An Introduction to E_0-Semigroups

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Abstract

This is a quick overview of E_0-semigroups from the product systems point of view. It was meant for a short series of talks at the Technion and Tel Aviv University, and it is a bit sketchy. For a more detailed introduction with full proofs, see the book by William Arveson, *Non-Commutative Dynamics and E-semigroups*.

Disclaimer: this writeup is still in flux. It may not be free from mistakes and may lead the reader to look further into the topic. If you find any mistakes please let me know.

Contents

1 Definition of E_0-semigroup

1.1 Basic Definition

Let $\mathfrak{H}$ be a separable Hilbert space.

Definition (Powers 88)**.** An E_0**-semigroup** is a semigroup $\{\alpha_t : t \geq 0\}$ of normal, $*$-endomorphisms of the algebra $B(\mathfrak{H})$ of all bounded operators over $\mathfrak{H}$, which preserve the unit

$$\alpha_t(I) = I, \qquad \forall t \geq 0$$

and such that for every $\xi, \eta \in \mathfrak{H}$, $X \in B(\mathfrak{H})$ the map

$$t \mapsto \langle \alpha_t(X)\xi, \eta \rangle \qquad \text{is continuous.}$$

1.2 Examples and Non-Examples

- **Example:** Let U_t be a strongly continuous group of unitary operators with $U_0 = I$ acting on a Hilbert space $\mathfrak{H}$:

  $$\alpha_t(X) = U_t X U_t^*$$

  is an E_0-semigroup, although here the endomorphisms are automorphisms.

- **NOT an example:** repeat with U_t (sot-cts) semigroup of isometries, e.g. the unilateral shift on $L^2(0,\infty)$ given by

$$(U_t\psi)(x) = \begin{cases} 0 & , x \leq t \\ \psi(x-t) & , x > t \end{cases}$$

  Not unital: $\alpha_t(I)$ is projection onto $L^2(t,\infty)$.

- **NOT an Example:** Can't build an example in $M_n(\mathbb{C})$. We need an endormorphism $M_n(\mathbb{C})$ but only the trivial ones exist, because the algebra is simple!

1.3 No "Wold" Decomposition

One may recall that for one-parameter strongly continuous semigroups of isometries there exists a generalizatio of the Wold decompostion. It suggests studying the space $H_\infty = \cap_{t>0} U_t H$ and $H_0 = H_\infty^\perp$. Then $U_t = V_t \oplus W_t$ on $H_0 \oplus H_\infty$. V_t is a pure shift and W_t is unitary. And in fact V_t is unitarily equivalent to a unilateral shift on $L^2(0,\infty;\mathfrak{C})$ for some $\mathfrak{C}$.

Nothing like this is true for E_0-semigroups. One can indeed repeat the construction:

$$M_\infty = \cap_{t>0}\alpha_t(M)$$

We can then set $N = M'_\infty$ (the commutant) and in the case when M_∞ is a factor, one has once again a decomposition

$$B(H) = M_\infty \vee N$$

by a pair of mutually commuting factors, α automorphism group on M_∞ and "pure" semigroup on N in the sense that

$$\cap_{t\geq 0}\alpha_t(N) = \mathbb{C}I$$

And, if M_∞ is a type I factor, we indeed have $B(H) = M_\infty \otimes N$, and $\alpha = \beta \otimes \gamma$.

Unfortunately M_∞ does not need to be a type I factor. . Furthermore, even if this were the case, not much is known about the classification of pure semigroups.

2 Motivation from Physics

2.1 Closed Quantum Systems:

2.1.1 Mathematical Framework:

- State space is a Hilbert space $\mathfrak{H}$

- Hamiltonian is a self-adjoint unbounded operator B over $\mathfrak{H}$

- Schrödinger picture of Time Evolution: Observables remain fixed, and a state ψ_0 evolves to $\psi(t)$ according Schrödinger's equation:

$$\frac{d\psi(t)}{dt} = iB\psi(t), \qquad \psi(0) = \psi_0$$

  or alternatively $\psi(t) = U_t\psi_0$ where $U_t = e^{itB}$ is a unitary one-parameter **group** (time-reversiblity).

