UHF slicing and classification of nuclear C*-algebras

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Workshop on C*-Algebras and Noncommutative Dynamics
Sde Boker, Israel
March 2013
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Let \( \mathcal{A} \) be the class of separable nuclear unital simple \( C^* \)-algebras satisfying

1. \( A \in \mathcal{A} \implies A \) is locally recursive subhomogeneous (RSH) where the RSH algebras can be chosen so that projections can be lifted along an \((\mathcal{F}, \eta)\)-connected decomposition,

2. \( A \in \mathcal{A} \implies T(A) \) has finitely many extreme points, each of which induce the same state on \( K_0(A) \).
Theorem (S.–Winter)

Let $A, B \in A$. Then

$$A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z} \iff \text{Ell}(A \otimes \mathcal{Z}) \cong \text{Ell}(B \otimes \mathcal{Z})$$
Theorem (S.–Winter)

Let $A, B \in A$. Then

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Corollary

Let $A, B \in A$ and suppose that $A$ and $B$ have finite decomposition rank. Then

$$A \simeq B \iff \Ell(A) \simeq \Ell(B)$$
Tensor with a UHF algebra to care of the lack of projections. UHF-stable classification can (often) be used to deduce $\mathcal{Z}$-stable classification (e.g. Winter, Lin).
Key tools

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2. Tracial approximation for $A \otimes Q$, for the universal UHF algebra $Q$ (i.e. $K_0(Q) \cong \mathbb{Q}$).
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We will show that $A \in \mathcal{A} \implies A \otimes Q$ is a tracially approximately interval algebra (TAI).
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1. Tensor with a UHF algebra to care of the lack of projections. UHF-stable classification can (often) be used to deduce $\mathcal{Z}$-stable classification (eg. Winter, Lin).

2. Tracial approximation for $A \otimes Q$, for the universal UHF algebra $Q$ (i.e. $K_0(Q) \cong \mathbb{Q}$).

We will show that $A \in \mathcal{A} \implies A \otimes Q$ is a tracially approximately interval algebra (TAI).

Then (Lin, 2009) $\implies$ classification.
Tracial approximation

$A$ is tracially approximately $S$:

$$F \subseteq \epsilon \left( \begin{array}{c} (1-p)A(1-p) \\ B \end{array} \right)$$

$p = 1_B, B \in S$
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$\tau(1-p) < \epsilon$ for every $\tau \in T(A)$
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$$
I = \{ (\bigoplus_{k=1}^{K} C([0,1]) \otimes M_{n_k}) \oplus (\bigoplus_{l=1}^{L} M_{n_l}) \}
$$
Main theorem

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Let $A \in \mathcal{A}$. Then $A \otimes Q$ is TAI.
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Let $A \in \mathcal{A}$. Then $A \otimes Q$ is TAI.

Recall: $\mathcal{A}$ is the class of separable nuclear unital simple $C^*$-algebras satisfying

1. $A \in \mathcal{A} \implies A$ is locally recursive subhomogeneous (RSH) where the RSH algebras can be chosen so that projections can be lifted along an $(\mathcal{F}, \eta)$-connected decomposition,

2. $A \in \mathcal{A} \implies T(A)$ has finitely many extreme points, each of which induce the same state on $K_0(A)$.
Recursive subhomogeneous $C^*$-algebras

[Phillips 2001] $B$ is RSH if it can be written as an iterated pullback

$$B = \left( \ldots \left( \left( C_0 \oplus C_1^{(0)} \oplus C_1 \right) \oplus C_2^{(0)} \oplus C_2 \right) \ldots \right) \oplus C_R^{(0)} C_R,$$

where

$$C_l = C(X_l) \otimes M_{n_l}$$

for some compact metrizable $X_l$ and

$$C_l^{(0)} = C(\Omega_l) \otimes M_{n_l}$$

for a closed subset $\Omega_l \subset X_l$. 
Recursive subhomogeneous $C^*$-algebras

