Decomposition rank of UHF-absorbing C*-algebras

Joint work with Hiroki Matui

12, Mar., 2013. Sde Boker
Murray-von Neumann equivalence

\( \mathcal{M} \): a finite von Neumann algebra,
\( p, q \): two projections in \( \mathcal{M} \).

If \( \tau(p) = \tau(q) \), for any tracial state \( \tau \) of \( \mathcal{M} \),
then there exists \( v \in \mathcal{M} \) such that

\[
    v^*v = p, \quad vv^* = q.
\]

This condition plays an essential role in the classification theorem of injective factors.
Murray-von Neumann equivalence and AFD

A. Connes proved that any injective factor with a separable predual is approximately finite dimensional (AFD), by using his deep study of automorphisms. And he classified injective factors of type II and type III\(\lambda\), \(\lambda \neq 1\).
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U. Haagerup gave an alternative proof (injectivity $\Rightarrow$ AFD) without using automorphisms, and classified the injective factor of type $\text{III}_1$.

S. Popa also gave another short proof by using excisions of amenable traces.

In Connes and Haagerup’s argument, they showed AFD by using a partial isometry $\nu$ which induces the Murray-von Neumann equivalence.
Murray-von Neumann equivalence for C*-algebras

Theorem (1980. Cuntz-Pedersen.)

Let $A$ be a C*-algebra, $p, q$ projections in $A$. If $\tau(p) = \tau(q)$ for any tracial state $\tau$ of $A$. Then $\exists v_i \in A, i = 1, 2, ..., N$, such that

$$\sum v_i^* v_i = p,$$  $$\sum v_i v_i^* = q.$$
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Lemma (2013. Matui-Sato)

Let $A$ be a $C^*$-algebra with strict comparison for projections, $p, q$ projections in $A$. If $\tau(p) = \tau(q)$ for any tracial state $\tau$ of $A$. Then $\exists v_i \in A \otimes M_n, i = 1, 2$, such that

$$\sum v_i^* v_i \approx_{4/n} p \otimes 1_n, \quad \sum v_i v_i^* \approx_{4/n} q \otimes 1_n.$$
Main theorem


Let $A$ be a unital separable, simple, $C^*$-algebra with a unique tracial state. Then $A$ is nuclear, with strict comparison, and is quasidiagonal $\iff \text{dr}(A) < \infty$, (in particular $\text{dr}(A) \leq 3$).
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Theorem (2013. H. Matui - Y. Sato)

Let $A$ be a unital separable, simple, C*-algebra with a unique tracial state. Then $A$ is nuclear, with strict comparison, and is quasidiagonal if and only if $\text{dr}(A) < \infty$, (in particular $\text{dr}(A) \leq 3$).

- If $A$ is in the above theorem, $A$ has strict comparison if and only if $A \otimes \mathbb{Z} \cong A$. 
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- **If $A$ is in the above theorem,**
  - $A$ has strict comparison $\iff A \otimes \mathcal{Z} \cong A$.
- **$A$ is quasidiagonal** $\iff A \hookrightarrow \prod M_{k_n} / \bigoplus M_{k_n}$, D. Voiculescu.
**Definition (2002. E. Kirchberg - W. Winter.)**

Let $A$ be a separable $C^*$-algebra. $A$ has decomposition rank at most $N$, $\text{dr}(A) \leq N$, if

- $\exists \varphi_n : A \rightarrow \bigoplus_{i=0}^{N} M_{k_i,n} : \text{c.p.c},$
- $\exists \psi_{i,n} : M_{k_i,n} \rightarrow A : \text{order zero (disjointness preserving), c.p.c}$

such that

\[ \sum \psi_{i,n} \text{ is also contractive,} \]

\[ \|(\sum \psi_{i,n}) \circ \varphi_n(a) - a\| \rightarrow 0, \quad \forall a \in A, \]

where we simply write $\left( \sum \psi_{i,n} \right)(\bigoplus x_i) := \sum \psi_{i,n}(x_i)$. 
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- $A \otimes \mathbb{Z} \cong A \implies A$ has strict comparison, 2001 M. Rørdam.
Main theorem


Let $A$ be a unital separable, simple, $C^*$-algebra with a unique tracial state. Then

- $A$ is nuclear, with strict comparison, and is quasidiagonal $\iff \text{dr}(A) < \infty$, (in particular $\text{dr}(A) \leq 3$).

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Toms - Winter conjecture

**Conjecture** (2009. A. Toms, W. Winter.)

Let $A$ be a unital separable simple nuclear finite $C^*$-algebra with infinite-dimension. Then the following are equivalent.

1. $A \otimes \mathbb{Z} \cong A$.  

(i) $A \otimes \mathbb{Z} \cong A$.  

2. $A$ has the strict comparison.

(iii) $\text{dr}(A) < \infty$

(i) $\Rightarrow$ (ii) 2001. M. Rørdam.

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Corollary of the main theorem

Suppose that $A$ is a unital separable simple nuclear $C^*$-algebra. Assume that $A$ is quasidiagonal and with a unique tracial state. Then (ii) $\Rightarrow$ (iii).
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Without Q.D.

Here we assume that the T. W. conjecture has been completely proved without Q.D. For a unital separable simple nuclear $C^*$-algebra $A$ it follows that $A \otimes \mathcal{Z}$ also absorbs $\mathcal{Z}$ ((i) in the T. W. conjecture.).
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\[ \therefore \text{ stably-finite and nuclearity } \sim \text{ quasidiagonality.} \]
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$\implies \text{dr}(A \otimes \mathcal{Z}) < \infty$ ((iii) in T.W. conjecture).

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$\therefore$ stably-finite and nuclearity $\rightsquigarrow$ quasidiagonality.

Problem (in Blackadar-Kirchberg)

Is any stably-finite $C^*$-algebra quasidiagonal?
Main theorem

<table>
<thead>
<tr>
<th>Theorem</th>
<th>(2013. H. Matui - Y. Sato)</th>
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