2.1.2 Automorphism Groups

- In the Heisenberg picture, states remain fixed, and an observable A_0 will evolve to $A(t)$:
$$A(t) = U_t^* A_0 U_t, \qquad A(0) = A_0$$

This leads to the study of groups of **automorphisms** of $B(\mathfrak{H})$. This is **completely understood**, including classification up to conjugacy:

Definition. We say that two E_0-semigroups α and β acting respectively on $B(\mathfrak{H})$ and $B(\mathfrak{K})$ are **conjugate** if there exists a *-isomorphism $\theta : B(\mathfrak{H}) \to B(\mathfrak{K})$ such that for all t,

$$\theta \circ \alpha_t = \beta_t \circ \theta$$

2.1.3 Classification up to conjugacy

Theorem (Wigner)**.** *Let $\{\alpha_t : t \in \mathbb{R}\}$ be a w^*-continuous one-parameter group of automorphisms of $B(\mathfrak{H})$. Then there exists a strongly continuous one-parameter unitary group $\{U_t : t \in \mathbb{R}\}$ such that for all $t \in \mathbb{R}$ and $X \in B(\mathfrak{H})$,*

$$\alpha_t(X) = U_t X U_t^*.$$

By Stone's theorem the unitary groups arising here are generated by unbounded self-adjoint operators. It follows that the classification up to conjucacy essentially reduces to the classification of these operators up to unitary equivalence – the well-known Hahn-Hellinger multiplicity theory.

2.2 Open Quantum Systems

2.2.1 Mathematical Framework:

Intrinsic description of a quantum system which interacts with the "external world" (system and "reservoir"). Think Schrödinger's cat and decoherence.

What about the time evolution α_t?

- Usually irreversible (=not automorphism!)
- Physical considerations suggest it is completely positive (=not multiplicative!)
- Additional assumption verified experimentally in many cases: approximately Markovian (=one-parameter semigroup).

2.2.2 Complete Positivity

Definition. A linear map between C^*-algs $\phi : A \to B$ induces a family of maps $\phi^{(n)} : M_n(A) \to M_n(B)$, $n \in \mathbb{N}$. For example

$$\phi^{(2)}((A_{ij})) = \begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) \\ \phi(A_{21}) & \phi(A_{22}) \end{bmatrix}$$

ϕ is Completely positive if $\phi^{(n)}$ is positive for every $n \geq 1$.

Examples.

- Any *-homomorphism of C^*-algebras is completely positive
- Any positive linear functional is completely positive
- If A is commutative, $\phi : A \to B(\mathfrak{H})$ positive is completely positive.
- An example that positivity does not imply complete positivity: the map $M_2(\mathbb{C}) \to M_2(\mathbb{C})$ given by
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.2.3 CP semigroups

Definition. A CP semigroup is a semigroup $\{\phi_t : t \geq 0\}$ of normal, unital completely positive maps $B(\mathfrak{K}) \to B(\mathfrak{K})$,

$$t \mapsto \langle \phi_t(x)\xi, \eta \rangle \quad \text{continuous,}$$

for all $\xi, \eta \in \mathfrak{K}, x \in B(\mathfrak{K})$.

- CP semigroups appear at first glance more abundant than E_0-semigroups: exist for $\dim(\mathfrak{K}) < \infty$!
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & e^{-t}b \\ e^{-t}c & d \end{bmatrix}$$

2.2.4 Dilations and Compressions

Definition. Given an E_0-semigroup α_t on $B(\mathfrak{H})$, and an increasing projection p, i.e. $\alpha_t(p) \geq p$ then
$$\phi_t(X) = p\alpha_t(X)p$$
is a CP semigroup of $pB(\mathfrak{H})p \simeq B(p\mathfrak{H})$. We say that ϕ_t **is a compression of** α_t or α_t **is a dilation of** ϕ_t.

Theorem (Bhat and Parthasarathy)**.** *Every CP semigroup can be dilated to an E_0 semigroup.*

- Analogy: CP semigroups are like states and E_0-semigroups like representations.