The $i$th stage $B_i$ is given by

$$B_i = B_{i-1} \oplus C^{(0)}_i, \quad C_i = \{(b, c) \in B_{i-1} \oplus C_i \mid \phi(b) = \rho(c)\}$$

where

$$\phi : B_{i-1} \to C^{(0)}_i$$

is a unital $\ast$-homomorphism, and

$$\rho : C_i \to C^{(0)}_i$$

is the restriction map.
The decomposition is not unique, so we keep track of it:

$$[B_l, X_l, \Omega_l, n_l, \phi_l]_{i=1}^R.$$
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We say that projections can be lifted along this decomposition if:

\[ \forall n \in \mathbb{N}, \forall l = 1, \ldots, R - 1 \text{ and for every projection } p \in B_l \otimes M_n, \text{ there exists a projection } \tilde{p} \in B_{l+1} \otimes M_n \text{ lifting } p. \]
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Proposition

If \( \dim(X_l) \leq 1 \) for \( l = 2, \ldots, R \) then projections can be lifted along

$$[B_l, X_l, \Omega_l, n_l, \phi_l]_{l=1}^R.$$
Idea of proof ($A \in \mathcal{A} \implies A$ TAI):

Given $\mathcal{F} \subset \subset A \otimes Q$, $\epsilon > 0$, need $C \in I$ with $\tau(1_C)$ bounded away from 0, $\forall \tau$,
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Assume \(\tau_0, \tau_1\) are the only extreme tracial states.
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Find a tracially large interval: Take $a \in (A \otimes Q)_+$ with $\tau_0(a) \approx 0$ and $\tau(a) \approx 1$ (Brown–Toms 2007), then take $C^*(a, 1)$.
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Must move this interval into position (w.r.t. \(\mathcal{F}\)): model an interval in \(B \otimes \mathcal{Q}\), use strict comparison.
Interval model: excisors and bridges

Definition

$B$ unital RSH with decomposition $[B_l, X_l, \Omega_l, n_l, \phi_l]$, $\mathcal{F} \subset B_1^+$, $\eta > 0$. An $(\mathcal{F}, \eta)$-excisor $(E, \rho, \sigma, \kappa)$ consists of
**Definition**

$B$ unital RSH with decomposition $[B_l, X_l, Ω_l, n_l, φ_l]$, $F \subset \subset B_1^+$, $η > 0$. An $(F, η)$-excisor $(E, ρ, σ, κ)$ consists of

1. a finite dimensional algebra $E = \bigoplus_{l=1}^{R} E_l$, 

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Interval model: excisors and bridges

Definition

$\mathcal{B}$ unital RSH with decomposition $[B_i, X_i, \Omega_i, n_i, \phi_i], \mathcal{F} \subset \subset B_1^+, \eta > 0$. An $(\mathcal{F}, \eta)$-excisor $(E, \rho, \sigma, \kappa)$ consists of

1. a finite dimensional algebra $E = \bigoplus_{i=1}^{R} E_i$,
2. a unital $\ast$-homomorphism $\rho = \bigoplus_{i=1}^{R} \rho_i : B \to \bigoplus_{i=1}^{R} E_i$
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$$\|\sigma(1_E)(b \otimes 1_Q) = \sigma \circ \rho(b)\| < \eta$$

for all $b \in \mathcal{F}$,
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4. for all $b \in \mathcal{F}$,

4. a unital $\ast$-homomorphism $\kappa : E \to Q$. 

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We say that \((E, \rho, \sigma, \kappa)\) is compatible with the RSH decomposition if each \(\rho_l\) factorizes through

\[
\begin{array}{ccc}
B & \xrightarrow{\rho_l} & E_l \\
\downarrow \psi_l & & \uparrow \check{\rho}_l \\
B_l & \xrightarrow{\check{\psi}_l} & C(\check{X}_l) \otimes M_{r_l}
\end{array}
\]

for some compact \(\check{X}_l \subset X_l \setminus \Omega_l\).
Interval model: excisors and bridges

**Definition**

An \((\mathcal{F}, \eta)\)-bridge between \((E_0, \rho_0, \sigma_0, \kappa_0)\) and \((E_1, \rho_1, \sigma_1, \kappa_1)\) consists of \(K \in \mathbb{N}\) and \((\mathcal{F}, \eta)\)-excisors \((E_j/K, \rho_j/K, \sigma_j/K, \kappa_j/K)\), \(j = 1, \ldots, K - 1\) satisfying

\[
\|\kappa_j/K \circ \rho_j/K(b) - \kappa_{j+1}/K \circ \rho_{j+1}/K(b)\| < \eta
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for all \(b \in \mathcal{F}\) and \(j = 0, \ldots, K - 1\).
Interval model: excisors and bridges

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\| \kappa_j/K \circ \rho_j/K(b) - \kappa((j+1)/K \circ \rho((j+1)/K(b)) \| < \eta
\]

for all \(b \in \mathcal{F}\) and \(j = 0, \ldots, K - 1\).

