginary quadratic extension, we define the *norm one torus relative to E/k*

$$T = T_{E/k} = \ker(\mathrm{N}_{E/k} \colon \mathrm{Res}_{E/k}\mathbb{G}_m \to \mathbb{G}_m),$$

which is a group scheme over k.

Proposition 5.2.6. *Let $\alpha \in E$ with minimal polynomial $t^2 + bt + c$ over k. Let U be the complement in $\mathbb{P}_k^1$ (with coordinate t) of the vanishing locus of t^2+bt+c. Then U is isomorphic to T.*

Proof. As $E = k(\alpha)$, and the norm of $x - y\alpha$ is $x^2 + bxy + cy^2$, we can express T as $\operatorname{Spec} k[x,y]/(x^2 + bxy + cy^2 - 1)$. This has projective closure $\operatorname{Proj}(k[x,y,z]/(x^2 + bxy +$

$cy^2 - z^2)$), which is a smooth projective conic. This conic has a point $(x, y, z) = (1, 0, 1)$, so it is isomorphic to $\mathbb{P}^1_k$. The complement of T is given by $0 = z^2 = x^2 + bxy + cy^2$, which corresponds to the vanishing locus of $t^2 + bt + c$ by setting $t = x/y$. ☐

Proposition 5.2.7. *For some integral model $\mathcal{T}$ of T, the set $\mathcal{T}(\mathcal{O}_k)$ is finite. Thus T is VSA for* $(\mathrm{f} - \mathrm{ab}, f, k)$ *by Lemma 5.2.2.*

Proof. The torus T has an integral model $\mathcal{T} = \ker(\mathrm{N}_{\mathcal{O}_E/\mathcal{O}_k} \colon \mathrm{Res}_{Oc_E/\mathcal{O}_k}\mathbb{G}_m \to \mathbb{G}_m)$. Thus $\mathcal{T}(\mathcal{O}_k)$ is the kernel of $\mathrm{N}_{\mathcal{O}_E/\mathcal{O}_k} \colon \mathbb{G}_m(\mathcal{O}_E) \to \mathbb{G}_m(\mathcal{O}_k)$.

The composition $\mathbb{G}_m(\mathcal{O}_k) \hookrightarrow \mathbb{G}_m(\mathcal{O}_E) \xrightarrow{\mathrm{N}_{\mathcal{O}_E/\mathcal{O}_k}} \mathbb{G}_m(\mathcal{O}_k)$ is $x \mapsto x^2$. Its image in the finitely generated abelian group $\mathbb{G}_m(\mathcal{O}_k)$ therefore has full rank, and therefore so does the image of $N \colon \mathbb{G}_m(\mathcal{O}_E) \to \mathbb{G}_m(\mathcal{O}_k)$. But E and k have the same number of Archimedean places, so $\mathbb{G}_m(\mathcal{O}_E)$ and $\mathbb{G}_m(\mathcal{O}_k)$ have the same rank. But a map between finitely generated abelian groups of the same rank whose image has full rank must have finite kernel. Therefore $\mathcal{T}(\mathcal{O}_k)$ is finite. ☐

Remark 5.2.8. One may alternatively prove Proposition 5.2.7 by noting that $\mathcal{O}_K$ is discrete in $k \otimes \mathbb{R}$ and that $T(k \otimes \mathbb{R})$ is compact.

Corollary 5.2.9. *If U is the complement in $\mathbb{P}^1_k$ of the vanishing locus of a quadratic polynomial $t^2 + bt + c$ with totally negative discriminant, then U is VSA for* $(\mathrm{f} - \mathrm{ab}, f, k)$*. In particular,* $A(\mathrm{f} - \mathrm{ab}, f, k)$ *holds.*

Proof. Let E be the splitting field of $t^2 + bt + c$. By Proposition 5.2.6, U is isomorphic to $T_{E/k}$, so by Proposition 5.2.7, U is VSA for $(\mathrm{f} - \mathrm{ab}, f, k)$. ☐

5.3 Finite Descent and the Section Conjecture

We now prove $A(\mathrm{f} - \mathrm{cov}, f, k)$ for any number field k assuming the section conjecture for $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$.

5.3.1 Grothendieck's Section Conjecture

The Kummer Map

For the remainder of this subsection, we let X denote a quasi-compact quasi-separated geometrically connected scheme over a field k. We fix a separable closure k_s of k, which produces a geometric point $\operatorname{Spec}(k_s) \to \operatorname{Spec}(k)$ of k. We identify $G_k := \operatorname{Gal}(k_s/k)$ with the étale fundamental group of $\operatorname{Spec} k$ based at this point. As in [Poo17], X^s denotes the base change X_{k_s}.

Definition 5.3.1. Let $\overline{a} \in X(k_s)$, which we may also view as an element of $X^s(k_s)$ and $\operatorname{Spec} k(k_s)$. By [Sta17, Tag 0BTX], there is an exact sequence

$$1 \to \pi_1(X^s, \overline{a}) \to \pi_1(X, \overline{a}) \to G_k \to 1,$$

known as the *fundamental exact sequence*

Definition 5.3.2. The set of sections $\mathscr{S}_{\pi_1(X/k),\overline{a}}$ is defined to be the set of (multiplicative) sections of the surjection $\pi_1(X, \overline{a}) \to \pi_1(\operatorname{Spec} k, \overline{a})$ modulo the action of $\pi_1(X^s, \overline{a})$ by conjugation. If we fix a section, we get an action of G_k on $\pi_1(X^s, \overline{a})$, and $\mathscr{S}_{\pi_1(X/k),\overline{a}}$ is isomorphic to the nonabelian continuous Galois cohomology pointed-set $H^1(k, \pi_1(X^s, \overline{a}))$ (c.f. "Generalized Sections" in 1.2 of [Sti13]).

Fact 5.3.3. As in Definition 23 of [Sti13], the sets $\mathscr{S}_{\pi_1(X/k),\overline{a}}$ for different choices of $\overline{a}$ are in canonical bijection, so we may write $\mathscr{S}_{\pi_1(X/k)}$ without ambiguity.

If k'/k is a field extension and X a k-scheme, $\mathscr{S}_{\pi_1(X/k')}$ denotes $\mathscr{S}_{\pi_1(X_{k'}/k')}$.

Definition 5.3.4 (Profinite Kummer Map)**.** Given a point $b \in X(k)$, there is a unique geometric basepoint $\overline{b}$ lying over b and compatible with the basepoint of $\operatorname{Spec} k$. Then b induces a pointed map of schemes $\operatorname{Spec} k \to X$, which induces a map of fundamental groups $G_k \to \pi_1(X, \overline{b})$ compatible with the projection $\pi_1(X, \overline{b}) \to G_k$, and hence an element of

$\mathscr{S}_{\pi_1(X/k),\bar{b}}$. By Fact 5.3.3, this gives us a well-defined map

$$\kappa = \kappa_{X/k} \colon X(k) \to \mathscr{S}_{\pi_1(X/k)},$$

which we call the *(profinite) Kummer map.*

As with Definition 5.3.2, $\kappa_{X/k'}$ denotes $\kappa_{X_{k'}/k'}$. As well, if S is a set of places of k, then we set $\kappa_S = \kappa_{X/\mathbb{A}_{k,S}} := \prod_{v \in S} \kappa_{X/k_v} \colon X(\mathbb{A}_{k,S}) \to \prod_{v \in S} X(hk_v)$, or $\kappa_{X/\mathbb{A}_k}$ when S is all places.

A map $f \colon X \to Y$ of quasi-compact quasi-separated geometrically connected k-schemes induces a map $f \colon \mathscr{S}_{\pi_1(X/k)} \to \mathscr{S}_{\pi_1(Y/k)}$ by choosing a compatible pair of geometric basepoints for X and Y (but is easily seen to be independent of that choice). This map is compatible under the Kummer map with $f \colon X(k) \to Y(k)$.

Remark 5.3.5. If we assume the existence of a (fixed) Galois-invariant basepoint $\overline{a}$, we can define $\kappa(b)$ as the class in $H^1(G_k, \pi_1(X^s, \overline{a}))$ of the G_k-equivariant torsor $\pi_1(X^s, \overline{a}, \overline{b})$.

Remark 5.3.6. In Section 6.1, we will reformulate the section conjecture in more homotopical language, which will have the advantage of obviating the need for basepoints.

Conjecture 5.3.7 (Grothendieck [Gro97])**.** *Let k be a number field and X/k a geometrically connected, smooth, proper curve of genus at least 2. Then the Kummer map $X(k) \to \mathscr{S}_{\pi_1(X/k)}$ is a bijection.*

Remark 5.3.8. This is false for all nontrivial abelian varieties, as shown in [CS15].

We will, however, need to extend this conjecture to the non-proper case, as in Chapter 18 of [Sti13]. To do so, we must introduce the notion of cuspidal sections.

Cuspidal Sections

Definition 5.3.9. A *cuspidal datum* (X, C, k) is a smooth geometrically connected variety X over a field k and a subset $C \subseteq \mathscr{S}_{\pi_1(X/k)}$, called the set of *cuspidal sections.*

Definition 5.3.10. A cuspidal datum (X, C, k) is said to satisfy the *surjectivity in the section conjecture* if $\mathscr{S}_{\pi_1(X/k)} \setminus C \subseteq \kappa(X(k))$.

Definition 5.3.11. Let $\overline{a}$ be a geometric basepoint, and assume that the fundamental exact sequence for $(X, \overline{a})$ has a section s. We define the centralizer $Z(s)$ as the set of elements of $\pi_1(X^s, \overline{a})$ that commute in $\pi_1(X, \overline{a})$ with $s(g)$ for all $g \in G_k$. If two sections are conjugate, their centralizers are conjugate. In particular, the property of having trivial centralizer is a property of elements of $\mathscr{S}_{\pi_1(X/k)}$.

Remark 5.3.12. As in Definition 5.3.2, a section s gives an action of G_k on $\pi_1(X^s, \overline{a})$. The centralizer of s is then isomorphic to the cohomology group $H^0(k, \pi_1(X^s, \overline{a}))$.

Definition 5.3.13. A cuspidal datum (X, C, k) is said to satisfy the *injectivity in the section conjecture* if

1. The map $\kappa\colon X(k) \to \mathscr{S}_{\pi_1(X/k)}$ is injective.

2. The sets $\kappa(X(k))$ and C are disjoint.

3. The centralizer of every element of $\kappa(X(k))$ is trivial

Remark 5.3.14. The reader might wonder what the last two conditions have to do with injectivity. Theorem 6.2.6 will show that this stronger definition of injectivity holds inductively in geometric fibrations, while the more naive version does not. We also note that this stronger version is known to hold for hyperbolic curves (Theorem 5.3.24).

Proposition 5.3.15. *Let k'/k be an extension of characteristic 0 fields and X as above. We choose a separable closure k'_s and an embedding of k_s in k'_s, which gives us a map $G_{k'} = \pi_1(\operatorname{Spec} k', k'_s) \to \pi_1(\operatorname{Spec} k, k'_s) \to \pi_1(\operatorname{Spec} k, k_s) = G_k$. Then there is a base change map*

$$\mathscr{S}_{\pi_1(X/k)} \to \mathscr{S}_{\pi_1(X/k')}$$

induced by precomposition with the map $G_{k'} \to G_k$ (defined by choosing a geometric basepoint for $X_{k'}$ but independent of basepoint). Furthermore, the diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(k') \\ {\scriptstyle \kappa_{X/k}}\downarrow & & \downarrow{\scriptstyle \kappa_{X/k'}} \\ \mathscr{S}_{\pi_1(X/k)} & \longrightarrow & \mathscr{S}_{\pi_1(X/k')} \end{array}$$

is commutative.

Proof. The base change map is Definition 27 of [Sti13]. Characteristic 0 is required to ensure that X_{k_s} and $X_{k'_s}$ have the same étale fundamental group. (Remark: This also works in positive characteristic if X is proper.)

Let $x \in X(k)$ and $\overline{x}$ an associated Galois-invariant basepoint of $X_{k'}$. We may apply $\pi_1(-, \overline{x})$ to the commutative diagram of schemes

$$\begin{array}{ccc} \operatorname{Spec} k' & \longrightarrow & \operatorname{Spec} k \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle x} \\ X_{k'} & \longrightarrow & X \end{array}$$

to obtain the commutative diagram of profinite groups

$$\begin{array}{ccc} G_{k'} & \longrightarrow & G_k \\ {\scriptstyle \kappa_{X/k'}(x)}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{X/k}(x)} \\ \pi_1(X_{k'}, \overline{x}) & \longrightarrow & \pi_1(X, \overline{x}) \end{array}$$

As the base change map is obtained by precomposition with the top horizontal arrow, the commutativity of the original diagram is clear. □

Definition 5.3.16. If (X, C, k) is a cuspidal datum and k'/k an extension of fields of characteristic 0, we say that a cuspidal datum $(X_{k'}, C', k')$ is compatible with (X, C, k) if the base change map sends C into C'.

Lemma 5.3.17. *With the notation of the previous definition, if $(X_{k'}, C', k')$ satisfies the injectivity in the section conjecture, then so does (X, C, k).*

Proof. As $X(k) \hookrightarrow X(k')$ and $\kappa_{X/k'}$ are injective, Proposition 5.3.15 tells us that $\kappa_{X/k}$ is injective. As well, if there were $x \in X(k)$ for which $\kappa_{X/k}(x) \in C$, then Proposition 5.3.15 tells us that $\kappa_{X/k'}(x) \in C'$, which is not the case.

Finally, suppose there were an element s of $\kappa(X(k))$ with nontrivial centralizer. Then the image s' of s in $\mathscr{S}_{\pi_1(X/k')}$ lies in $\kappa(X(k'))$, so by injectivity for $(X_{k'}, C', k')$, we know that the centralizer of s' is trivial. But a nontrivial element of the centralizer of s is also a nontrivial element of the centralizer of s', so s must have trivial centralizer. □

Curves

Definition 5.3.18. Let U be a smooth geometrically connected curve over a field k of characteristic 0, and let $U \hookrightarrow X$ be its unique compactification. Let $Y = X \setminus U$ and $y \in Y(k)$. We fix a Henselization $\mathcal{O}^h_{X,y}$ and set $X_y = \operatorname{Spec} \mathcal{O}^h_{X,y}$. We let $U_y = X_y \times_X U$. By 18.3 of [Sti13], there is a short exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}(U_y)^\times/\mathcal{O}(X_y)^\times, \widehat{\mathbb{Z}}(1)) \to \pi_1(U_y) \to G_k \to 0,$$

and $\mathscr{S}_{\pi_1(U_y/k)}$ is the set of sections of the surjection $\pi_1(U_y) \to G_k$ modulo conjugation by the kernel. The map $U_y \to U$ induces a map $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}$, whose image C_y is known as the *packet of cuspidal sections at* y, originally defined by Grothendieck. The set C_U of all *cuspidal sections of the curve* U is defined to be

$$\bigcup_{y \in Y(k)} \operatorname{Im}\left(\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}\right) \subseteq \mathscr{S}_{\pi_1(U/k)}.$$

When we speak of cuspidal sections of a curve U/k, we always mean (U, C_U, k) unless otherwise stated.

Lemma 5.3.19. *Let U/k be as in Definition 5.3.18, and let k'/k be a field extension. Then (U, C_U, k) is compatible with $(U_{k'}, C_{U_{k'}}, k')$ in the sense of Definition 5.3.16.*

Proof. Let X be the compactification of U over k, and let $y \in (X \setminus U)(k)$. We can view y as an element of $(X \setminus U)(k')$. The map $\mathcal{O}(U_y)^\times/\mathcal{O}(X_y)^\times \to \mathcal{O}((U_{k'})_y)^\times/\mathcal{O}((X_{k'})_y)^\times$ is an isomorphism, so the map $(U_{k'})_y \to U_y$ induces an isomorphism on geometric fundamental groups. This produces a base change map $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1((U_{k'})_y/k')}$. A diagram chase as in

Proposition 5.3.15 shows that the diagram

$$\begin{array}{ccc} \mathscr{S}_{\pi_1(U_y/k)} & \longrightarrow & \mathscr{S}_{\pi_1((U_{k'})_y/k')} \\ \downarrow & & \downarrow \\ \mathscr{S}_{\pi_1(U/k)} & \longrightarrow & \mathscr{S}_{\pi_1(U_{k'}/k')} \end{array}$$

commutes, which proves the desired compatibility.