3 Examples of E_0-semigroups: CCR flows

3.1 Bosonic Fock Spaces

- **Symmetric Fock space:** $\mathcal{F}_+(\mathfrak{K}) = \oplus_{n=0}^{\infty} \mathfrak{K}^{\circledcirc n}$ where

$$v_1 \circledcirc v_2 \circledcirc \cdots \circledcirc v_n = \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

- Define the **exponential vectors**:

$$\exp v = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^{\otimes n}$$

- $\langle \exp v, \exp w \rangle = e^{\langle v,w \rangle}$
- $\mathcal{F}_+(\mathfrak{K}) = \overline{\text{span}}\{\exp(v) : v \in \mathfrak{K}\}$
- Given two Hilbert spaces $\mathfrak{K}_1, \mathfrak{K}_2$, there is a unique unitary map:

$$\mathcal{F}_+(\mathfrak{K}_1) \otimes \mathcal{F}_+(\mathfrak{K}_2) \to \mathcal{F}_+(\mathfrak{K}_1 \oplus \mathfrak{K}_2)$$
$$\exp(u) \otimes \exp(v) \mapsto \exp(u \oplus v)$$

- The **Second Quantization** of an isometry $V : \mathfrak{K} \to \mathfrak{K}$ is the isometry $\Gamma(V)$ defined uniquely by the expression

$$\Gamma(V)\exp(f) = \exp(Vf)$$

- For each $f \in \mathfrak{K}$, define the **Weyl unitary** $W(f)$ as follows:

$$W(f)\exp v = e^{-\frac{1}{2}\|f\|^2 - \langle v,f \rangle} \exp(f + v)$$

3.2 CCR algebras and CCR functor

- **Weyl form of the Canonical Commutation Relations:**

$$W(f)W(g) = e^{i\,\mathrm{Im}\langle f,g \rangle} W(f+g), \qquad \forall f, g \in \mathfrak{K}$$

- For any subspace $\mathfrak{X} \subset \mathfrak{K}$, define the **CCR algebra**:

$$\mathcal{A}(\mathfrak{X}) = C^*(W(f) : f \in \mathfrak{X})$$

- $\mathfrak{X} \subseteq \mathfrak{K} \Rightarrow \mathcal{A}(\mathfrak{X}) \subseteq \mathcal{A}(\mathfrak{K})$
- Any isometry $V : \mathfrak{K} \to \mathfrak{K}$ induces a unique unital *-endomorphism $\alpha_V : \mathcal{A}(\mathfrak{K}) \to \mathcal{A}(\mathfrak{K})$ satisfying

$$\alpha_V(W(f)) = W(Vf)$$

- This gives a from Hilbert space, isometries to CCR algebras and endomorphisms.
- The algebra $\mathcal{A}(\mathfrak{K})$ is w*-dense in $B(\mathcal{F}_+(\mathfrak{K}))$.

3.3 CCR flow

- Take $\mathfrak{K} = L^2(0, \infty; \mathfrak{C})$ where $\mathfrak{C}$ is a separable Hilbert space. Let $\mathfrak{H} = \mathcal{F}_+(\mathfrak{H})$.
- Apply the functor to the shift semigroup of isometries $S_t : L^2(0, \infty; \mathfrak{C}) \to L^2(0, \infty; \mathfrak{C})$ to obtain one-parameter semigroup of *-endomorphisms α_{S_t} – but of $\mathcal{A}(\mathfrak{K})$.
- For each t, the map α_{S_t} extends to a normal *-endomorphism $\alpha_t^{\mathfrak{C}}$ on $B(\mathfrak{H})$.
- $\alpha^{\mathfrak{C}}$ is called a CCR-flow of rank c, where $c = \dim \mathfrak{C}$ and which can be $1, 2, 3, \ldots, \aleph_0$.

4 Product Systems

4.1 Basic Questions

- How do we distinguish examples of E_0-semigroups?
- We must do Classification Theory – but with respect to what equivalence relation?
- Even for single endomorphisms conjugacy is not appropriate (Bratteli, Jørgensen, Price: Borel structure is not countably separated)
- Spectacular theory of automorphism groups of general von Neumann algebras suggests a different equivalence relation: cocycle conjugacy.