In this case, write \((E_0, \rho_0, \sigma_0, \kappa_0) \sim_{(\mathcal{F}, \eta)} (E_1, \rho_1, \sigma_1, \kappa_1)\).
$(\mathcal{F}, \eta)$-connected decomposition

$$[B_l, X_l, \Omega_l, n_l, \phi_l]_{l=1}^R$$ the RSH decomposition.

For every $l = 1, \ldots, R$ and every $x \in X_l$ we can define an $(\mathcal{F}, \eta)$-excisor.
(\mathcal{F}, \eta)-connected decomposition

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For every \(l = 1, \ldots, R\) and every \(x \in X_l\) we can define an \((\mathcal{F}, \eta)\)-excisor.

The decomposition is \((\mathcal{F}, \eta)\)-connected if, for any \(l = 1, \ldots, R\) and any \(x, y \in X_l\), we can always find an \((\mathcal{F}, \eta)\)-bridge between their corresponding \((\mathcal{F}, \eta)\)-excisors.
With a suitable calculus for $(\mathcal{F}, \eta)$-excisors, we can find $(E_0, \rho_0, \sigma_0, \kappa_0)$ and $(E_1, \rho_0, \sigma_0, \kappa_0)$ with $\tau_i(\sigma(1_{E_i}))$ large and $\tau_i(\sigma_j(1_{E_j}))$ small, $i \neq j \in \{0, 1\}$.
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It remains to find an $(\mathcal{F}, \eta)$-bridge through the decomposition.

To do this, we use linear algebra based on equations which we can read off the RSH decomposition.
Interval model: large trace

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To do this, we use linear algebra based on equations which we can read off the RSH decomposition.

This is where we require that projections can be lifted and that each tracial state induces the same state on $K_0$. 
What we get...

With two extreme tracial states \( \tau_0, \tau_1 \), we find \((\mathcal{F}, \eta)\)-excisors

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(E_0, \rho_0, \sigma_0, \kappa_0) \sim_{(\mathcal{F}, \eta)} (E_1, \rho_1, \sigma_1, \kappa)
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What we get...

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such that

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Now use strict comparison to move the elements in a partition of unity of the actual interval under this model.
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Now use strict comparison to move the elements in a partition of unity of the actual interval under this model. This will be an interval which is large in trace, and the condition

$$\| \sigma(1_E)(b \otimes 1_Q) = \sigma \circ \rho(b) \| < \eta$$

for all $b \in \mathcal{F}$, for $(\mathcal{F}, \eta)$-excisors allows us to properly approximate elements in the finite subset.
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$\sim A \otimes Q$ is TAI.
Main theorem

**Theorem (S.–Winter)**

Let \( A, B \in \mathcal{A} \). Then

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A \otimes \mathbb{Z} \cong B \otimes \mathbb{Z} \iff \text{Ell}(A \otimes \mathbb{Z}) \cong \text{Ell}(B \otimes \mathbb{Z}).
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Where $\mathcal{A}$ is the class of separable nuclear unital simple $C^*$-algebras satisfying

1. $A \in \mathcal{A} \implies A$ is locally recursive subhomoeneous (RSH) where the RSH algebras can be chosen so that projections can be lifted along an $(\mathcal{F}, \eta)$-connected decomposition,

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Some consequences

1. Elliott 1996 – Simple approximately SH algebras constructed by attaching 1-dimensional spaces to the circle. Theorem $\implies$ classification when restricted to finitely many extreme tracial states, each inducing same $K_0$-state.

2. Lin–Matui 2005 – $A := C(X \times T) \rtimes \mathbb{Z}$. Restricting to finitely many traces each inducing same state on $K_0$, theorem $\implies A\{x\} \otimes \mathbb{Q}$ is TAI, then (S.-Winter 2010) $\implies A \otimes \mathbb{Q}$ TAI $\implies$ classification.
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1. Elliott 1996 – Simple approximately SH algebras constructed by attaching 1-dimensional spaces to the circle. Theorem $\Rightarrow$ classification when restricted to finitely many extreme tracial states, each inducing same $K_0$-state.

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