□

Definition 5.3.20. Let U be a connected smooth curve over a separably closed field k. Let $U \hookrightarrow X$ be the unique embedding of X into a projective geometrically connected regular curve. Let $Y := X \backslash U$, which is a zero dimensional scheme of some degree n. Let g denote the genus of X. We define the *Euler characteristic* $\chi(U)$ of U to be $2 - 2g - n$.

Definition 5.3.21. Let U be a geometrically connected smooth curve over an arbitrary field k. We set $\chi(U) = \chi(U^s)$.

Definition 5.3.22. Let U/k be a geometrically connected smooth curve (not necessarily projective). Then we say U is *hyperbolic* if and only if $\chi(U) < 0$.

Remark 5.3.23. In characteristic 0, U is hyperbolic if and only if $\pi_1(U^s)$ is non-abelian.

Theorem 5.3.24. *Let k be a subfield of a finite extension of $\mathbb{Q}_p$, and U/k a geometrically connected smooth hyperbolic curve. Then U satisfies the injectivity in the section conjecture.*

Proof. By Lemma 5.3.17 and Lemma 5.3.19, it suffices to prove the theorem for finite extensions of $\mathbb{Q}_p$.

The injectivity of the Kummer map follows from Corollary 74 and Proposition 75 of [Sti13]. The second part follows from Theorem 250 of [Sti13]. The fact that $H^0(k, \pi_1(U^s, \overline{a})) = 0$ is Proposition 104 of [Sti13].

□

The following is an extension of Conjecture 5.3.7 to non-proper curves:

Conjecture 5.3.25 (Grothendieck [Gro97])**.** *Let k be a number field and X/k a geometrically connected, smooth hyperbolic curve (not necessarily proper). Then the surjectivity in the section conjecture holds for (X, C_X, k).*

5.3.2 VSA via the Section Conjecture

We now explain the relationship between the section conjecture and very strong approximation:

Proposition 5.3.26. *Let S be a nonempty set of finite places of a number field k and X a variety over k. Suppose X satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for all $v \in S$ (and some compatible choice of cuspidal data). Then X satisfies VSA for* $(\mathrm{f-cov}, S, k)$

Proof. Let $P \in X(\mathbb{A}_{k,S})^{\mathrm{f-cov}}$, and let $P' \in X(\mathbb{A}_k)^{\mathrm{f-cov}}$ project to P. For each place v of k, let $s_v :\ = \kappa_{X/k_v}(P'_v) \in \mathscr{S}_{\pi_1(X/k_v)}$. Then P' satisfies (i) of Theorem 2.1 of [HS12] for U the trivial group and S (in the notation of [HS12]) the empty set. It therefore satisfies (iii) of the same theorem; i.e., (s_v) is image of some $s \in \mathscr{S}_{\pi_1(X/k)}$ under the bottom horizontal arrow of the following diagram:

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ \Big\downarrow{\scriptstyle \kappa_{X/k}} & & \Big\downarrow{\scriptstyle \kappa_{X/\mathbb{A}_k}} \\ \mathscr{S}_{\pi_1(X/k)} & \longrightarrow & \prod_v \mathscr{S}_{\pi_1(X/k_v)} \end{array}$$

If s were cuspidal, s_v it would be for each v. But s_v is not cuspidal for every finite v by the second part of injectivity in the section conjecture, so s is not cuspidal. By surjectivity in the section conjecture, there is an element x of $X(k)$ mapping to s. It has the same image as P under $\prod_v \kappa_{X/k_v}$. But Theorem 5.3.24 tells us that κ_S is injective, so P' equals x at all $v \in S$. Thus, P equals x and therefore lies in $X(k)$. □

This now allows us to answer Question 4.3.2.

Corollary 5.3.27. *Conjecture 5.3.25 implies that the property $A(\mathrm{f-cov}, S, k)$ holds for any number field k and nonempty set S of finite places.*

Proof. By Conjecture 5.3.25 and Theorem 5.3.24, $\mathbb{P}^1 \setminus \{0,1,\infty\}$ satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for all finite places v of k (which holds for a compatible choice of cuspidal data by Lemma 5.3.19). Proposition 5.3.26 then implies that $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is VSA for $(\mathrm{f}-\mathrm{cov}, S, k)$. By Theorem 5.1.1, $A(\mathrm{f}-\mathrm{cov}, S, k)$ holds. □

5.4 Finite Abelian Descent for all k

Let k be a number field. Lemma 5.2.1 states that the only reason tori do not satisfy VSA for $(\mathrm{f}-\mathrm{ab}, f, k)$ is that the rational points are not closed in the adelic points. The following result therefore suggests that $\mathbb{P}^1 \setminus \{0,1,\infty\}$ might play the role of tori over any number field k:

Proposition 5.4.1. *Let $X = \mathbb{P}^1 \setminus \{0,1,\infty\}_k$. The set $X(k)$ is closed in $X(\mathbb{A}_k^f)$.*

Proof. Let $\alpha \in X(\mathbb{A}_k^f) \setminus X(k)$. We find a neighborhood of α that does not intersect $X(k)$.

There must be some finite set S of places of k for which $\alpha \in U_S := \prod_{v\in S} X(k_v) \times \prod_{v\notin S} X(\mathcal{O}_v)$. But

$$X(k) \cap U_S = X(\mathcal{O}_{k,S})$$

is finite by Siegel's Theorem, so $U_S \setminus X(k)$ is an open set containing α. □

This leads us to conjecture:

Conjecture 5.4.2. $\mathbb{P}^1\setminus\{0,1,\infty\}_k$ *is VSA for* $(\mathrm{f}-\mathrm{ab}, f, k)$*, so* $A(\mathrm{f}-\mathrm{ab}, f, k)$ *for any number field k.*

In fact, a weaker version of this follows from a conjecture of Harari and Voloch:

Proposition 5.4.3. *Conjecture 2 of [HV10] implies that the answer to Question 4.3.1 is yes for the finite abelian descent obstruction over any number field k.*

Proof. By the proof of Theorem 5.1.3, we can decompose any variety into affine open subsets that embed in some power of $(\mathbb{P}^1 \setminus \{0,1,\infty\})_k$. Therefore, it suffices to prove the result for

U/k for which there exists $f\colon U \to X^n$ for some positive integer n and $X = (\mathbb{P}^1 \setminus \{0,1,\infty\})_k$. Suppose that $U(\mathbb{A}_k)^{\mathrm{f-ab}}$ is nonempty; we wish to show that U has a k-point.

Let $\alpha \in U(\mathbb{A}_k)^{\mathrm{f-ab}}$. We want to show that each coordinate of $f(\alpha)$ is in $X(\mathbb{A}_k)^{\mathrm{Br}}$. If X were projective, this would follow from Corollary 7.3 of [Sto07]. But X is not, so we must instead prove this using the étale homotopy obstruction of [HS13].

We $f(\alpha) \in X^n(\mathbb{A}_k)^{\mathrm{f-ab}} = X^n(\mathbb{A}_k)^{\mathbb{Z}h,1}$ by Theorem 9.103 of [HS13]. It follows by Lemma 4.2.10 that every coordinate $f_i(\alpha)$ $(1 \le i \le n)$ of $f(\alpha)$ is in $X(\mathbb{A}_k)^{\mathbb{Z}h,1}$. As X^s is connected, and $H_2(X^s) = 0$, we have $X(\mathbb{A}_k)^{\mathbb{Z}h,1} = X(\mathbb{A}_k)^{\mathbb{Z}h}$ by Theorem 9.152 of [HS13]. Theorem 9.116 of [HS13] tells us that $X(\mathbb{A}_k)^{\mathbb{Z}h} = X(\mathbb{A}_k)^{\mathrm{Br}}$. Therefore, $f_i(\alpha) \in X(\mathbb{A}_k)^{\mathrm{Br}}$. In particular, it pairs to 0 with every element of $\mathrm{Br}(X)$.

Let S be a finite set of places of k large enough so that X and f are defined over $\mathcal{O}_{k,S}$, and $f(\alpha)$ is integral over $\mathcal{O}_{k,S}$. Let $\mathcal{X} = (\mathbb{P}^1 \setminus \{0,1,\infty\})_{\mathcal{O}_{k,S}}$. By forgetting the v-components for $v \in S$, we get a point $f_i(\alpha)^S$ of $X(\mathbb{A}_k^S)^{\mathrm{f-ab}} \cap \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v)$ for each i. The pairing of any element of $B_S(X)$ (in the notation of [HV10]) with $f_i(\alpha)^S$ is the same as its pairing with $f_i(\alpha)$, for elements of $B_S(X)$ pair trivially with every element of $X(k_v)$; in particular, $f_i(\alpha)^S$ is orthogonal to all of $B_S(X)$. But the conjecture implies that $f_i(\alpha)^S$ comes from an element of $\mathcal{X}(\mathcal{O}_{k,S})$. It follows that $f(\alpha)_v$ is in $X^n(k)$ for every $v \notin S$. Thus for any $v \notin S$, we find that $f(\alpha)_v \in f(U(k))$. □

Remark 5.4.4. It does not seem that Conjecture 2 of [HV10] implies Conjecture 5.4.2, for the former requires using only integral points, which in turn means that we must throw out a different set S of primes depending on the point in question.

Chapter 6

The Fibration Method

In this chapter, we describe an alternative proof of a positive answer to Question 4.3.2 for $(\mathrm{f}-\mathrm{cov}, f, k)$ that depends on the section conjecture but works over any number field k. The proof essentially uses the fibration method, due originally to M. Artin and first used to prove many important properties of étale cohomology. We begin by reformulating the section conjecture in a form that works better in fibrations. We also prove some important auxiliary results on very strong approximation in fibration sequences (Proposition 6.2.10), which will be useful in Chapter 7, and on cuspidal sections for higher dimensional varieties (Theorem 6.4.6).

6.1 The Homotopy Section Conjecture

We will now reformulate the section conjecture in terms of homotopy fixed points before showing what happens to the section conjecture in fibration sequences in Section 6.2. There are three main reasons why homotopy fixed points work better than fundamental group sections for use in fibration sequences: (1) If the base has a nontrivial π_2, the ordinary section conjecture does not hold inductively in elementary fibrations (2) by working with homotopy types instead of fundamental groups, one need not deal with basepoints (3) the homotopy obstruction deals with connected components when the variety is not geometrically connected.

However, in most of our applications, the space is an étale $K(\pi, 1)$, making (1) irrelevant. This begs the question why we did not use profinite groupoids instead of homotopy types, which would adequately deal with (2) while allowing us to dispose of Propositions 6.1.16 and 6.3.10. Aside from the natural inclination to prove results in the most general setup possible, we justify this choice by the fact that we also found it easier to cite references on homotopy fixed points of profinite spaces than of profinite groupoids.

Unless stated otherwise, we let k denote an arbitrary field. Let $\mathcal{S}$ denote the category of simplicial sets.

Definition 6.1.1. The category $\hat{\mathcal{S}}$ of *profinite spaces* is the category of simplicial objects in the category of profinite sets.

Definition 6.1.2. In Definition 4.4 of [Fri82], Friedlander defines the étale topological type $Et(X)$ of a scheme X, which is an object of $\mathrm{Pro}(\mathcal{S})$ and represents the étale homotopy type of Artin-Mazur ([AM69]). This is functorial in the scheme X.

Definition 6.1.3. In Section 2.7 of [Qui08], Quick defines the profinite completion functor $\mathrm{Pro}(\mathcal{S}) \to \hat{\mathcal{S}}$ and defines $\hat{E}t(X)$ to be the profinite completion of $Et(X)$. The profinite completion of pro-spaces is defined by first taking profinite completion level-wise to get a pro-object in the category of profinite spaces and then taking the inverse limit in the category of profinite spaces.

The cohomology of this profinite space with finite coefficients recovers the étale cohomology of X, and the fundamental group of this space is the profinite étale fundamental group when X is geometrically unibranch.

Definition 6.1.4. If X is a k-scheme, the *homotopy fixed points* $\hat{E}t(X^s)^{hG_k}$ (Definition 2.22 of [Qui11]) are defined as follows. If G is a profinite group, one may use the bar construction to define EG and BG as profinite spaces. Then $\hat{E}t(X^s)^{hG_k}$ is the space of G_k-equivariant maps from EG_k to $\hat{E}t(X^s)$. (Compare with 9.3.2 of [HS13], 8.2 of [BS15], and introduction to [Pál15]). By the discussion immediately following Proposition 2.14 of [Qui08], this has the structure of a simplicial set. As is $\hat{E}t(X^s)$, this is functorial in the k-scheme X.

Definition 6.1.5. As in [HS13], [BS15], and [Pál15], we denote $\pi_0(\hat{E}t(X^s)^{hG_k})$ by $X(hk)$ and refer to it as the *homotopy sections* of X/k. This is again functorial in the k-scheme X.

If k'/k is a field extension and X a k-scheme, $X(hk')$ still denotes $\pi_0(\hat{E}t(X^s)^{hG_{k'}})$.

Definition 6.1.6. The *homotopy profinite Kummer map* $\kappa_{X/k}\colon X(k) \to X(hk)$ of Section 3.2 of [Qui11] is defined as follows. The set $\operatorname{Spec} k(hk)$ has one element, so by functoriality, an element of $X(k)$ gives rise to an element of $X(hk)$. Compare with 9.3.2 of [HS13], 8.2 of [BS15], and the introduction to [Pál15]. We write κ_X or even κ when k, X are understood. We also write κ_S, $\kappa_{X/\mathbb{A}_{k,S}}$, and $\kappa_{X/\mathbb{A}_k}$ as in the case of Definition 5.3.4.

The section conjecture has been reformulated in homotopy-theoretic terms; see Section 3.2 of [Qui11] and Theorem 9.7(b) of [Pál15], or Section 2.6 of [Sti13] for a summary. However, these sources lack an analogue of cuspidal sections, without which one can only restrict to projective curves or conjecture something like the "homotopy section property" (HSP) of [Pál15], a local-to-global statement. We will fix this gap. In fact, HSP follows from our versions of the section conjecture (although the former is expected to hold more generally), as proven below in Lemma 6.2.9. We now define the analogue of cuspidal sections:

Definition 6.1.7 (Analogue of Definition 5.3.9)**.** A *homotopy cuspidal datum* (X, C, k) is a (smooth) variety X over k and a subset $C \subseteq X(hk)$, called the set of *cuspidal fixed points.*

Remark 6.1.8. In practice (e.g. to reprove Corollary 5.3.27), we will only use cuspidal fixed points for varieties of dimension > 1 when they arise from good neighborhoods as in Definition 6.3.11. However, in [Cor17], we begin the study of a general notion of cuspidal sections for higher dimensional varieties similar to Definition 5.3.18.

Definition 6.1.9 (Analogue of Defintion 5.3.10)**.** A homotopy cuspidal datum (X, C, k) is said to satisfy the *surjectivity in the homotopy section conjecture* if $X(hk) \setminus C \subseteq \kappa(X(k))$.

Definition 6.1.10 (Analogue of Definition 5.3.13)**.** A homotopy cuspidal datum (X, C, k) is said to satisfy the *injectivity in the homotopy section conjecture* if

1. The map $\kappa\colon X(k) \to X(hk)$ is injective.

2. The sets $\kappa(X(k))$ and C are disjoint.

3. Every connected component of $\hat{E}t(X^s)^{hG_k}$ in the image of the Kummer map is simply connected.

6.1.1 Basepoints and Homotopy Groups

While the statements of the homotopical versions of the section conjecture avoid the use of basepoints, we sometimes need to use them. In particular, in order to discuss homotopy groups, we must discuss basepoints for $\hat{E}t(X)$.