4.2 Cocycle Conjugacy

Definition. We say that α, β E_0-semigroups on $B(\mathfrak{H})$ are **cocycle equivalent** if there is $(U_t)_{t\geq 0}$ strongly continuous family of unitary operators,

$$U_{t+s} = U_t \alpha_t(U_s), \qquad t, s \geq 0$$

and $\beta_t(X) = U_t \alpha(X) U_t^*$.

If α, β act on different Hilbert spaces, we say they are **cocycle conjugate** if there exists a conjugacy θ so that α_t and $\theta \circ \beta_t \circ \theta^{-1}$ are cocycle equivalent.

Theorem. *Any two automorphism groups $\alpha_t(X) = V_t X V_t^*$ and $\beta_t(X) = W_t X W_t^*$ are cocycle conjugate.*

Proof. May assume on same Hilbert space, so take $U_t = W_t V_t^*$. See the appendix. □

Generators and perturbations

- Generator δ defined by $\lim_{t\to 0} \| \frac{\alpha_t(X)-X}{t} - \delta(X) \| = 0$ densely defined.
- Given $H \in B(\mathfrak{H})$ self-adjoint,

  $$\delta'(X) = \delta(X) + i[H, X]$$

  defines a generator for another E_0-semigroup β_t which is cocycle conjugate to α_t, implemented by U_t as follows:

  $$U_0 = I, \qquad \frac{dU_t}{dt} = iU_t \alpha_t(H)$$

- **Heuristics:** *Cocycle conjugacy is perturbation by "unbounded inner derivation".*

4.3 Definition of Product systems

Definition. We say that a Borel space is standard if it is Borel-isomorphic to a Borel subset of a complete separable metric space.

Definition. A **(continuous tensor) product system** of Hilbert spaces is a structure $(E, p, \langle\cdot,\cdot\rangle, \cdot)$ such that:

1. E is a standard Borel space
2. $p : E \to (0, \infty)$ is a surjective measurable map
3. For every $t > 0$, $E_t = p^{-1}(\{t\})$ is a separable Hilbert space under $\langle\cdot,\cdot\rangle$
4. The vector space operations and inner product are measurable.
5. E is isomorphic as a measurable Hilbert bundle to $(0, \infty) \times \ell^2(\mathbb{N})$
6. The multiplication operation $E \times EtoE$ is associative, measurable, and it "acts like tensoring":
   - $E(t + s) = \overline{\operatorname{span}}\, E(t) \cdot E(s)$
   - $< u \cdot x, v \cdot y >=< u, v >< x, y >$, for $u, v \in E(t)$ and $x, yinE(s)$

Remark. Think "discrete" tensor product system, where $\mathfrak{H}$ is a Hilbert space, and $E = \cup_n (n, E_n)$ where

$$E_n = H^{\otimes^n}$$

and the product is given by $vw = v \otimes w$. Much easier since there is no "continuity" involved.

Remark. Notice that if E is a product system, then $dimE_t$ is constant for all $t > 0$, and it is equal to 1 or $aleph_0$.

4.4 An Example: Exponential Product Systems

Let $\mathfrak{C}$ be a separable Hilbert space. For every $t > 0$, let $\mathfrak{K}_t = L^2(0, \infty; \mathfrak{C})$, and let $\mathfrak{H}_t = \mathcal{F}_+(\mathfrak{K}_t)$.

Definition. Let $U_t : L^2(0, \infty; \mathfrak{C}) \to L^2(t, \infty; \mathfrak{C})$ denote the unitary map corresponding to the unilateral shift to the shifted space.

Definition. Define the **exponential product system** $E^{\mathfrak{C}} = \cup_{t>0}(t, \mathfrak{H}_t)$. Define the product as the unitary map corresponding to the equivalence

$$\mathfrak{H}_{t+s} = \mathcal{F}_+(\mathfrak{K}_{t+s}) = \mathcal{F}_+(\mathfrak{K}_t \oplus U_t\mathfrak{K}_s) \simeq \mathfrak{H}_t \otimes \mathfrak{H}_s$$

More precisely

$$\exp f \cdot \exp g = \exp(f + U_t g), \qquad f \in \mathfrak{H}_t, g \in \mathfrak{H}_s$$

The Borel structure arises from $E^{\mathfrak{C}} \subseteq (0, \infty) \times B(\mathfrak{H}_\infty)$.