On p.37 of [Fri82], it is stated that a geometric point a of X gives rise to a basepoint of $Et(X)$, and this then gives rise to a basepoint of $\hat{E}t(X)$, which we call b_a.

If this geometric point is a rational point, it is invariant under G_k, so our basepoint of $\hat{E}t(X)$ is invariant under G_k, i.e. we have a pointed G_k-space. Sometimes X does not have any rational points, and yet we want a G_k-invariant basepoint. This is provided by the following construction:

Construction 6.1.11. Let $a \in X(hk)$. This is a homotopy class of G_k-equivariant maps $EG_k \to \hat{E}t(X^s)$. By taking a homotopy pushout of this map along the map from EG_k to a point, we obtained a new model of $\hat{E}t(X^s)$ as a profinite space with G_k-action with a G_k-invariant basepoint. Any such basepoint is denoted by b_a.

Definition 6.1.12. In Definition 2.15 of [Qui08] and Definition 2.12 of [Qui11], Quick defines the homotopy groups of a pointed profinite space. If X is a scheme, and a is a geometric point or a homotopy section of X, we denote $\pi_i(\hat{E}t(X), b_a)$ by $\pi_i^{\acute{e}t}(X, a)$ or simply $\pi_i(X, a)$ (c.f. Notation and Conventions). If a is a Galois-invariant geometric point or a homotopy section of a scheme X/k, this basepoint is fixed by the action of G_k, so there is a natural action of G_k on $\pi_i(X^s, a)$.

Remark 6.1.13. Suppose that X is geometrically unibranch (e.g., normal), Noetherian, and connected. By Proposition 2.33 of [Qui08], the homotopy groups of $\hat{E}t(X)$ are isomorphic (as pro-groups) to the pro-homotopy groups of the profinite completion of the Artin-Mazur homotopy type of X. By Theorem 11.2 of [AM69], this profinite completion is already isomorphic to the (non-completed) Artin-Mazur homotopy type as a pro-homotopy type. But $Et(X)$ represents the Artin-Mazur homotopy type (c.f. Corollary 6.3 of [Fri82]), so its homotopy groups are isomorphic to all of the above.

More generally, if X is not assumed to be connected (but still Noetherian), then it is a finite disjoint union of connected schemes. Thus the same result holds, as all constructions behave well with respect to finite coproducts.

Remark 6.1.14. If X is connected, then $\pi_0(\hat{E}t(X))$ contains one element, so the homotopy groups are independent of the choice of basepoint up to isomorphism. This isomorphism is, however, only unique modulo the action of the fundamental group. This is also a subtlety in using different models for $\hat{E}t(X^s)$, as in Construction 6.1.11. Furthermore, any two basepoints of $\hat{E}t(X^s)$ arising from the same element of $X(hk)$ are homotopic in the G_k-equivariant category, so the resulting actions on $\pi_i(X^s)$ are equivalent. This latter fact is important in the statement of Theorem 6.1.18. The reader may check that this is does not pose a problem in any of the theorems we prove.

Definition 6.1.15. A k-variety X has the *étale $K(\pi, 1)$-property* or *is an étale $K(\pi, 1)$* if X is geometrically connected, and $\pi_i(X^s, a) = 0$ for $i \geq 2$ and some (equivalently, any) geometric point a.

Proposition 6.1.16. *Let X be a geometrically connected variety over k. Then there is a map $\mathrm{Trunc}_1 \colon X(hk) \to \mathscr{S}_{\pi_1(X/k)}$ (functorial in X) whose composition with the homotopy profinite Kummer map is the ordinary profinite Kummer map. Furthermore, if X is an étale $K(\pi, 1)$, the following are true:*

- *The map Trunc_1 is a bijection*
- *A cuspidal datum (X, C, k) is the same as a homotopy cuspidal datum (X, C, k)*
- *X satisfies the surjectivity in the section conjecture (5.3.10) if and only if it satisfies the surjectivity in the homotopy section conjecture (6.1.9)*

- *X satisfies the injectivity in the section conjecture (5.3.13) if and only if it satisfies the injectivity in the homotopy section conjecture (6.1.10)*

Remark 6.1.17. Related facts are discussed in Section 3.2 of [Qui11], Proposition 12.6(b) of [Pál15], and Section 2.6 of [Sti13], especially Equation (2.13).

Before proving Proposition 6.1.16, we must introduce the descent spectral sequence for homotopy fixed points. It is essentially Theorem 2.16 of [Qui11] for the pointed profinite G_k-space $\hat{Et}(X^s)$:

Theorem 6.1.18 (Theorem 2.16 of [Qui11])**.** *For each $a \in X(hk)$, there is a spectral sequence*

$$E_2^{p,q} = H^p(G_k; \pi_q(X^s, a)) \Rightarrow \pi_{q-p}(\hat{Et}(X^s)^{hG_k}, a),$$

known as the descent spectral sequence*. The basepoint a of $\hat{Et}(X^s)^{hG_k}$ naturally results from the G_k-invariant basepoint b_a of Construction 6.1.11, and the G_k-action on $\pi_q(X^s, a)$ is well-defined by Remark 6.1.14.*

Proof of Proposition 6.1.16. If $X(hk)$ is empty, then the map Trunc_1 automatically exists, and it is functorial because there is a unique map from an empty set. It follows that Trunc_1 is automatically a bijection. In this case, $X(k)$ and C are empty, so the surjectivity and injectivity in either version of the section conjecture always hold.

We now suppose that there is a homotopy fixed point $a \in X(hk)$. As $\pi_0(X_s, a)$ and hence $H^0(G_k; \pi_0(X^s, a))$ is trivial, the spectral sequence produces a map $X(hk) \to \mathscr{S}_{\pi_1(X/k)}$.

When $X = \operatorname{Spec} k$, this map is clearly an isomorphism. Functoriality of the spectral sequence and the definition of the Kummer map imply that for arbitrary X, the map $X(hk) \to \mathscr{S}_{\pi_1(X/k)}$ respects the Kummer map.

We now suppose that X is an étale $K(\pi, 1)$. As $\pi_i(X^s)$ contains one element for $i \neq 1$, the spectral sequence tells us that $\pi_{1-p}(\hat{Et}(X^s)^{hG_k}, a) \cong H^p(G_k; \pi_1(X^s, a))$. For $p = 1$, this tells us that Trunc_1 is a bijection. (Alternatively, we could avoid the spectral sequence by noting that an étale $K(\pi, 1)$ is the classifying space of its fundamental group and then applying Proposition 2.9 of [Qui15].)

The second bullet-point of the proposition follows from the first. The third bullet-point follows from the compatibility of the Kummer maps, as do the first two parts of injectivity (the fourth bullet-point).

For third part of injectivity, we suppose that a is in the image of the Kummer map. Then the descent spectral sequence tells us that $\pi_1(\hat{E}t(X^s)^{hG_k}, a) \cong H^0(G_k; \pi_1(X^s, a))$. The group $\pi_1(X^s, a)$ has G_k-action arising from the section a, so $H^0(G_k; \pi_1(X^s, a))$ is simply the centralizer of $\mathrm{Trunc}_1(a)$.

□

Proposition 6.1.19. *Let X be a finite scheme over k. Suppose that X_{red} is geometrically reduced (e.g., k is perfect). Then X satisfies the injectivity and surjectivity in the homotopy section conjecture.*

Proof. We first suppose that X is reduced. Such a scheme must be a disjoint union $\amalg_i \mathrm{Spec}(k_i)$, where k_i/k is a finite separable extension. Then $X^s = \amalg_i \amalg_{j=1}^{[k_i:k]} \mathrm{Spec}(k_s)$, and G_k acts transitively on $\amalg_{j=1}^{[k_i:k]} \mathrm{Spec}(k_s)$ for each i. Thus $\pi_0(X^s)^{G_k}$ is the set of all i for which $k_i = k$, which is the same as $X(k)$. But $\pi_n(X^s)$ is trivial for $n \geq 1$, so by Theorem 6.1.18, this is also $X(hk)$. We have $X(k) = X(hk)$, and it is easy to check that this equality comes from the Kummer map, so we are done in this case.

We now reduce to the reduced case by considering the map $X_{\mathrm{red}} \to X$. This map is a universal homeomorphism, and it remains so after base change to k_s, so the map $X^s_{\mathrm{red}} \to X^s$ induces an equivalence of étale sites. It follows that $\hat{E}t(X^s_{\mathrm{red}}) \to \hat{E}t(X^s)$ is an equivalence, so the induced map $X_{red}(hk) \to X(hk)$ is an isomorphism. Furthermore, the map $X_{\mathrm{red}}(k) \to X(k)$ is an isomorphism, so we are done. □

6.1.2 Obstructions to the Hasse Principle

Proposition 6.1.20 (Analogue of Proposition 5.3.15)**.** *If k'/k is an extension of fields of characteristic* 0*, there is a base change map*

$$X(hk) \to X(hk')$$

Furthermore, the diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(k') \\ {\scriptstyle \kappa_{X/k}}\downarrow & & \downarrow{\scriptstyle \kappa_{X/k'}} \\ X(hk) & \longrightarrow & X(hk') \end{array}$$

is commutative.

Proof. Proposition 5.4 in [Pál15] says that algebraically closed extension of base field in characteristic 0 does not change the étale homotopy type. The result is then clear because a map of groups gives a map on homotopy fixed points.

□

Let X be a scheme over a field k. Then $\hat{Et}(X^s)$ can be represented as a pro-space $\{X_\alpha\}_\alpha$ with G_k-action, with each X_α π-finite (c.f. the proof of Lemma 6.2.4). As $\hat{Et}(X^s) = \operatorname{holim}_\alpha X_\alpha$, and homotopy limits commute with homotopy fixed points, we have $X(hk) = \pi_0(\operatorname{holim}_\alpha X_\alpha^{hG_k})$.

We now recall that for a number field k there is an étale homotopy obstruction to rational points defined on p.314 of [HS13], denoted by $X(\mathbb{A}_k)^h$. These are defined using another version of homotopy fixed points ([HS13], Definition 3.3), which we denote by $X(hk)_{HS}$. For a pro-space $\{X_\alpha\}$ with G_k-action, it is given by taking $\lim_\alpha \pi_0(X_\alpha^{hG_k})$ (while [HS13] uses the $X_\alpha^{hG_k}$ of [Goe95], it is easy to see that this is the same as that of [Qui11] by considering the descent spectral sequence). There is a natural map $X(hk) \to X(hk)_{HS}$, giving us a diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ {\scriptstyle \kappa_{X/k}}\downarrow & & \downarrow{\scriptstyle \kappa_{X/\mathbb{A}_k}} \\ X(hk) & \xrightarrow{\text{loc}} & \prod_v X(hk_v) \\ \downarrow & & \downarrow \\ X(hk)_{HS} & \xrightarrow{\text{loc}} & \prod_v X(hk_v)_{HS}, \end{array}$$

Then $X(\mathbb{A}_k)^h$ is the subset of the upper-right object of this diagram whose image in the lower-right object is contained in the image of the bottom horizontal arrow.

Lemma 6.1.21. *The map $X(hk) \to X(hk)_{HS}$ is surjective, and the map $X(hk_v) \to X(hk_v)_{HS}$ is an isomorphism for each v.*

Proof. As X is defined over a number field, we can express $\hat{Et}(X^s)$ as a pro-space $\{X_\alpha\}$, where α ranges over a countable cofiltering category. It follows that we may replace it by a tower, so we may apply Proposition VI.2.15 of [GJ09] to get an exact sequence

$$* \to \lim_{\alpha}{}^1 \pi_1 X_\alpha^{hG_k} \to X(hk) \to X(hk)_{HS} \to *$$

This implies the desired surjectivity.

Replacing k by k_v, we note that we can assume (by Theorem 11.2 of [AM69]) that each X_α has finite homotopy groups. As G_{k_v} is topologically finitely generated, this implies by Theorem 6.1.18 that $\pi_1 X_\alpha^{hG_k}$ is finite for each α. But this implies that our $\lim_{\alpha}{}^1$ vanishes, so we get the desired isomorphism. □

Proposition 6.1.22. *In the notation of the previous diagram, $X(\mathbb{A}_k)^h$ is $\kappa_{X/\mathbb{A}_k}^{-1}(\mathrm{Im}(\mathrm{loc}))$.*

Proof. This follows immediately from Lemma 6.1.21 and the definition of $X(\mathbb{A}_k)^h$. □

Proposition 6.1.23 (Analogue of Proposition 5.3.26)**.** *Let k be a number field, S a nonempty set of places of k, and X a smooth k-variety. If X satisfies the surjectivity in the homotopy section conjecture over k and the injectivity in the homotopy section conjecture over k_v for all $v \in S$, then it is VSA for (h, S, k). If X is geometrically connected, then the same is true for $(\text{ét} - \mathrm{Br}, S, k)$.*

Proof. Suppose $\alpha \in X(\mathbb{A}_{k,S})^h$ is the projection of $\alpha' \in X(\mathbb{A}_k)^h$. By the same argument as in Proposition 5.3.26, we find that α' comes from a rational point of X at every $v \in S$, i.e. α is a rational point. The last statement follows from Theorem 9.1 of [HS13]. □

6.2 Homotopy Section Conjecture in Fibrations

In this section, "Kummer map," "cuspidal datum," and "section conjecture " will mean the homotopy versions as explained in Section 6.1. Of course, for an étale $K(\pi, 1)$, Proposition 6.1.16 says that these are equivalent to the corresponding definitions in Section 5.3.1.

Definition 6.2.1 (Definition 1.1 of [Fri82]). A *special geometric fibration* is a morphism $f\colon X \to S$ of schemes fitting into a diagram:

$$\begin{array}{ccccc} X & \overset{j}{\hookrightarrow} & \overline{X} & \overset{i}{\hookleftarrow} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}} & \swarrow{\scriptstyle g} & \\ & & S & & \end{array}$$

satisfying the following conditions:

1. i is a closed embedding

2. j is a open immersion which is dense in every fiber of $\overline{f}$, and $X = \overline{X} - Y$

3. $\bar{f}$ is smooth and proper

4. Y is a union of schemes $Y_1, \cdots, Y_m$, with Y_i of pure relative codimension c_i in $\overline{X}$ over S, with the property that every intersection $Y_{i_1} \cap \cdots \cap Y_{i_k}$ is smooth over S of pure codimension $c_{i_1} + \cdots + c_{i_k}$.

More generally, a *geometric fibration* is a map $f\colon X \to S$ admitting a Zariski covering $\{V_i \to S\}$ such that $f_{V_i}\colon X \times_S V_i \to V_i$ is a special geometric fibration for all i.

Lemma 6.2.2. *Let $f : X \to S$ be a geometric fibration and $U \to S$ a morphism of schemes. Then the base change*

$$f_U : X \times_S U \to U$$

is a geometric fibration.

Proof. This is clear because all of the properties in Definition 6.2.1 are stable under base extension. □

Definition 6.2.3. Let k be a field and S a k-scheme. Let $f\colon X \to S$ be a geometric fibration. Suppose we are given cuspidal data (S, C_S, k) for S and (X_s, C_{X_s}, k) for the fiber X_s above every $s \in S(k)$. Then we define the *cuspidal datum* $(X, C_{X,f}, k)$ *induced by* f as follows.

For $s \in S(k)$, let X_s denote the fiber of f above s and $\iota_s \colon X_s \to X$ the inclusion. The map f induces a map $f : X(hk) \to S(hk)$, and the map ι_s induces a map $\iota_s \colon X_s(hk) \to X(hk)$. We define $C_{X,f}$ by

$$C_{X,f} = f^{-1}(C_S) \bigcup \left(\bigcup_{s \in S(k)} \iota_s(C_{X_s}) \right) \subseteq X(hk).$$

In particular, if C_S and C_{X_s} are all empty, which is typical for smooth proper curves, then $C_{X,f}$ is empty.

Lemma 6.2.4. *Let $f \colon X \to S$ be a map of k-schemes. Suppose that $f \colon X \to S$ is a geometric fibration of Noetherian normal schemes with connected geometric fibers, and k has characteristic* 0*. Then the sequence*

$$\hat{E}t(X_s^s) \to \hat{E}t(X^s) \to \hat{E}t(S^s)$$

is a fibration sequence of profinite etale homotopy types (in the model structure of [Qui08]).

Proof. It follows by Theorem 11.5 of [Fri82] (also c.f. Theorem 3.7 of [Fri73]) that the sequence $\hat{E}t(X_s) \to \hat{E}t(X) \to \hat{E}t(S)$ is a fibration sequence, or equivalently that

$$\begin{array}{ccc} \hat{E}t(X_s) & \longrightarrow & \hat{E}t(X) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \hat{E}t(S) \end{array}$$

is a homotopy pullback diagram. We need to show that the profinite completion of this homotopy pullback diagram remains a homotopy pullback diagram.

We may assume this is a diagram of fibrant cofibrant objects in $\mathrm{Pro}(\mathcal{S})$, with the Isaksen model structure. By Theorem 11.2 of [AM69], each is isomorphic to a levelwise π-finite space (after possibly replacing each by its Postnikov tower, which is invariant under Quick's profinite completion). A π-finite space is fibrant in $L_{K^\pi} \mathrm{Pro}(\mathcal{S})$, in the notation of Chapter 7 of [BHH17], by Proposition 7.2.10 of loc.cit. As a limit of fibrant objects, it is itself fibrant, i.e., this is a diagram of objects fibrant for $L_{K^\pi} \mathrm{Pro}(\mathcal{S})$. In particular, this means that it is in the image of the inclusion $(L_{K^\pi} \operatorname{Pro}(\mathcal{S}))_\infty \hookrightarrow \operatorname{Pro}(\mathcal{S})_\infty$ of infinity categories, which is induced by the identity functor on model categories. This inclusion of infinity categories is conservative and preserves limits, so it detects limits. In particular, the above diagram is a pullback diagram in $(L_{K_\pi} \operatorname{Pro}(S))_\infty$, or equivalently, a homotopy pullback diagram in the model category $L_{K_\pi} \operatorname{Pro}(S)$. Therefore, by Theorem 7.4.8 of [BHH17], applying Ψ_{K_π} to this diagram yields a homotopy pullback diagram in $\operatorname{Pro}(S_\tau)$, which by Proposition 7.4.1 of the same paper is the category $\hat{S}$ of profinite spaces, with Quick's model structure.

We are done if we can identify the map $\Psi = \Psi_{K^\pi}$ (these are the same functor on ordinary categories, just refer to different model structures) with the profinite completion functor of Definition 6.1.3. There is an adjunction $\Psi \colon \operatorname{Pro}\mathcal{S} \rightleftarrows \hat{S} : \Phi$ as well as an adjunction $\mathrm{const} \colon \mathcal{S} \rightleftarrows \operatorname{Pro}(\mathcal{S}) : \lim$. By construction, $\lim \circ \Phi$ takes a profinite space X to an object of $\operatorname{Pro}(\mathcal{S}_\tau)$ whose limit is X, then takes the limit of this pro-object as a pro-space. In particular, this is the forgetful functor from profinite spaces to simplicial sets of p.587 of [Qui08], and therefore $\Psi \circ \mathrm{const}$ is Quick's profinite completion from $\mathcal{S}$ to $\hat{\mathcal{S}}$.

It remains to show that Ψ can be obtained by applying Ψ levelwise and taking the limit. We therefore wish to show that Ψ preserves filtered limits. For this, let $\{X_j\}$ be a filtered system in $\operatorname{Pro}(\mathcal{S})$. For T an object of $\mathcal{S}_\tau$, we have $\operatorname{Hom}(\lim_j \Psi(X_j), T) = \operatorname{colim}_j \operatorname{Hom}(\Psi(X_j), T)$ because T is cocompact. But this latter is $\operatorname{colim}_j \operatorname{Hom}(X_j, \Phi(T))$, which in turn is $\operatorname{Hom}(\lim_j X_j, \Phi(T)) = \operatorname{Hom}(\Psi(\lim_j X_j), T)$ because $\phi(T)$ is in $\mathcal{S}$ and therefore cocompact. As $\mathcal{S}_\tau$ cogenerates $\operatorname{Pro}(\mathcal{S}_\tau)$, this shows that $\lim_j \Psi(X_j) = \Psi(\lim_j X_j)$ in $\operatorname{Pro}(\mathcal{S}_\tau)$, and we are done. $\square$

Lemma 6.2.5. *Let $f \colon X \to S$ be a map of k-schemes. Let $a \in X(hk)$ and $s \in S(k)$ such that $f(a) = \kappa(s)$. Let X_s be the fiber of f above s (a geometrically connected variety). Suppose that the sequence $\hat{E}t(X_s^s) \to \hat{E}t(X^s) \to \hat{E}t(S^s)$ is a fibration sequence of profinite etale homotopy types (e.g., by Lemma 6.2.2v and Lemma 6.2.4, if f is a geometric fibration of Noetherian normal schemes).*

Then a comes from an element of $X_s(hk)$, also denoted by a, and there is a long exact *sequence*

$$\pi_1(\hat{E}t(X_s^s)^{hG_k}, a) \to \pi_1(\hat{E}t(X^s)^{hG_k}, a) \to \pi_1(\hat{E}t(S^s)^{hG_k}, \kappa(s)) \to X_s(hk) \to X(hk) \to S(hk),$$

where the latter three are pointed sets based at a and $\kappa(s)$.

Proof. As homotopy fixed points commutes with homotopy limits, the sequence $\hat{E}t(X_s^s)^{hG_k} \to \hat{E}t(X^s)^{hG_k} \to \hat{E}t(S^s)^{hG_k}$ is a fibration sequence of simplicial sets. This implies that we get an element of $X_s(hk)$ mapping to a.

We then choose a basepoint b_a of $\hat{E}t(X_s^s)^{hG_k}$ in the connected component represented by $a \in X_s(hk)$, which makes $\hat{E}t(X_s^s)^{hG_k} \to \hat{E}t(X^s)^{hG_k} \to \hat{E}t(S^s)^{hG_k}$ a fibration sequence of pointed simplicial sets. The exact sequence then follows by the long exact sequence of homotopy groups for a pointed fibration. □

Theorem 6.2.6. *Let $f\colon X \to S$ satisfy the hypotheses of Lemma 6.2.5. If S and the fibers of f above k-points of S satisfy the injectivity in the section conjecture, then so does X.*

Proof. We refer to 1, 2, and 3 of Definition 6.1.10.

1 for X: Let $a, b \in X(k)$, and suppose $\kappa_X(a) = \kappa_X(b)$. This implies $f(\kappa(a)) = f(\kappa(b))$, so by 1 for S, $f(a) = f(b)$. Let $s = f(a)$. Then a, b are both in $X_s(k)$. By Lemma 6.2.5 and 3 for S, the map $\iota_s\colon X_s(hk) \to X(hk)$ is injective. Thus $\kappa_{X_s}(a) = \kappa_{X_s}(b)$, so 1 for X_s tells us that $a = b$.

2 for X: Let $a \in X(k)$ and $s = f(a)$. Then $f(\kappa_X(a)) = \kappa_S(s)$, so it is not in C_S by 2 for S, hence not in $f^{-1}(C_S)$. As in the previous paragraph, the map $\iota_s\colon X_s(hk) \to X(hk)$ is injective. As $a \in X_s(k)$, $\kappa_{X_s}(a) \notin C_{X_s}$ by 2 for X_s, so $\kappa_X(a) \notin \iota_s(C_{X_s})$. Suppose $s' \in S(k) \setminus \{s\}$. Then any element of $\iota_{s'}(F_{s'}(hk))$, in particular any element of $\iota_{s'}(C_{F_{s'}})$, maps to $\kappa_S(s')$ under f. By 1 for S, $\kappa_S(s') \neq \kappa_S(s)$, so $\kappa_X(a) \notin \iota_{s'}(C_{F_{s'}})$. Together, this implies that $\kappa(a) \notin C_{X,f}$.

3 for X: Let $a \in X(k)$ and $s = f(a)$, so $s \in X_s(k)$. We choose a basepoint of $\hat{E}t(X_s^s)^{hG_k}$ in the connected component $\kappa_{X_s}(a)$. Then 3 for X_s and S and Lemma 6.2.5 tell us that $\pi_1(\hat{E}t(X^s)^{hG_k})$ is trivial on the connected component $\kappa_X(a)$. □

Theorem 6.2.7. *Let $f\colon X \to S$ satisfy the hypotheses of Lemma 6.2.5. If S and the fibers of f above k-points of S satisfy the surjectivity in the section conjecture, then so does X.*

Proof. Let $a \in X(hk) \setminus C_{X,f}$. Then $f(a) \notin C_S$, so $f(a) = \kappa(s)$ for $s \in S(k)$. Then Lemma 6.2.5 gives us a long exact sequence. Thus a comes from $X_s(hk)$ and is not in C_{X_s}. By the surjectivity in the section conjecture for X_s, it comes from a k-point of X_s. □

We finally prove a general result about VSA for the étale homotopy obstruction that will be useful in Chapter 7. Before proving it, we need one definition.

Definition 6.2.8 (Notation 12 of [Pál15])**.** If X is a scheme over a number field k, the Selmer set $\mathrm{Sel}(X/k)$ is the subset of $X(hk)$ whose image under loc is in the image of $X(\mathbb{A}_k)$ in the following diagram:

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ \downarrow & & \downarrow \\ X(hk) & \xrightarrow{\text{loc}} & \prod_v X(hk_v), \end{array}$$

Clearly $\kappa(X(k)) \subseteq \mathrm{Sel}(X/k)$.

Lemma 6.2.9. *Suppose X is a variety over a number field k that satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for some place v of k. Then $\kappa(X(k)) = \mathrm{Sel}(X/k)$.*

Proof. As mentioned in Definition 6.2.8, $\kappa(X(k)) \subseteq \mathrm{Sel}(X/k)$. Now suppose $s \in \mathrm{Sel}(X/k)$. It suffices to show that s is not cuspidal. But if it were cuspidal, its base change to k_v would be cuspidal, contradicting injectivity over k_v, so it is not. □

Theorem 6.2.10. *Let $f\colon X \to S$ be a geometric fibration of smooth varieties over a number field k with connected geometric fibers, let T be a nonempty set of places of k. Suppose that the follows conditions hold:*

1. *S is VSA for (h, T, k).*

2. *One of the following is true:*

   (a) *S satisfies the surjectivity in the section conjecture over k and the injectivity in the section conjecture over k_v for some v.*

   (b) *The image of $S(k)$ in $S(hk)$ under κ is $\mathrm{Sel}(S/k)$, and S satisfies the injectivity in the section conjecture over k_v for some $v \in T$.*

   (c) *The localization map $\mathrm{loc}_T \colon S(hk) \to \prod_{v \in T} S(hk_v)$ is injective on $\mathrm{Sel}(S/k)$.*

3. *For all finite $v \notin T$, κ_{S/k_v} is injective (vacuous if $T = f$).*

4. *For all finite v, every connected component of $\hat{E}t(S^s)^{hG_{k_v}}$ in the image of the Kummer map is simply connected (e.g., by Proposition 104 of [Sti13] and Proposition 6.1.16, S is a smooth geometrically connected curve not isomorphic to $\mathbb{P}^1$).*

5. *For every real place v of k, every $a \in S(k)$, and every $b \in X_a(k_v)$, the map $\pi_1(X(k_v), b) \to \pi_1(S(k_v), a)$ is surjective and $\pi_1(X_a((k_v)_s))$ is trivial (where π_1 denotes the topological fundamental group under the v-adic topology).*

6. *For every $a \in S(k)$, the fiber X_a of f above a is VSA for (h, T, k).*

Then X is VSA for (h, T, k). One may replace (h, T, k) by $(\text{ét} - \mathrm{Br}, T, k)$ whenever the variety in question is smooth and geometrically connected.

Proof. We first note that 2a implies 2b, which implies 2c. The first implication is Lemma 6.2.9. For the second implication, suppose $\alpha, \beta \in \mathrm{Sel}\, S/k$ with $\mathrm{loc}_v(\alpha) = \mathrm{loc}_v(\beta)$. Then there are $a, b \in S(k)$ such that $\alpha = \kappa(a)$ and $\beta = \kappa(b)$. Therefore $a, b \in S(k_v)$ have the same image under κ_{X/k_v}, so injectivity tells us that $a = b$. This implies that $\alpha = \beta$, so 2c holds.

Now let $\alpha \in X(\mathbb{A}_k)^h$ project to $\alpha' \in X(\mathbb{A}_{k,S})$, and let $\beta \in X(hk)$ such that $\mathrm{loc}(\beta) = \kappa(\alpha)$. Then $\kappa(f(\alpha)) = \mathrm{loc}(f(\beta))$, so $f(\alpha) \in S(\mathbb{A}_k)^h$. By 1, there is $a \in S(k)$ such that $a = f(\alpha)_v$ for all $v \in T$.

For $v \in T$, we have $\mathrm{loc}_v(f(\beta)) = \kappa_{S/k_v}(f(\alpha)_v) = \kappa_{S/k_v}(a) = \mathrm{loc}_v(\kappa_{S/k}(a))$. Thus $\mathrm{loc}_T(f(\beta)) = \mathrm{loc}_T(\kappa_{S/k}(a))$. It's clear that $f(\beta), \kappa_{S/k}(a) \in \mathrm{Sel}(S/k)$, so 2c implies that $f(\beta) = \kappa_{S/k}(a)$.

This implies by Lemma 6.2.5 that there exists $\gamma \in X_a(hk)$ such that $i_a(\gamma) = \beta$, where i_a denotes the inclusion $X_a \hookrightarrow X$.

We now know that $\kappa_{S/k_v}(f(\alpha)_v) = \mathrm{loc}_v(\beta) = \kappa_{S/k_v}(a)$ for all v. By 3, we know that $a = f(\alpha)_v$ for all finite v. For v infinite, Theorem 1.2 of [Pál15] implies that a and $f(\alpha)_v$ are in the same connected component of $S(k_v)$. As f is an elementary fibration and therefore a fiber bundle for the v-adic topology, we can modify α so that $f(\alpha)_v = a$ and $\kappa(\alpha)$ remains the same. We do this at all infinite places, so that $f(\alpha) = a$, yet still $\kappa(\alpha) = \mathrm{loc}(\beta)$.

We now have $\alpha \in X_a(\mathbb{A}_k)$ such that $i_a(\mathrm{loc}(\gamma)) = i_a(\kappa_{X_a/\mathbb{A}_k}(\alpha))$. We need to show that $i_a : X_a(hk_v) \to X(hk_v)$ is injective for all v.

By Theorem 6.2.5, 4 implies injectivity for all finite v. At all complex v, the geometric connectedness assumption on X_a implies that $X_a(hk_v)$ is a point, so injectivity follows.

Let v be a real place. By Theorem 1.2 of [Pál10], the map $X_a(k_v)_\bullet \to X_a(hk_v)$ is a bijection, and by Theorem 1.2 of [Pál15], the map $X(k_v)_\bullet \to X(hk_v)$ is injective. It therefore suffices to prove that $X_a(k_v)_\bullet \to X(k_v)_\bullet$ is injective. But $X_a(k_v) \to X(k_v) \to S(k_v)$ is a fibration sequence of real analytic varieties, so the condition on topological fundamental groups guarantees the desired injectivity.

We conclude that $\mathrm{loc}(\gamma) = \kappa_{X_a/\mathbb{A}_k}(\alpha)$. We conclude by 6 that there is $c \in X_a(k)$ such that $c = \alpha'$.

□

6.3 Elementary Fibrations

This section gives us a tool for finding open subsets of arbitrary smooth geometrically connected varieties that satisfy the section conjecture.

The following definition is similar to one from [sga73].

Definition 6.3.1 (Definition 11.4 of [Fri82])**.** An *elementary fibration* is a geometric fibration whose fibers are geometrically connected affine curves.