4.5 Concrete Product Systems

Definition. We always consider $B(\mathfrak{H})$ as endowed with the Borel structure arising from the w*-topology. This coincides with strong, σ-strong, weak, etc. And it is a standard Borel structure.

Definition. By a **concrete product system** we mean a Borel subset $\mathcal{E}$ of $(0,\infty)\times B(\mathfrak{H})$ which has the following properties. The map $p : \mathcal{E} \to (0,\infty)$ is the projection $p(t,T) = t$ and it is surjective. Furthermore,

1. For each $t > 0$, $\mathcal{E}_t = p^{-1}(t)$ is a norm-closed subspace of $B(\mathfrak{H})$ with the property that B^*A is a scalar for all $A, B \in E_t$.
2. For every $t, s > 0$, $\mathcal{E}_{t+s}$ is the closed linear span of $\{AB : A \in \mathcal{E}_t, B \in \mathcal{E}_s\}$.
3. $\mathcal{E}$ is trivializable, i.e. it is isomorphic as a measurable Hilbert bundle to $(0,\infty)\times \ell^2$.

Theorem. *Let $\varphi : B(\mathfrak{H}) \to B(\mathfrak{H})$ be a *-endomorphism (not necessarily unital). Then there exists a (possibly infinite) sequence of isometries $V_1, V_2, \ldots$ with mutually orthogonal ranges such that*

$$\varphi(X) = \sum_{n=1}^{\infty} V_n X V_n^*$$

and $\varphi(X)V_n = V_n X$ for all n.

Definition. We say that α is an **E-semigroup** if it satisfies all the properties of an E_0-semigroup with the exception that α_t is not required to be unital. (Although we require it to be nonzero).

Theorem (Arveson)**.** *Let α be an E-semigroup on $B(\mathfrak{H})$, and define $E_\alpha = \cup_t (t, E_\alpha(t))$ where For each t,*

$$E_\alpha(t) = \{A \in B(\mathfrak{H}) : \alpha_t(X)A = AX, \ \forall X \in B(\mathfrak{H})\}$$

Then E_α is a concrete product system.

Proof. (Sketch).

- For each $A, B \in E_\alpha(t)$, and any $X \in B(\mathfrak{H})$, notice that

  $$A^*BX = XB^*A \implies A^*B = (A,B)I$$

  Where $(,)$ is a sesquilinear form. In fact, it is an inner product because $A^*A = (A,A)I$, and $\|A^*A\| = \|A\|^2 = (A,A)$.

  Because the norm is the same as $B(H)$, this space is complete so it is a Hilbert space.

- Define the product $(t,A)\cdot(s,B) \mapsto (t+s, AB)$. The theorem above guarantees that $E(t+s) = \overline{\text{span}} E(t)E(s)$. The other property of multiplication is trivial.

- Endow E_α with the structure from its inclusion into $(0,\infty)\times B(\mathfrak{H})$. Trivialization is technical and we refer the reader to the references.

□

we have a converse for this theorem:

Theorem (Arveson). *Let $\mathcal{E}$ be a concrete product system on $B(\mathfrak{H})$. Then there exists an E-semigroup α such that $\mathcal{E} = \mathcal{E}_\alpha$.*

Proof. (Sketch). For each $\mathcal{E}_t$ pick an orthonormal basis $V_1, V_2, \dots$ and set

$$\alpha_t(X) = \sum_n V_n X V_n^*$$

This converges strongly and it is easy to check the algebraic properties. Measurabilty is a bit more technical. □

4.6 CCR flows and Exponential Product Systems

Definition. Let E, F be two product systems. We say that $\theta : E \to F$ is an **isomorphism (of product systems)** if

1. $\theta(E_t) = F_t$
2. $\langle \theta(x), \theta(y) \rangle = \langle x, y \rangle$
3. $\theta(xy) = \theta(x)\theta(y)$
4. θ is measurable

We characterize the product systems associated to the CCR flows:

Theorem (Arveson). *Let $\mathfrak{C}$ be a separable Hilbert space and let $\alpha^{\mathfrak{C}}$ be the corresponding CCR flow on the space $\mathcal{F}_+(L^2(0,\infty;\mathfrak{C}))$. There exists a unique isomorphism of product systems*

$$\phi : E^{\mathfrak{C}} \to \mathcal{E}_{\alpha^{\mathfrak{C}}}$$

which for $g \in E^{\mathfrak{C}}_t = \mathcal{F}_+(L^2(0,t;\mathfrak{C}))$ and $h \in L^2(0,\infty;\mathfrak{C})$, is given by:

$$\phi(\exp g)\exp(h) = \exp(g + S_t h)$$

where S_t is the shift semigroup on $L^2(0,\infty;\mathfrak{C})$.

Proof. (Sketch). We show only that $\phi(g)$ is indeed in the intertwiner space for α_t. For the remainder, see the references. Notice that for every $f \in L^2(0,\infty;\mathfrak{C})$

$$\alpha_t(W(f))\phi(\exp g)\exp h = W(S_t f)\exp(g + S_t h) = e^{-\frac{1}{2}\|S_t f\|^2 - \langle g + S_t h, S_t f\rangle}\exp(S_t f + g + S_t h)$$

But S_t is an isometry, and also $\langle g + S_t h, S_t f\rangle = \langle h, f\rangle$, hence

$$\alpha_t(W(f))\phi(\exp g)\exp h = e^{-\frac{1}{2}\|f\|^2 - \langle h, f\rangle}\exp(f + S_t(f + h)) = \phi(\exp g)W(f)\exp h$$

So that $\alpha_t(X)\phi(\exp g) = \phi(\exp g)X$ for all X by taking limits. □

4.7 Product Systems are complete invariants

Theorem (Arveson). *Two E_0-semigroups α and β are cocycle conjugate if and only if $E_\alpha \simeq E_\beta$.*

Proof. (Sketch)

($\Rightarrow$) Suppose α and β are cocycle conjugate. A simple checking shows that we may assume that they are cocycle equivalent and

$$\beta_t(X) = U_t \alpha_t(X) U_t^*$$

where U_t is the cocycle. Define a map $\theta : E_\alpha \to E_\beta$ as follows. If $A \in E_\alpha(t)$,

$$\theta_t(A) = U_t A$$

- it is well-defined

$$\beta_t(X)\theta_t(A) = U_t\alpha_t(X)U_t^* U_t A = U_t A X = \theta_t(A)X$$

- it preserves fibers by def
- it is obviously unitary fiberwise, since U_t unitary
- it preserves multiplication

$$\theta_t(A)\theta_s(B_= U_t A U_s B = U_t \alpha_t(U_s)AB = U_{t+s}AB = \theta_{t+s}(AB)$$

- it is measurable because U_t is continuous.

($\Leftarrow$) Suppose $\theta : E_\alpha \to E_\beta$ is an isomorphism. For each t, by the theorem on endomorphisms,

$$\alpha_t = \sum V_n(t) X V_n(t)^* \qquad \text{no continuity on } V_n(t) \text{ !!}$$

In fact, the V_n is an o.n. basis. Set

$$W_n(t) = \theta_t(V_n(t))$$

then

$$\beta_t = \sum W_n(t) X W_n(t)^*$$

Define

$$U_t = \sum_n W_n(t) V_n(t)^*$$

Then U_t is a cocycle. □

5 Classification Theory

5.1 Units

Definition. A **unit** for an E_0-semigroup α on $B(\mathfrak{H})$ is a strongly continuous one-parameter semigroup of operators T_t such that $T_0 = 1$ and

$$\alpha_t(X)T_t = T_t X, \qquad \forall t \geq 0, \forall X \in B(\mathfrak{H})$$

A **unit** for a product system E is a measurable section u of E such that $u_{t+s} = u_t u_s$ and it is not the trivial section $u \equiv 0$.

Remark. It is a nontrivial fact that units of E_0-semigroups are in 1-1 correspondence with the units of their corresponding concrete product systems. The difficult fact is that $u_t \to 1$ in its concrete product system realization, since this does not make sense in terms of abstract product systems. More in the references.