Remark 6.3.2. Assuming f is a special geometric fibration, Y defines a relatively ample divisor of $\overline{X}$, which means that $\overline{f}$ is automatically projective.

Remark 6.3.3. In Definition 6.2.3, if $f\colon X \to S$ is an elementary fibration, it is understood unless mentioned otherwise that C_{X_s} is taken as in Definition 5.3.18.

Definition 6.3.4. In the notation of the previous definition, if the geometric fibers of f are hyperbolic (Definition 5.3.22), we say that f is a *hyperbolic elementary fibration.*

Lemma 6.3.5. *Let k be a perfect field. Then a map $X \to \operatorname{Spec} k$ is a (hyperbolic) elementary fibration if and only if X is a (hyperbolic) smooth geometrically irreducible curve.*

Proof. It is clear that if $X \to \operatorname{Spec} k$ is a (hyperbolic) elementary fibration then X is a (hyperbolic) smooth geometrically irreducible curve. To show the converse, note that every smooth curve X/k has a unique smooth compactification $\bar{X}$. We take $\bar{f} : \bar{X} \to \operatorname{Spec} k$ and $Y := \bar{X}\backslash X$ and note that Y is smooth and finite and thus étale over $\operatorname{Spec} k$. □

Lemma 6.3.6. *Let $f : X \to S$ be a (hyperbolic) elementary fibration and $U \to S$ a map of schemes. Then the base change*

$$f_U : X \times_S U \to U$$

is a (hyperbolic) elementary fibration.

Proof. This is an immediate corollary of Lemma 6.2.2 and the fact that every geometric fiber of f_U is also a geometric fiber of f. □

Lemma 6.3.7. *Let X, S be two smooth geometrically connected varieties over an infinite field k, and let $f : X \to S$ be an elementary fibration over k. Let x be a closed point of X. Then there exist nonempty open subsets $U_X \subset X, U_S \subset S$, with $x \in U_X$, such that $f(U_X) \subset U_S$, and the restriction $f|_{U_X} : U_X \to U_S$ is an hyperbolic elementary fibration.*

Proof. The problem is local on S, so we may assume that S is quasi-projective and that f is a special geometric fibration with compactification $\overline{f}\colon \overline{X} \to S$. By Remark 6.3.2, $\overline{f}$ is projective, so $\overline{X}$ is quasi-projective by EGA II.5.3.4(ii) ([DG67]), meaning we can embed $\overline{X}$ into $\mathbb{P}_k^N$ for some $N \in \mathbb{N}$.

Let F denote the fiber of $\overline{f}$ over $f(x)$. We need to find a hyperplane section Y' of $\overline{X}$ that is smooth, intersects F transversely (i.e., with smooth scheme-theoretic intersection), and does not intersect $Y \cap F$ or x. The last condition defines a subscheme of the space of hyperplanes containing a dense open subset, and Bertini's Theorem (Théorème 6.3 of [Jou83]) implies that the first two conditions do as well. Since k is infinite, such a hyperplane exists over k, providing our Y'.

It follows that there is a neighborhood of $f(x)$ in S over which the map $Y \cup Y' \to S$ is étale. Since Y' is closed in $\overline{X}$, its projection to S is proper, so this map is finite étale.

We now replace Y by $Y \cup Y'$. Either the fibers of f are now hyperbolic, in which case we are done, or the fibers have two punctures, in which case we repeat our procedure to finally obtain a hyperbolic elementary fibration around x. □

Lemma 6.3.8. *Let x be a closed point of a smooth geometrically connected variety X over an infinite field k. Then there is an open subset U of X containing x and an elementary fibration $f\colon U \to S$ for some smooth geometrically connected variety S over k.*

Proof. The proof of Lemma 6.3 in [SS16] explains how to extend the argument of [sga73] to the case of a closed point over an infinite field. □

6.3.1 Good Neighborhoods

Definition 6.3.9. An S-scheme X is called a *(hyperbolic) good neighborhood* if there exists a sequence of S-schemes

$$X = X_n, ..., X_0 = S$$

and (hyperbolic) elementary fibrations

$$f_i : X_i \to X_{i-1}, i = 1, ..., n.$$

Following [Hos14], we refer to such a sequence of S-schemes as a *sequence of parametrizing morphisms.*

Proposition 6.3.10. *Let $f\colon X \to S$ be an elementary fibration over a field of characteristic 0. If S is an étale $K(\pi, 1)$, then so is X. In particular, a good neighborhood is an étale $K(\pi, 1)$. Under this condition, the results of Section 6.2 hold for the classical section conjecture.*

Proof. Lemma 2.7(a) of [SS16] implies that the geometric fibers are étale $K(\pi, 1)$'s. The result then follows from the long exact sequence of Theorem 11.5 of [Fri82], noting that Remark 6.1.13 implies that different notions of étale homotopy group coincide for normal schemes. □

Definition 6.3.11. Let $f\colon X \to S$ be a good neighborhood with a sequence of parametrizing morphisms. Suppose we are given a cuspidal datum (S, C, k) for S. Then we define the *cuspidal datum $(X, C_{X,f}, k)$ induced by f and the sequence of parametrizing morphisms* to be the cuspidal datum induced successively by the elementary fibrations.

Remark 6.3.12. By an abuse of notation, we do not include the sequence of parametrizing morphisms in the notation for $C_{X,f}$, but we believe this should not lead to too much confusion. In Conjecture 6.4.1, we conjecture that $C_{X,f}$ does not depend on the sequence of parametrizing morphisms.

Corollary 6.3.13. *Let X be a hyperbolic good neighborhood over a sub-p-adic field k. Then $(X, C_{X,f}, k)$ satisfies the injectivity in the (homotopy) section conjecture (for any sequence of parametrizing morphisms).*

Proof. By Proposition 6.1.16 and repeated application of Proposition 6.3.10, the ordinary and homotopy section conjectures are equivalent. The result then follows by repeated application of Theorem 6.2.6 along with Theorem 5.3.24. □

Corollary 6.3.14. *Let X be a hyperbolic good neighborhood. If Conjecture 5.3.25 is true over k, then $(X, C_{X,f}, k)$ satisfies the surjectivity in the (homotopy) section conjecture (for any sequence of parametrizing morphisms).*

Proof. As in Corollary 6.3.13, the ordinary and homotopy section conjectures are equivalent. The result follows by repeated application of Theorem 6.2.7. □

Lemma 6.3.15. *Every smooth geometrically connected variety X over an infinite field k has an open cover by hyperbolic good neighbourhoods over k.*

Proof. Let x be a closed point of X. We proceed by induction on the dimension of X. If X has dimension 0, it is $\operatorname{Spec} k$, so we are done. Otherwise, Lemma 6.3.8 implies that there is a neighborhood U of x and an elementary fibration $f\colon U \to S$ over k. By Lemma 6.3.7, we can replace U and S so that f is now a hyperbolic elementary fibration. Finally, by the induction hypothesis, there is an open neighborhood V of $f(x)$ that is a hyperbolic good neighborhood over k. By Lemma 6.3.6, the pullback of f from S to V is still a hyperbolic elementary fibration, and this pullback is our desired hyperbolic good neighborhood. □

We can now answer Question 4.3.2 for $(\mathrm{f}-\mathrm{cov}, S, k)$ while circumventing Theorem 5.1.3:

Corollary 6.3.16. *Let X be a smooth geometrically connected variety over a number field k, and assume Conjecture 5.3.25. Then X has a Zariski open cover $X = \bigcup_i U_i$ such that U_i is VSA for $(\mathrm{f}-\mathrm{cov}, S, k)$ for any nonempty set S of finite places of k.*

Proof. By Proposition 6.3.15, X has an open cover by hyperbolic good neighborhoods U_i. By Corollary 6.3.14, U_i satisfies the surjectivity in the section conjecture over k, and by Corollary 6.3.13, U_i satisfies the injectivity in the section conjecture over k_v for all finite places v of k. By Proposition 5.3.26, U_i satisfies VSA for $(\mathrm{f}-\mathrm{cov}, S, k)$. □

6.4 Cuspidal Sections in Arbitrary Dimension

Conjecture 6.4.1. *Let $f\colon X \to S$ be a good neighborhood and (S, C, k) a cuspidal datum for S. Then the cuspidal datum $(X, C_{X,f}, k)$ induced by f and a sequence of parametrizing morphisms is independent of the sequence of parametrizing morphisms.*

If we assume the section conjecture (Conjecture 5.3.25), we may then define $C_{X,f}$ as the complement in $X(hk)$ of $\kappa(X(k))$, which is clearly independent of the sequence of parametrizing morphisms. We therefore expect Conjecture 6.4.1 to be true, at least for finitely generated fields of characteristic 0.

In order to attempt to answer this question, it seems natural to construct a general definition of cuspidal sections for arbitrary smooth varieties (similar to Definition 5.3.18) and then prove that this construction is stable under Definition 6.2.3. We survey some current ideas of the author in this direction.

To do this, we must work with ordinary rather than homotopy cuspidal data, as in Section 5.3.1. Doing so allows us to ignore boundary components of codimension greater than one, which would make it easier to prove a claim such as Claim 6.4.6. While this might not seem completely ideal, it suffices for the application to Conjecture 6.4.1, by Proposition 6.3.10.

6.4.1 Cuspidal Sections Associated to a Compactification

Definition 6.4.2. Let $U \subseteq X$ be an open immersion of good k-schemes. Let $Y = X \setminus U$ and $y \in Y(k)$. We fix a Henselization $\mathcal{O}^h_{X,y}$ and set $X_y = \operatorname{Spec} \mathcal{O}^h_{X,y}$. We let $U_y = X_y \times_X U$, which is known as the *scheme of nearby points to U at y.*

In this case, U_y is a good k-scheme, so we may write $\mathscr{S}_{\pi_1(U_y/k)}$.

Definition 6.4.3. The map $U_y \to U$ induces a map $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}$, whose image C_y is known as the *packet of cuspidal sections at y.*

Definition 6.4.4. Given a smooth variety U over a field k, we refer to an open inclusion $U \hookrightarrow X$ into a smooth proper variety X with complement Y a smooth normal crossings divisor in X as a *good compactification.*

If $U \subseteq X$ is a good compactification, we let U_y be the scheme of nearby points to U at y. By 18.3 of [Sti13], there is a short exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}(U_y)^\times/\mathcal{O}(X_y)^\times, \widehat{\mathbb{Z}}(1)) \to \pi_1(U_y) \to G_k \to 0.$$

Definition 6.4.5. Let U be a smooth geometrically connected variety over a field k of characteristic 0, and let $U \hookrightarrow X$ be a good compactification. Let $Y = X \setminus U$ and $y \in Y(k)$.

We define the set $C_{U,X}$ of all *cuspidal sections of the variety* U as

$$C_{U,X} \colon = \bigcup_{y \in Y(k)} C_y \subseteq \mathscr{S}_{\pi_1(U/k)}.$$

In a forthcoming paper, we intend to prove the following.

Claim 6.4.6. *The subset $C_{U,X} \subseteq \mathscr{S}_{\pi_1(U/k)}$ is independent of X, and we denote it by C_U.*

6.4.2 Cuspidal Sections in Fibrations

Conjecture 6.4.7. *Let $f\colon X \to S$ be a special geometric fibration of smooth étale $K(\pi, 1)$ varieties, let C_S, C_{X_s}, and C_X be as in Definition 6.4.5, and let*

$$C_{X,f} = f^{-1}(C_S) \bigcup \left(\bigcup_{s \in S(k)} \iota_s(C_{X_s}) \right) \subseteq \mathscr{S}_{\pi_1(X/k)}.$$

Then $C_{X,f} = C_X$.

The above conjecture clearly implies Conjecture 6.4.1. In the same forthcoming paper, we also intend to prove:

Claim 6.4.8. *With the notation of Conjecture 6.4.7, $C_X \subseteq C_{X,f}$.*

Chapter 7

Examples

7.1 Poonen's Counterexample

In [Poo10], Poonen found the first example of a variety with no rational points that does not satisfy VSA for the étale-Brauer obstruction. In this section, we review the construction of this variety and then explicitly find an open cover with empty étale-Brauer set.

7.1.1 Conic Bundles

We now present some general notions about conic bundles, as described in §4 of [Poo10]. We base our notation on [Poo10] and then add some notation of our own.

Let k be a field. From now on, let ε equal 1 if k has characteristic 2 and 0 otherwise.

Let B be a nice k-variety. Let $\mathcal{L}$ be a line bundle on B. Let $\mathcal{E}$ be the rank 3 locally free sheaf

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}$$

on B. Let $a \in k^\times$, and let $s \in \Gamma(B, \mathcal{L}^{\otimes 2})$ be a nonzero global section. Consider the section

$$\varepsilon \oplus 1 \oplus (-a) \oplus (-s) \in \Gamma(B, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \operatorname{Sym}^2 \mathcal{E}),$$

where the first $\mathcal{O}$ corresponds to the product of the first two summands of $\mathcal{E}$, and the last three terms $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}^{\otimes 2} \subset \operatorname{Sym}^2 \mathcal{E}$ are the symmetric squares of the three individual summands of $\mathcal{E}$. The zero locus of $\varepsilon \oplus 1 \oplus (-a) \oplus (-s)$ in $\mathbb{P}\mathcal{E}^v$ is a projective geometrically integral k-variety $X = X(B, \mathcal{L}, a, s)$ with a morphism $\alpha : X \to B$.

Definition 7.1.1. We call an element

$$(\mathcal{L}, s, a) \in \operatorname{Pic} B \times \Gamma(B, \mathcal{L}^{\otimes 2}) \times k^\times,$$

where $s \neq 0$, a *conic bundle datum on* B and X *the total space of* $(\mathcal{L}, s, a)$. We denote $X = Tot_B(\mathcal{L}, s, a)$.

If U is a dense open subscheme of B with a trivialization $\mathcal{L}|_U \cong \mathcal{O}_U$, and we identify $s|_U$ with an element of $\Gamma(U, \mathcal{O}_U)$, then the affine scheme defined by $y^2 + \varepsilon yz - az^2 = s|_U$ in $\mathbb{A}^2_U$ is a dense open subscheme of X. We therefore refer to X as the conic bundle given by $y^2 + \varepsilon yz - az^2 = s$.

In the special case where $B = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(2)$, and the homogeneous form $s \in \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ is separable, X is called the Châtelet surface given by $y^2 + \varepsilon yz - az^2 = s(x)$, where $s(x) \in k[x]$ denotes a dehomogenization of s.

Returning to the general case, we let Z be the subscheme $s = 0$ of B. We call Z the degeneracy locus of the conic bundle $(\mathcal{L}, s, a)$. Each fiber of α above a point of $B - Z$ is a smooth plane conic, and each fiber above a geometric point of Z is a union of two projective lines crossing transversally at a point. A local calculation shows that if Z is smooth over k, then X is smooth over k.

7.1.2 Poonen's Variety

Let k be a global field, let $a \in k^\times$, and let $\tilde{P}_\infty(x), \tilde{P}_0(x) \in k[x]$ be relatively prime separable degree 4 polynomials such that the (nice) Châtelet surface $\mathcal{V}_\infty$ given by $y^2 + \varepsilon yz - az^2 = \tilde{P}_\infty(x)$ over k satisfies $\mathcal{V}_\infty(\mathbb{A}_k) \neq \emptyset$ but $\mathcal{V}_\infty(k) = \emptyset$. Such Châtelet surfaces always exist : see [[Poo10], Proposition 5.1 and 11] if the characteristic of k is not 2 and [Vir12] otherwise. If $k = \mathbb{Q}$, one may use the original example from [Isk71] with $a = -1$ and $\tilde{P}_\infty(x) := (x^2 - 2)(3 - x^2)$.

Let $P_\infty(w,x)$ and $P_0(w,x)$ be the homogenizations of $\tilde{P}_\infty$ and $\tilde{P}_0$. Let $\mathcal{L} = \mathcal{O}(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and define

$$s_1 := u^2 P_\infty(w,x) + v^2 P_0(w,x) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{L}^{\otimes 2}),$$

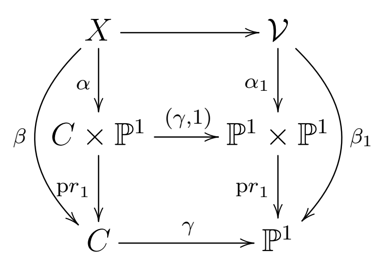
where the two copies of $\mathbb{P}^1$ have homogeneous coordinates $(u:v)$ and $(w:x)$, respectively. Let $Z_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero locus of s_1. Let $F \subset \mathbb{P}^1$ be the (finite) branch locus of the first projection $Z_1 \to \mathbb{P}^1$. i.e.,

$$F := \left\{(u:v) \in \mathbb{P}^1 | u^2 P_\infty(w,x) + v^2 P_0(w,x) \text{ has a multiple root}\right\}.$$

Let $\alpha_1 : \mathcal{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the conic bundle given by $y^2 + \varepsilon yz - az^2 = s_1$, a.k.a. the conic bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the datum $(O(1,2), a, s_1)$.

Composing α_1 with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ yields a morphism $\beta_1 : \mathcal{V} \to \mathbb{P}^1$ whose fiber above $\infty := (1:0)$ is the Châtelet surface $\mathcal{V}_\infty$ defined earlier.

Now let C be a nice curve over k such that $C(k)$ is finite and nonempty. Choose a dominant morphism $\gamma\colon C \to \mathbb{P}^1$, étale above F, such that $\gamma(C(k)) = \{\infty\}$. Define $X := \mathcal{V} \times_{\mathbb{P}^1} C$ to be the fiber product with respect to the maps $\beta_1\colon \mathcal{V} \to \mathbb{P}^1, \gamma\colon C \to \mathbb{P}^1$, and consider the morphisms α and β as in the diagram:



The variety X is the one constructed in §6 of [Poo10]. The same paper proves that $X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$ (Theorem 8.2) and $X(k) = \emptyset$ (Theorem 7.2). We present the proof that $X(k) = \emptyset$ because it is short and simple.

Proposition 7.1.2. *If X is the variety constructed above, then $X(k) = \emptyset$.*

Proof. Assume $x_0 \in X(k)$; we have $c_0 := \beta(x_0) \in C(k)$, but then $x \in \beta^{-1}(c_0)$. By the construction of X, $\beta^{-1}(c_0)$ is isomorphic to $\beta_1^{-1}(\gamma(c_0)) = \beta_1^{-1}(\infty) \cong \mathcal{V}_\infty$, but $\mathcal{V}_\infty(k) = \emptyset$ by construction. $\square$

Note that X can also be considered as the variety corresponding to the datum $(\mathcal{O}(1,2), a, s_1)$ pulled back via $(\gamma, 1)$ to $C \times \mathbb{P}^1$.

7.1.3 VSA Stratifications and Open Covers

We now show, under some conditions, that there is a finite stratification $\mathscr{X}$ of X with $X(\mathbb{A}_k)^{\mathscr{X},\text{ét},\text{Br}} = \emptyset$.

Proposition 7.1.3. *Suppose k is a number field. Let J be the Jacobian of C, and suppose that $J(k)$ is finite and that the Tate-Shafarevich group of J has trivial divisible subgroup. Then there is a stratification $X = X_1 \cup X_2$ such that $X_1(\mathbb{A}_k)^{\text{Br}} = \emptyset$ and $X_2(\mathbb{A}_k)^{\text{Br}} = \emptyset$.*

Proof. Let U denote the complement in C of $C(k)$. Let X_1 be $\beta^{-1}(U)$, and let X_2 be $\beta^{-1}(C(k))$.

By Theorem 8.1 of [Sto07], we have

$$C(k) = C(\mathbb{A}_k)^{\text{Br}}_{\bullet}.$$

In particular, we find that $C(k) = C(\mathbb{A}_k^f)^{\text{Br}}$, and therefore that $U(\mathbb{A}_k^f)^{\text{Br}} = U(\mathbb{A}_k)^{\text{Br}} = \emptyset$. This implies that $X_1(\mathbb{A}_k)^{\text{Br}} = \emptyset$ as well.

We also know by construction that X_2 is a union of copies of $\mathcal{V}_\infty$. As $\mathcal{V}_\infty(\mathbb{A}_k)^{\text{Br}}) = \emptyset$, we find that $X_2(\mathbb{A}_k)^{\text{Br}} = \emptyset$. $\square$

In this subsection, we present an explicit proof that X, as constructed in Section 7.1.2, has a Zariski open cover with empty étale-Brauer set using Theorem 6.2.10. Section 7.2 will then prove a stronger result, namely that there is a finite ramified cover with empty étale-Brauer set.

We let k be a number field in this section. From now on, we shall assume that $C(k)$ is finite. Let U denote the complement in C of $C(k)$.

We now have the following:

Theorem 7.1.4. *Suppose that every nonempty open subvariety of C satisfies Conjecture 5.3.25. Then there is an open subvariety $S \subseteq C$ such that $S \cup U = C$, and $X_S := S \times_C X$ satisfies VSA for $(\text{ét} - \text{Br}, f, k)$.*

Proof. Let $F' := \gamma^{-1}(F)$, and let $C' := C \setminus F'$. Then β is smooth over C'.

Let v be a real place of k. As C is smooth, $C'(k_v)$ is a disjoint union of components each homeomorphic to a line or circle. For each circle component, we wish to remove a closed non-rational point of C' that meets that component.

To do this, we need to show that each component contains a k_s-point that is not a k-point. Let k_v^{alg} denote the algebraic closure of k in k_v. By the completeness of the theory of real closed fields, any first-order formula true of k_v is also true of k_v^{alg}. By Theorems 2.4.4 and 2.4.5 of [BCR98], the property that a point lies in a given connected component is a semi-algebraic condition. As well, the property that a point is not equal to any element of $C'(k)$ is also a semi-algebraic condition. As each component has uncountably many k_v-points and hence at least one point not in $C'(k)$, each component has a k_v^{alg}-point not in $C(k)$. We choose one for each component and remove the corresponding closed points of C'.

We let S denote the resulting open subvariety of C. It is clear that $S(k) = C(k)$, so that $S \cup U = C$.

It suffices to verify the hypotheses of Theorem 6.2.10 for the projection $f = \beta\,|_{X_S} : X_S \to S$ and $T = f$.

By Lemma 4.2.7, S is VSA for (h, f, k), so 1 holds.

By Conjecture 5.3.25 and Theorem 5.3.24, 2a holds.

3 and 4 follow immediately because $T = f$ and because S is a smooth geometrically connected curve, respectively.

By construction, every connected component of $S(k_v)$ is simply connected. As well, every geometric fiber is a Châtelet surface, so $\pi_1(X_a((k_v)_s))$ is trivial, and hence 5 is verified.

Finally, every fiber over a rational point of S is isomorphic to $\mathcal{V}_\infty$. But we assumed that $\mathcal{V}_\infty(k) = \emptyset$, so by [CTSSD87a, CTSSD87b], we have $\mathcal{V}_\infty(\mathbb{A}_k)^{\mathrm{Br}} = \mathcal{V}_\infty(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. This shows that 6 holds. □

Corollary 7.1.5. *Under the hypotheses and notations of Theorem 7.1.4, the variety X has an open cover $X = X_S \cup X_U$ such that X_S and $X_U := U \times_C X$ both have empty étale-Brauer set.*

Proof. We have already shown that $X_S(k) \subseteq X(k) = \emptyset$, so by Theorem 7.1.4, we have $X_S(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. By Conjecture 5.3.25 and Proposition 5.3.26 applied to U, we find that $U(\mathbb{A}_k)^{\mathrm{f-cov}} = \emptyset$, which implies that $X_U(\mathbb{A}_k)^{\mathrm{f-cov}} = X_U(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. □

Remark 7.1.6. For the proof of Corollary 7.1.5, one may avoid using Conjecture 5.3.25 for U by assuming that the Jacobian of C has finitely many rational points and Tate-Shafarevich group with trivial divisible subgroup. For then, as in Proposition 7.1.3, $U(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. In particular, U satisfies VSA for (Br, f, k) and hence for $(\text{ét} - \mathrm{Br}, f, k)$, and the fact that $U(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$ implies that $X_U(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$ as well.

7.2 The Brauer-Manin obstruction applied to ramified covers

In this section, we introduce the notion of the ramified étale-Brauer obstruction. In a sense, this lies between the obstructions from Section 4.3.1 and the ordinary étale-Brauer obstruction. We also formulate a conjecture analogous to Theorems 4.3.3 and 4.3.4. In Section 7.2.2, we then explain how to apply this to Poonen's example.

7.2.1 Quasi-Torsors and the Ramified Etale-Brauer Obstruction

We shall now define slight generalizations of the concepts of torsors and the étale Brauer-Manin obstruction.

Definition 7.2.1. Let X be a variety over a field k, G a finite smooth k-group, and $D \subseteq X$ an a closed subvariety. A *G-quasi-torsor over X unramified outside D* is a map $\pi : Y \to X$ and a G-action on Y respecting π such that

1. π is a surjective and quasi-finite morphism.

2. The pullback of π from X to $X \setminus D$ is a G-torsor over $X \setminus D$.

We call $d = |G|$ the degree of Y.

Given $\rho \colon Z \to X$ an arbitrary morphism of k-varieties, the pullback $\rho^{-1}(Y)$ is a G-quasi-torsor unramified outside $\rho^{-1}(D)$.

Let $\pi : Y \to X$ be a G-quasi-torsor over X unramified outside D. As in the case of a usual G-torsor, one can twist $\pi : Y \to X$ by any $\sigma \in H^1(k, G)$ and get a G^σ-quasi-torsor $\pi^\sigma : Y^\sigma \to X$, also unramified outside D.

If we assume that $D(k) = \emptyset$, then as in Section 4.2.3, we get:

$$X(k) = \bigcup_{\sigma \in H^1(k,G)} \pi^\sigma(Y^\sigma(k))$$

If k is a global field, it follows that:

$$X(k) \subset X(\mathbb{A}_k)^{\pi,\mathrm{Br}} := \bigcup_{\sigma \in H^1(k,G)} \pi^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{Br}}).$$

Definition 7.2.2. Letting π range over all quasi-torsors over X unramified outside D, we define the *(D-)ramified étale-Brauer obstruction* by

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D} := \bigcap_{\pi} X(\mathbb{A}_k)^{\pi,\mathrm{Br}} \subset X(\mathbb{A}_k)$$

It follows from Section 4.2.3 that $(X \setminus D)(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subseteq X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim D}$.

Until now, we have left the fact that $D(k) = \emptyset$ as a black box. The idea behind this construction is that D has smaller dimension, and it therefore might be easier to show that D has empty étale-Brauer set. More generally, one might, e.g., find a subvariety E of D, prove that $D(\mathbb{A}_k)^{\text{ét},\mathrm{Br}\sim E} = \emptyset$, and finally prove that $E(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = \emptyset$. We formalize this as follows.

Definition 7.2.3. We say that *the absence of rational points is explained by the ramified étale-Brauer obstruction* if there exists a stratification $X = \coprod_i X_i$ and for each i, we have $X_i(\mathbb{A}_k)^{\text{ét},\text{Br}\sim D_i} = \emptyset$, where $D_i := \overline{X_i} \setminus X_i$.

This is similar to the method of considering the étale-Brauer sets of dense open subsets, for $X(\mathbb{A}_k)^{\text{ét},\text{Br}\sim D} = \emptyset$ implies that $(X \setminus D)(\mathbb{A}_k)^{\text{ét},\text{Br}} = \emptyset$. In particular, if the absence of rational points is explained by the ramified étale-Brauer obstruction, then the answer to Question 4.3.1 is yes. However, one crucial difference is that the ramified étale-Brauer obstruction only requires proving emptiness of Brauer sets of *proper* varieties.

Inspired by Theorem 4.3.3, we conjecture that the absence of rational points is always explained by the ramified étale-Brauer obstruction and that one may furthermore take a single quasi-torsor for each stratum:

Conjecture 7.2.4. *For any number field k and variety X/k with $X(k) = \emptyset$, there exists a stratification $X = \coprod_i X_i$ and for each i, a quasi-torsor Y_i over the closure $\overline{X_i}$ of X_i such that Y_i restricts to a torsor over X_i and $Y_i^\sigma(\mathbb{A}_k)^{\text{Br}} = \emptyset$ for all twists Y_i^σ of Y_i.*

This conjecture still does not imply that there is an algorithm to determine whether $X(k)$ is empty, as one may need to consider infinitely many twists of a quasi-torsor Y_i. Nonetheless, we now show that it is computable in the example of [Poo10] (in particular, see Remark 7.2.8 to see why we need only finitely many twists). In particular, we know of no counterexamples to Conjecture 7.2.4.

7.2.2 Quasi-torsors in Poonen's Example

In this subsection, we show (under some conditions) that for the variety X defined in Section 7.1.2, one can choose a divisor $D \subseteq X$ such that $D(\mathbb{A}_k)^{\text{Br}} = \emptyset$, and $X(\mathbb{A}_k)^{\text{ét},\text{Br}\sim D} = \emptyset$.

More specifically, we assume as in Remark 7.1.6 that the Jacobian of C has finitely many rational points and trivial divisible subgroup of its Tate-Shafarevich group, which implies by Theorem 8.1 of [Sto07] that

$$C(k) = C(\mathbb{A}_k)^{\text{Br}}_\bullet$$

and

$$U(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset.$$

As before, let $F' := \gamma^{-1}(F) \subseteq C$ and $C' := C \backslash F'$. Note that C' is a non-projective curve. Now let $D := \beta^{-1}(F')$. Note that $\infty \notin F$, so that $C(k) \cap F' = \emptyset$. As D maps to U and $U(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, we have $D(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.

We will now spend the rest of Section 7.2.2 proving:

Theorem 7.2.5. *With notations as above, the absence of rational points on X is explained by the ramified étale-Brauer obstruction if k is a global field with no real places (i.e., a function field or a totally imaginary number field).*

Now X is a family over C of conic bundles over $\mathbb{P}^1$. The fiber over any element of $C(k)$ is isomorphic to the Châtelet surface $\mathcal{V}_\infty$. All the fibers over C' are smooth conic bundles.

Let $E' \subset (\mathbb{P}^1 \backslash F) \times (\mathbb{P}^1)^4$ be the curve defined by

$$u^2 P_\infty(w_i, x_i) + v^2 P_0(w_i, x_i) = 0, 1 \leq i \leq 4$$

$$(w_i : x_i) \neq (w_j : x_j), i \neq j, 1 \leq i, j \leq 4,$$

where $(u : v)$ are the projective coordinates of $\mathbb{P}^1 \backslash F$ and $(w_i : x_i), 1 \leq i \leq 4$ are the projective coordinates of the four copies of $\mathbb{P}^1$. Since $\tilde{P}_\infty(x)$ and $\tilde{P}_0(x)$ are separable and coprime, we have that E' is a smooth connected curve and that the first projection $E' \to \mathbb{P}^1 \backslash F$ gives E' the structure of an étale Galois covering of $\mathbb{P}^1 \backslash F$ with automorphism group $G = S_4$ that acts on the fibres by permuting

$$(w_i : x_i), 1 \leq i \leq 4.$$

Since every birationality class of curves contains a unique projective smooth member, one can construct an S_4-quasi-torsor over $E \to \mathbb{P}^1$ unramified outside F which gives E' when restricted to $\mathbb{P}^1 \backslash F$.