Example. The CCR flows have units. For example, the family defined by

$$U_t = \Gamma(S_t)$$

where S_t is the unilateral shift. Just check that

$$\alpha_t(W(f))\Gamma(S_t)\exp(g) = W(S_t f)\exp(S_t g) = \Gamma(S_t)W(f)\exp g$$

Definition. An E_0-semigroup is said to be **spatial** if it possesses a unit.

Theorem. *If two E_0-semigroups α and β are cocycle conjugate, α is spatial if and only if β is spatial.*

Proof. (Sketch). Let $\beta_t(X) = U_t\alpha_t(X)U_t^*$, $U_{t+s} = U_t\alpha_t(U_s)$. If T_t is a unit for α_t, then U_tT_t is a unit for β. □

Theorem (Powers '87)**.** *There exists at least one non-spatial E_0-semigroup.*

5.2 The Arveson Index

Assume that α is a spatial E_0-semigroup.

- Given two units T_t, S_t, for every $X \in B(\mathfrak{H})$:

  $$T_t^* S_t X = T_t^* \alpha_t(X) S_t = [\alpha_t(X^*)T_t]^* S_t = [T_t X^*]^* S_t = X T_t^* S_t$$

  So there exists $c(t)$ such that $T_t^* S_t = c(t)I$.

- In fact, $c(t+s) = c(t)c(s)$ and since $c(0) = 1$ and $c(t)$ is continuous, we conclude that there exists a complex number $\gamma(S,T)$ such that

  $$T_t^* S_t = e^{t\gamma(S,T)} I$$

Definition. We denote $\mathcal{U}_\alpha$ the set of units of α, and $\gamma : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$ is called **covariance function** of α.

- For every $t > 0$, the function $e^{t\gamma(S,T)}$ is positive definite in the following sense: for $T_i \in \mathcal{U}_\alpha$ and $\lambda_i \in \mathbb{C}$ for $i = 1 \dots n$,

  $$\sum_{i,j=1}^{n} e^{t\gamma(T_i,T_j)} \lambda_i \overline{\lambda_j} \geq 0$$

  Indeed,

  $$\left(\sum_{i,j=1}^{n} e^{t\gamma(T_i,T_j)} \lambda_i \overline{\lambda_j}\right) I = \sum_{i,j=1}^{n} T_i(t)^* T_j(t) \overline{\lambda_i} \lambda_j = \left(\sum_{i=1}^{n} \lambda_i T_i(t)\right)^* \left(\sum_{j=1}^{n} \lambda_j T_j(t)\right)$$

- It follows by a theorem of Guichardet that γ is conditionally positive definite, i.e., it determines a positive semidefinite sesquilinear form on the space

$$\mathbb{C}_0(\mathcal{U}_\alpha) = \{f : \mathcal{U}_\alpha \to \mathbb{C} \text{ with finite support and } \sum_{u \in \mathcal{U}_\alpha} f(u) = 0\}$$

$$\langle f, g \rangle = \sum_{S,T \in \mathcal{U}_\alpha} \gamma(S,T) f(S) \overline{g(T)}$$

Definition (Arveson)**.** If α is a spatial E_0-semigroup, let $H(\mathcal{U}_\alpha)$ be the Hilbert space obtained from the quotient.

Theorem (Arveson)**.** *If α is a spatial E_0-semigroup then the Hilbert space $H(\mathcal{U}_\alpha)$ is separable.*

Definition (Arveson Index)**.** If α is a spatial E_0-semigroup, define its index as follows:

$$\text{index}(\alpha) = \dim H(\mathcal{U}_\alpha)$$

If α is non-spatial, set $\text{index}(\alpha) = 2^{\aleph_0}$. In particular, the index takes values in the set $\{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$.

Remark. Imitating this construction for the units of a product system, it possible to define the index of a product system. There is a technical issue with measurability vs. continuity, but we refer the reader to the references. Of course $\text{index}\, \alpha = \text{index}\, \mathcal{E}_\alpha$.