The k-twists of $E \to \mathbb{P}^1$ are classified by $H^1(k, S_4)$ which (since the action of Γ_k on S_4 is trivial) coincides with the set $\mathrm{Hom}(\Gamma_k, S_4)/\sim$ of homomorphisms up to conjugation. More concretely, for every homomorphism $\phi : \Gamma_k \to S_4$, define E_ϕ to be the k-form of E with Galois action that restricts to the action

$$\sigma : ((u : v), ((w_1 : x_1), (w_2 : x_2), (w_3 : x_3), (w_4 : x_4))) \mapsto$$

$$((u : v), ((w_{\phi_\sigma(1)} : x_{\phi_\sigma(1)}), (w_{\phi_\sigma(2)} : x_{\phi_\sigma(2)}), (w_{\phi_\sigma(3)} : x_{\phi_\sigma(3)}), (w_{\phi_\sigma(4)} : x_{\phi_\sigma(4)})))^\sigma$$

on E'.

For every $\phi : \Gamma_k \to S_4$, we set $C_\phi := C \times_{\mathbb{P}^1} E_\phi$ (relative to $\gamma : C \to \mathbb{P}^1$ and the first projection $\pi_\phi \colon E_\phi \to \mathbb{P}^1$) and $X_\phi := X \times_C C_\phi$ (relative to $\beta : X \to C$ and the projection $C_\phi \to C$).

Since the maps $\gamma : C \to \mathbb{P}^1$ and $E \to \mathbb{P}^1$ have disjoint ramification loci, all C_ϕ are geometrically integral, and so are all the X_ϕ.

Then X_ϕ is a complete family of twists of a quasi-torsor over X of degree 24 unramified outside D. As we already know that $D(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, it suffices for the proof of Theorem 7.2.5 to show that

$$X_\phi(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$$

for every $\phi \in H^1(\Gamma_k, S_4)$. We devote the rest of Section 7.2.2 to proving this fact.

Reduction to X_{ϕ_∞}

Lemma 7.2.6. *For every $\phi \in H^1(k, S_4)$, we have $C_\phi(k) = C_\phi(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet$.*

Proof. By Corollary 7.3 of [Sto07], we have $C_\phi(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C_\phi(\mathbb{A}_k)_\bullet^{\mathrm{f-ab}}$ and $C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\mathrm{f-ab}}$. Then Proposition 8.5 of [Sto07] and the fact that $C(k) = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$ implies the result. □

Denote by $\phi_\infty \in H^1(k, S_4)$ the map $\Gamma_k \to S_4$ defined by the Galois action on the four roots of P_∞.

Lemma 7.2.7. *Let $\phi \in H^1(\Gamma_k, S_4)$ be such that $\phi \neq \phi_\infty$. Then $C_\phi(k) = \emptyset$.*

Proof. Since $\phi \neq \phi_\infty$ we get that $E_\phi(k) \cap \pi_\phi^{-1}(\infty) = \emptyset$. Since $\gamma(C(k)) = \infty$, we get that $C_\phi(k) = \emptyset$. □

Let $\rho_\phi \colon X_\phi \to C_\phi$ denote the map defined earlier. For every $\phi \in H^1(k, S_4)$, we have

$$\rho_\phi(X_\phi(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet) \subseteq C_\phi(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet = C_\phi(k) = \emptyset,$$

so $X_\phi(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ for $\phi \neq \phi_\infty$.

Remark 7.2.8. It is Lemma 7.2.7 along with VSA for C coming from our condition on the Tate-Shafarevich group that lets us avoid all but one twist. In fact, this has a conceptual explanation. While infinitely many twists might have adelic points (as $\beta^{-1}(C')$ is not proper), all possible elements of the Brauer set lie in the fiber over $C(k)$, which is in fact proper. There are therefore finitely many twists that might have adelic points in this fiber.

The proof that $X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$

In this subsection, we shall prove that if k has no real places, then $X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}} = X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}}_\bullet = \emptyset$.

Let $p \in C_{\phi_\infty}(k)$. The fiber $\rho_{\phi_\infty}^{-1}(p)$ is isomorphic to the Châtelet surface $\mathcal{V}_\infty$. We shall denote by $\rho_p : \mathcal{V}_\infty \to X_{\phi_\infty}$ the corresponding natural isomorphism onto the fiber $\rho_{\phi_\infty}^{-1}(p)$. Recall that $\mathcal{V}_\infty$ satisfies $\mathcal{V}_\infty(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.

Lemma 7.2.9. *Let k be a global field with no real places. Let $x \in X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}}_\bullet$. Then there exists $p \in C_{\phi_\infty}(k)$ such that $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A}_k)_\bullet)$.*

Proof. From functoriality and Lemma 7.2.6 we get

$$\rho_{\phi_\infty}(x) \in \rho_{\phi_\infty}(X_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}}_\bullet) \subset C_{\phi_\infty}(\mathbb{A}_k)^{\mathrm{Br}}{}_\bullet = C_{\phi_\infty}(k)$$

We denote $p = \rho_{\phi_\infty}(x) \in C'_{\phi_\infty}(k)$. Now it is clear that in all but maybe the infinite places $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A}_k))$. Hence it remains to deal with the infinite places, which by assumption are all complex. But since both X_{ϕ_∞} and $\mathcal{V}_\infty$ are geometrically integral, taking connected components reduces $X(\mathbb{C})$ and $\mathcal{V}_\infty(\mathbb{C})$ to a single point. $\square$

Lemma 7.2.10. *Let $p \in C_{\phi_\infty}(k)$ be a point. Then the map*

$$\rho_p^* : \mathrm{Br}(X_{\phi_\infty}) \to \mathrm{Br}(\mathcal{V}_\infty)$$

is surjective.

We will prove Lemma 7.2.10 in Section 7.2.2.

Lemma 7.2.11. *Let k be global field with no real places. Then $X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$.*

Proof. Assume that $X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \neq \emptyset$. Let $x \in X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$. By Lemma 7.2.9 there exists a $p \in C_{\phi_\infty}(k)$ such that $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A}_k)_\bullet)$. Let $y \in \mathcal{V}_\infty(\mathbb{A}_k)_\bullet$ be such that $\rho_p(y) = x$. We shall show that $y \in \mathcal{V}_\infty(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

Indeed, let $b \in \mathrm{Br}(\mathcal{V}_\infty)$. By Lemma 7.2.10 there exists a $\tilde{b} \in \mathrm{Br}(X'_{\phi_\infty})$ such that $\rho_p^*(\tilde{b}) = b$. Now

$$(y, b) = (y, \rho_p^*(\tilde{b})) = (\rho_p(y), \tilde{b}) = (x, \tilde{b}) = 0$$

But by assumption $x \in X_{\phi_\infty}(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$, so we have $(y, b) = (x, \tilde{b}) = 0$. Thus we have $y \in \mathcal{V}_\infty(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = \emptyset$ which is a contradiction. □

The surjectivity of ρ_p^*

In this subsection, we shall prove the statement of Lemma 7.2.10. We switch gears for a moment and let $\alpha\colon X \to B$ be an arbitrary conic bundle given by datum $(\mathcal{L}, s, a)$.

Lemma 7.2.12. *The generic fiber X_η^s of $X^s \to B^s$ is isomorphic to $\mathbb{P}^1_{\kappa(B^s)}$, where $\kappa(B^s)$ is the field of rational functions on B^s .*

Proof. It is a smooth plane conic and it has a rational point since a is a square in $k_s \subset \kappa(B^s)$. □

Lemma 7.2.13. *Denote the generic point of B by η. Let Z be the degeneracy locus. Assume that Z^s is the union of the irreducible components $Z^s = \bigcup_{1 \le i \le r} Z_i$. Then there is a natural exact sequence of Galois modules.*

$$\begin{array}{ccccccccc} 0 \longrightarrow \bigoplus \mathbb{Z}Z_i & \xrightarrow{\rho_1} & \operatorname{Pic} B^s \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^- & \xrightarrow{\rho_2} & \operatorname{Pic} X^s & \underset{\rho_3}{\overset{\rho_4}{\rightleftarrows}} & \operatorname{Pic} {X^s}_\eta & \longrightarrow 0 \\ & & & & & \searrow^{deg} & \| & \\ & & & & & & \mathbb{Z} & \end{array}$$

where ρ_4 is a natural section of ρ_3.

Proof. Call a divisor of X^s vertical if it is supported on prime divisors lying above prime divisors of B^s, and horizontal otherwise. Denote by $Z_i^\pm$ the divisors that lie over Z_i and defined by the additional condition that $y = \pm\sqrt{a}z$. Now define ρ_1 by

$$\rho_1(Z_i) = (-Z_i, Z_i^+, Z_i^-)$$

and ρ_2 by

$$\rho_2(M, 0, 0) = \alpha^* M$$

$$\rho_2(0, Z_i^+, 0) = Z_i^+$$

$$\rho_2(0, 0, Z_i^-) = Z_i^-$$

Let ρ_3 be the map induced by $X_\eta^s \to X^s$. Each ρ_i is Γ_k-equivariant. Given a prime divisor D on X_η^s, we take $\rho_4(D)$ to be its Zariski closure in X^s. It is clear that $\rho_3 \circ \rho_4 = \mathrm{Id}$, so ρ_3 is indeed surjective.

The kernel of ρ_3 is generated by the classes of vertical prime divisors of X. In fact, there is exactly one above each prime divisor of B, except that above each $Z_i \in \operatorname{Div} B^s$ we have both $Z_i^+, Z_i^- \in \operatorname{Div} X^s$. This proves exactness at $\operatorname{Pic} X^s$.

Now, since $\alpha : X^s \to B^s$ is proper, a rational function on X^s with a vertical divisor must be the pullback of a rational function on B^s. Using the fact that the image of ρ_2 only contains vertical divisors, we prove exactness at

$$\operatorname{Pic} B^s \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^-$$

The injectivity of ρ_1 is then trivial. $\square$

We switch gears once again and let X be as in Poonen's example.

Lemma 7.2.14. *Let $p \in C_{\phi_\infty}(k)$ and $\rho_p : \mathcal{V}_\infty \to X_{\phi_\infty}$ be the corresponding map as above. Then the map of Galois modules*

$$\rho_p^* : \mathrm{Pic}(X^s_{\phi_\infty}) \to \mathrm{Pic}(\mathcal{V}^s_\infty)$$

has a section.

Proof. Consider the map $\phi_p : \mathbb{P}^1 \to \mathbb{P}^1 \times C_{\phi_\infty}$ defined by $x \mapsto (x, p)$. It is clear that the map $\rho_p : \mathcal{V}_\infty \to X_{\phi_\infty}$ comes from pulling back the conic bundle datum defining X_{ϕ_∞} over $\mathbb{P}^1 \times C_{\phi_\infty}$ by this map. Let $B = \mathbb{P}^1 \times C_{\phi_\infty}$, and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus \mathbb{Z}Z_i & \longrightarrow & \mathrm{Pic}\, B^s \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^- & \longrightarrow & \mathrm{Pic}\, X^s_{\phi_\infty} & \underset{deg}{\rightleftarrows} & \mathbb{Z} & \longrightarrow 0 \\
& s_1 \uparrow\downarrow & & s_2 \uparrow\downarrow & & \downarrow \rho_p^* & & \| & \\
0 \longrightarrow & \bigoplus \mathbb{Z}W_i & \longrightarrow & \mathrm{Pic}\, \mathbb{P}^1_{k_s} \oplus \bigoplus \mathbb{Z}W_i^+ \oplus \bigoplus \mathbb{Z}W_i^- & \longrightarrow & \mathrm{Pic}\, \mathcal{V}^s_\infty & \underset{deg}{\rightleftarrows} & \mathbb{Z} & \longrightarrow 0
\end{array}$$

where Z is the degeneracy locus of X_{ϕ_∞} over B, W is the degeneracy locus of $\mathcal{V}_\infty$ over $\mathbb{P}^1$, and $Z^s = \bigcup_{1 \le i \le r} Z_i$ and $W^s = \bigcup_{1 \le i \le r} W_i$ are decompositions into irreducible components. The existence of a section for ρ_p^* follows by diagram chasing and the existence of the compatible sections s_1 and s_2.

Every W_i ($1 \le i \le 4$) is a point that corresponds to a different root $(w_i : x_i)$ of the polynomial $P_\infty(x, w)$. We can choose $Z_i \subset B^s$ to be Zariski closure of the zero set of $w_i x - x_i w$, and similarly $Z_i^\pm \subset X^s_{\phi_\infty}$ to be Zariski closure of the zero set of $y \pm \sqrt{a}z, w_i x - x_i w$.

Now we define: $Z_i = s_1(W_i)$ and $Z_i^\pm = s_2(W_i^\pm)$ and the map $s_2 : \mathrm{Pic}\, \mathbb{P}^1_{k_s} \to \mathrm{Pic}\, B^s$ is defined by the unique section of the map $\phi_p : \mathbb{P}^1 \to \mathbb{P}^1 \times C_{\phi_\infty}$.

It is clear that s_1 and s_2 are indeed group-theoretic sections. To prove that s_1 and s_2 also respect the Galois action, we can write

$$p = (c, ((x_1^0 : w_1^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0))) \in C(k) \times_{\mathbb{P}^1(k)} E_{\phi_\infty}(k),$$

and since $\gamma(C(k)) = \{\infty\}$, the four points $\{(x_1^0 : w_1^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0)\}$ are exactly the four different roots of $P_\infty(x, w)$. $\square$

Lemma 7.2.15 (Lemma 7.2.10)**.** *Let $p \in C_{\phi_\infty}(k)$. Then the map*

$$\rho_p^* : \mathrm{Br}(X_{\phi_\infty}) \to \mathrm{Br}(\mathcal{V}_\infty)$$

is surjective.

Proof. Denote by $s_p : \mathrm{Pic}\,\overline{(\mathcal{V}_\infty)} \to \mathrm{Pic}(X^s_{\phi_\infty})$ the section of

$$\rho_p^* : \mathrm{Pic}(X^s_{\phi_\infty}) \to \mathrm{Pic}(\mathcal{V}^s_\infty)$$

It is clear that s_p induces a section of the map

$$\rho_p^{**} : H^1(k, \mathrm{Pic}(X^s_{\phi_\infty})) \to H^1(k, \mathrm{Pic}(\mathcal{V}^s_\infty))$$

By the Hochschild–Serre spectral sequence for X, we have:

$$H^1(k, \mathrm{Pic}(X^s)) = \ker[\mathrm{Br}X \to \mathrm{Br}X^s]/\mathrm{Im}\,[\mathrm{Br}k \to \mathrm{Br}X]$$

Letting

$$\mathrm{Br}_1(X) := \ker[\mathrm{Br}X \to \mathrm{Br}X^s],$$

we get that the map $\rho_p^* : \mathrm{Br}_1(X_{\phi_\infty}) \to \mathrm{Br}_1(\mathcal{V}_\infty)$ is surjective. But since $\mathcal{V}^s_\infty$ is a rational surface (it is a Châtelet surface), we have $\mathrm{Br}\mathcal{V}^s_\infty = 0$, and thus $\mathrm{Br}_1(\mathcal{V}_\infty) = \mathrm{Br}(\mathcal{V}_\infty)$. So we get that $\rho_p^* : \mathrm{Br}(X_{\phi_\infty}) \to \mathrm{Br}(\mathcal{V}_\infty)$ is surjective. $\square$

Chapter 8

Appendices to Part II

8.1 Appendix A: Obstructions Without Functors

As the main novelty of our methods is to introduce decompositions into subvarieties, one might wonder what happens if one tries to build local-global obstructions using only such decompositions. We extend the notation from above by omitting ω:

$$X(\mathbb{A}_{k,S})^{\mathscr{X}} = \bigcup_i X_i(\mathbb{A}_{k,S})$$

$$X(\mathbb{A}_{k,S})^{\mathscr{C}} = \bigcap_{\mathscr{X} \in \mathscr{C}} X(\mathbb{A}_{k,S})^{\mathscr{X}}.$$

For any ω, we then have:

$$X(k) \subseteq X(\mathbb{A}_{k,S})^{\mathscr{X},\omega} \subseteq X(\mathbb{A}_{k,S})^{\mathscr{X}}, X(\mathbb{A}_{k,S})^{\omega}.$$

The main point of this appendix is the following, open-ended question:

Question 8.1.1. For a variety X, describe $X(\mathbb{A}_{k,S})^{\mathcal{STRAT}}$ and $X(\mathbb{A}_{k,S})^{\mathcal{OPEN}}$.

In particular, for $X = \mathbb{A}^1$, this provides two interesting subsets of the adele ring $\mathbb{A}_{k,S}$. Is either a subring?

As a first step, we note that neither of these subsets is always equal to the set of rational points. We prove the following result modulo a lemma, then provide the lemma:

Proposition 8.1.2. *Let X/k be integral. If X/k has a generic k_v-point for every $v \in S$ (e.g., if X is a k-rational variety), we have $X(k) \not\subseteq X(\mathbb{A}_{k,S})^{\mathcal{STRAT}} \subseteq X(\mathbb{A}_{k,S})^{\mathcal{OPEN}}$.*

Proof. It suffices to find a k-algebra homomorphism $k(X) \to \mathbb{A}_{k,S}$, for such a homomorphism gives an $\mathbb{A}_{k,S}$-point of X that extends to every nonempty open in X. As the existence of such a homomorphism depends only on $k(X)$, it suffices to assume that X is affine and smooth. To do this, we need to find a point $\alpha \in X(\mathbb{A}_{k,S})$ such that $f(\alpha) \in \mathbb{A}_{k,S}^{\times}$ for every nonzero $f \in \mathcal{O}(X)$.

By spreading out, there exists a finite type model $\mathcal{X}$ of X over $\mathcal{O}_k$. As every element of $\mathcal{O}(X)$ is a $k^{\times}$-multiple of an element of $\mathcal{O}(\mathcal{X})$, it suffices to check $f(\alpha) \in \mathbb{A}_{k,S}^{\times}$ for every nonzero $f \in \mathcal{O}(\mathcal{X})$. This is equivalent to saying that $f(\alpha_v) \neq 0$ for all v, and $f(\alpha_v) \in \mathcal{O}_v \setminus \mathfrak{m}_v$ for all but finitely many v.

We choose an enumeration of $\mathcal{O}(\mathcal{X}) \setminus \{0\}$ by the positive integers, where to $i \in \mathbb{Z}_{>0}$ we associate $f_i \in \mathcal{O}(\mathcal{X})$. We thus wish to choose $\alpha \in \mathcal{X}(\mathbb{A}_{k,S})$ such that $f_i(\alpha) \in \mathbb{A}_{k,S}^{\times}$ for all $i \in \mathbb{Z}_{>0}$.

For $m \geq 1$ let $R_m = \{f_i\}_{1 \leq i \leq m}$. By Lemma 8.1.4, there is some constant, which we denote by C_m, such that for all v of norm at least C_m, there is a smooth $\mathbb{F}_v$-point of $\mathcal{X}$ at which no element of R_m vanishes. We choose each C_m to be as small as possible for that m, so that $C_m \leq C_{m+1}$ for all m. We then set $D_m = m + C_m$, so that $D_m < D_{m+1}$, and $\lim_{m \to \infty} D_m = \infty$. Starting with $m = 1$, for each place v with norm in $[D_m + 1, D_{m+1}]$, we choose $a_v \in \mathcal{X}(\mathbb{F}_v)$ such that $f_i(a_v) \neq 0$ for all $1 \leq i \leq m$. It follows that for every m, the value of $f_m(a_v)$ is nonzero in $\mathbb{F}_v$ for v of norm greater than D_m.

For every $v \in S$ of norm $> D_1$, we choose an $\alpha_v \in \mathcal{X}(\mathcal{O}_v) \subseteq X(k_v)$ reducing to a_v modulo $\mathfrak{m}_v$ and which maps to the generic point of X. This exists by Lemma 8.1.3.

For every $v \in S$ of norm $\leq D_1$, we choose an arbitrary $\alpha_v \in X(k_v)$ that maps to the generic point of X.

We set $\alpha = \prod_{v \in S} \alpha_v$. Then for every nonzero $f \in \mathcal{O}(X)$, the value $f(\alpha_v)$ is nonzero for all v and invertible in $\mathcal{O}_v$ for all but finitely many v. That is, $f(\alpha)$ is invertible in $\mathbb{A}_{k,S}$, so α

gives us our desired homomorphism from $k(X)$ to $\mathbb{A}_{k,S}$. □

Lemma 8.1.3. *Let $\mathcal{X}$ be a finite type scheme over $\mathcal{O}_k$ with irreducible generic fiber X of dimension d. Let a be a smooth point in $\mathcal{X}(\mathbb{F}_v)$ for a finite place v of k. Then there is an $\mathcal{O}_v$-point α of $\mathcal{X}$ lifting a whose generic fiber is a generic point of $\mathcal{X}$.*

Proof. Let $x_1, \cdots, x_d$ be local parameters at a. Then Hensel's lemma says that there is a bijection between lifts of a to $\alpha \in \mathcal{X}(\mathcal{O}_v)$ and $(\mathcal{O}_v)^d$, given by

$$\alpha \mapsto (x_1(\alpha)/\pi_v, \cdots, x_d(\alpha)/\pi_v).$$

Let us choose an α associated to a set of d elements of $\mathcal{O}_v$ that are algebraically independent over k. Tensoring over $\mathcal{O}_k$ with k, we get a homomorphism $\mathcal{O}(X) \to \mathcal{O}_v \otimes_{\mathcal{O}_k} k = k_v$ whose image in k_v generates a field of transcendence degree d. This implies that the map $\mathcal{O}(X) \to k_v$ is injective; i.e., the point is generic. □

Lemma 8.1.4. *Let $\mathcal{X}$ be a finite type integral scheme over $\mathcal{O}_k$ with generic fiber of dimension $d \geq 1$ that is geometrically integral and affine. Let R be a finite subset of $\mathcal{O}(\mathcal{X}) \setminus \{0\}$. Then for almost all places v of k, there is a smooth $\mathbb{F}_v$-point of $\mathcal{X}$ at which no element of R vanishes.*

Proof. By replacing R with the one-element set containing the product of all elements of R, we may assume that R has a single element, call it f.

By the chart in Appendix C of [Poo17], the properties "flat," "affine," and "geometrically integral fibers" satisfy spreading out. We can therefore find a set T of places of k so that the fiber of $\mathcal{X}$ over $\mathcal{O}_{k,T}$ has geometrically integral fibers and is flat and affine. By flatness, all of its fibers have the same dimension, which must be r. We replace $\mathcal{X}$ by its fiber over $\mathcal{O}_{k,T}$ at the expense of modifying finitely many places. The zero set of f is a closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ with generic fiber of dimension $d - 1$. We can enlarge T so that $\mathcal{Y}$ is flat, ensuring that the dimension is constant. Let $\mathcal{Z}$ denote the open subscheme of $\mathcal{X}$ obtained by removing $\mathcal{Y}$. Its fibers are geometrically irreducible of dimension d. We need to show that $\mathcal{Y}(\mathbb{F}_v) \nsubseteq \mathcal{X}(\mathbb{F}_v)$, or that $\mathcal{Z}(\mathbb{F}_v) \neq \emptyset$, for almost all v.

Let us apply the Lang-Weil bounds, i.e., Theorem 7.7.1(iv) of [Poo17], with $Y = \operatorname{Spec} \mathcal{O}_{k,T}$ and $X = \mathcal{Z}$. Noting that $\mathcal{Z}_{\mathbb{F}_v}$ is geometrically integral, $\mathcal{Z}$ has a smooth $\mathbb{F}_v$-point for q_v sufficiently large, so the result is true. □

8.2 Appendix B: Reformulation in Terms of Cosheaves

Definition 8.2.1. Let C be a site. Then a *precosheaf* $\mathcal{F}$ on C is a functor $\mathcal{F}\colon C \to \mathbf{Set}$.

Definition 8.2.2. Let C be a site. Then a *cosheaf* $\mathcal{F}$ on C is a precosheaf such that for any $U \in \mathrm{ob}(C)$ and covering $\{U_i\}_i$ of U, the natural map

$$\operatorname{colim}\left(\coprod_{i,j} \mathcal{F}(U_i \times_U U_j) \rightrightarrows \coprod_i \mathcal{F}(U_i)\right) \xrightarrow{g} \mathcal{F}(U)$$

is an isomorphism.

Definition 8.2.3. In the notation of Definition 8.2.2, if g is only assumed to be injective, we call C a *separated precosheaf.*

For any scheme X, we let X_{Zar} denote the (big or small) Zariski site of X.

Definition 8.2.4. Let R be a k-algebra and X a scheme over k. There is a precosheaf $\mathcal{F}_R$ on X_{Zar} associating to every $U \in \mathrm{ob}(X_{\mathrm{Zar}})$ the set $U(R)$.

Lemma 8.2.5. *The precosheaf $\mathcal{F}_R$ is always separated. If R is a local ring (e.g. a field), then $\mathcal{F}_R$ is a cosheaf.*

Proof. Since the lemma is about any scheme X, it suffices to consider Zariski open covers $\{U_i\}_i$ of the whole scheme X.

Consider any two elements of the domain of g whose images in $\mathcal{F}_R(X)$ are equal. Let $x_1, x_2 \in \coprod_i \mathcal{F}_R(U_i)$ be representatives for these. Suppose $x_1 \in \mathcal{F}_R(U_i)$ and $x_2 \in \mathcal{F}_R(U_j)$. The universal property of fiber products of schemes then gives us a map $x_1 \times x_2\colon \operatorname{Spec} R \to U_i \times_X U_j$. Thus x_1 and x_2 each come from an element of $\coprod_{i,j} \mathcal{F}(U_i \times_U U_j)$, so they correspond to the same element of the domain of g. It follows that g is injective, so $\mathcal{F}_R$ is separated.

Now suppose that R is a local ring. Let $x : \operatorname{Spec} R \to X$. Then the closed point of $\operatorname{Spec} R$ maps to some physical point of X, which must be contained in U_i for some i. Thus the preimage of U_i under x contains the closed point of $\operatorname{Spec} R$, so it is all of $\operatorname{Spec} R$. It follows that x factors as $\operatorname{Spec} R \to U_i \to X$, i.e., comes from $\mathcal{F}_R(U_i)$. As x was arbitrary, the map g from Definition 8.2.2 is surjective. As g is surjective and injective, it is bijective, so $\mathcal{F}_R$ is a cosheaf. □

Example 8.2.6. Let $X = \mathbb{P}^1_{\mathbb{Q}}$. If S contains at least two elements, then $\mathcal{F}_{\mathbb{A}_{k,S}}$ is not a cosheaf. Indeed, let $v_1, v_2 \in S$. Let $\alpha \in X(\mathbb{A}_{k,S})$ have coordinate 0 in the v_1-component and ∞ in the v_2-component. Then α is a $\mathbb{A}_{k,S}$-point of neither $X \setminus \{0\}$ nor $X \setminus \{\infty\}$, so the cosheaf condition is violated.

Definition 8.2.7. Let (ω, S, k) be an obstruction datum and X/k a scheme. By Proposition 4.3.8, there is a precosheaf $\underline{X(\mathbb{A}_{k,S})^{\omega}}$ associating to $U \in \operatorname{ob}(X_{\mathrm{Zar}})$ the set

$$U(\mathbb{A}_{k,S})^{\omega}.$$

Furthermore, in the terminology of Section 4.3.1, we denote by $\underline{X(\mathbb{A}_{k,S})^{\mathscr{C},\omega}}$ the precosheaf associating to U the set

$$U(\mathbb{A}_{k,S})^{\mathscr{C},\omega}.$$

Definition 8.2.8. By Theorem 2.1(a) of [Pra16], the inclusion functor from cosheaves on C to precosheaves on C has a right adjoint, known as *cosheafification*.

Proposition 8.2.9. *Suppose that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a separated precosheaf (e.g., suppose the answer to Question 4.3.2 is yes for (ω, S, k)). Suppose X is a variety over k or $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a cosheaf. Then the cosheafification of $\underline{X(\mathbb{A}_{k,S})^{\omega}}$ is $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$.*

Proof. We first must show that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a cosheaf. Let $U \in \operatorname{ob}(X_{\mathrm{Zar}})$, and let $\{U_i\}_{i \in I}$ be a finite open cover of U. We need to show that the associated g from Definition 8.2.2 is surjective and injective. We start by showing that it is surjective.

We therefore need to show that any $\alpha \in U(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ is in $U_i(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$ for some i. If we are not already assuming this fact, we are assuming that X is a variety. As k is countable, there are countably many Zariski open subsets of X and therefore countably many finite open covers of U. We enumerate all finite open covers of U as $\mathcal{W}_k = \{W_{jk}\}_j$ for $k \in \mathbb{Z}_{>0}$. For each positive integer k, we let $\mathcal{T}_k = \{T_{jk}\}_j$ be the common refinement of $\mathcal{W}_l$ for $1 \leq l \leq k$, so that $\mathcal{T}_{k+1}$ is always a refinement of $\mathcal{T}_k$, and the system is still cofinal. Finally, we let $V_{ijk} = U_i \cap T_{jk}$, and we let $\mathcal{V}_k$ denote the open cover $\{V_{ijk}\}_{ij}$ (where k is fixed).

As a result, the system $\mathcal{V}_k$ of open covers satisfies the following properties:

- The system is cofinal as k ranges over the positive integers
- The covering $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$
- For each i and k, the collection $\{V_{ijk}\}_j$ is a covering of U_i

Furthermore, for fixed i, the open covers $\{V_{ijk}\}_j$ form a cofinal system of open covers of U_i as k varies, and each such covering is a refinement of the previous one.

For every $k \in \mathbb{Z}_{>0}$, let $I_k \subseteq I$ be the set of $i \in I$ for which there exists j such that $\alpha \in V_{ijk}(\mathbb{A}_{k,S})$. Because $\alpha \in U(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$, the set I_k is nonempty for all k. Furthermore, $I_{k+1} \subseteq I_k$, as $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$. As I is finite, the set $\bigcap_k I_k$ is nonempty. Choose i in this intersection.

We conclude that $\alpha \in U_i(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$, since the open covers $\{V_{ijk}\}_j$ form a cofinal system of open covers of U_i. It follows that g is surjective.

Finally, g is injective by assumption. It follows that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is in fact a cosheaf.

Now let $\mathcal{G}$ be a cosheaf. It suffices to show that every precosheaf map $f\colon \mathcal{G} \to \underline{X(\mathbb{A}_{k,S})^{\omega}}$ comes from a unique map $\mathcal{G} \to \underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$. Uniqueness follows from the injectivity of $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}} \to \underline{X(\mathbb{A}_{k,S})^{\omega}}$. It is simply necessary to show that the image of any such f lands in $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$.

Let $U \in \mathrm{ob}(X_{\mathrm{Zar}})$, and let $s \in \mathcal{G}(U)$. We want to show that $f(s) \in U(\mathbb{A}_{k,S})^{\omega}$ actually lies in the subset $U(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}$. For this, let $\{U_i\}$ be a covering of U. We want to show that $f(s) \in U_i(\mathbb{A}_{k,S})^{\omega}$ for some i. By the cosheaf property, there exists i such that s is the corestriction of some element $s_i \in \mathcal{G}(U_i)$. But then $f(s_i)$ is the element of $U_i(\mathbb{A}_{k,S})^{\omega}$ we seek, so we are done.

Under the assumption about Question 4.3.2, Lemma 8.2.5 implies that $\underline{X(\mathbb{A}_{k,S})^{\mathcal{OPEN},\omega}}$ is a cosheaf. □

Our main results can therefore be thought of as a description of the cosheafification of $\mathcal{F}_{(\mathrm{f-ab},f,k)}$ or $\mathcal{F}_{(\mathrm{f-cov},f,k)}$.

Furthermore, the interest of Question 8.1.1 is based on the fact that $\mathcal{F}_{\mathbb{A}_{k,S}}$ is not a cosheaf.

Finally, in this language, this cosheafification contains $\mathcal{F}_k$ precisely because $\mathcal{F}_k$ maps to $\mathcal{F}_{\mathbb{A}_{k,S}}$ and is a cosheaf.

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