Theorem (Arveson)**.**

- *The index is a cocycle conjugacy invariant.*
- *The index is additive with respect to tensor products:*

$$\text{index}(\alpha \otimes \beta) = \text{index}(\alpha) + \text{index}(\beta)$$

5.3 The Index of the CCR flows

In order to compute the index of a CCR flow we must find all of its units, the covariant function and the corresponding Hilbert space for the exponential product systems.

Theorem (Arveson)**.** *All units of the product exponential product system $E^{\mathfrak{C}}$ are of the form*

$$u_{\lambda,c}(t) = e^{t\lambda} \exp(\chi_{(0,t)} \otimes c)$$

for $\lambda \in \mathbb{C}, c \in \mathfrak{C}$. Furthermore,

$$\gamma(u_{\lambda_1,c_1},\ u_{\lambda_2,c_2}) = \lambda_1 + \overline{\lambda_2} + \langle c_1, c_2 \rangle$$

therefore $H(\mathcal{U}_{E^{\mathfrak{C}}}, \gamma)$ is canonically isomorphic with $\mathfrak{C}$. In particular, $\text{index}\, E^{\mathfrak{C}} = \dim \mathfrak{C}$.

It follows that two exponential product systems are isomorphic, or two CCR flows are cocycle conjugate, if and only if they have the same index.

5.4 Decomposable Product Systems

Definition. Let E be a product system. A vector $v \in E(t)$ is decomposable if for every $0 < s < t$ there exist $a \in E(s)$ and $b \in E(t-s)$ such that

$$v = ab$$

Remark. Units provide examples of decomposable vectors for every $t > 0$.

Definition. Let E be a product system, and for every $t > 0$, let $D(t)$ be the decomposable vectors of $E(t)$. We say that E is **decomposable** if $E(t) = \overline{\text{span}}\, D(t)$ for every $t > 0$.

Theorem (Arveson)**.** *If E is a decomposable product system, then it has units and it is isomorphic to the exponential product system of same index or the product system of an automorphism group.*

Theorem (Powers '98)**.** *There exists an E_0 semigroup of index one whose product system is not decomposable.*

5.5 Classification into types

Definition. We shall say that an E_0-semigroup belongs to a type I, II or III according to the following conditions:

Type I: $\mathcal{E}_\alpha$ is decomposable

Type II: α is spatial but $\mathcal{E}_\alpha$ is not decomposable.

Type III: α has no units.

In general when α is spatial with index n, we add the index as a subscript to the type, for example type I_n or II_n.

Remark. Observe there are examples of E_0-semigroups of all types. E_0-semigroups of type I are completely classified up to cocycle conjugacy by their index. This is not true for type II (unpublished work of R. T. Powers).

6 Representations of Product Systems

6.1 Defintion of Representation

Definition (Arveson)**.** Give a product system E, we say that a map $\phi : E \to B(H)$ is a **representation of product systems** if it satisfies:

1. $\phi(u)^*\phi(v) = \langle u, v\rangle I$
2. $\phi(u)\phi(v) = \phi(uv)$
3. ϕ is measurable

Remark. Fiberwise linearity follows automatically from these conditions, by considering for u, v in one fiber, $T = \phi(\alpha u+\beta v) - \alpha\phi(u) - \beta\phi(v)$ and showing it satisfies $\phi(w)^*T = 0$ for all w, hence T which has range in $\phi(E_t)H$ must be zero.

Theorem (Arveson)**.** *Let $\phi : E \to B(\mathfrak{H})$ be a representation of an abstract product system E, and define*

$$\mathcal{E} = \{(t, A) : t > 0, A \in \phi(E_t)\}$$

and endow it with the Borel structure from the inclusion into $(0,\infty) \times B(\mathfrak{H})$.

Then $\mathcal{E}$ is a concrete product system which is isomorphic to E.

Remark. Suppose a representation $\phi : E \to B(\mathfrak{H})$ produces a concrete product system arising from an E-semigroup α.. Observe that in this case we have that:

$$\phi(E_t)\mathfrak{H} = \alpha_t(I)\mathfrak{H}$$

This is because given an o.n. basis V_n for E_t, we have $\phi(E_t)\mathfrak{H} = \phi(V_1)\mathfrak{H} \oplus \phi(V_2)\mathfrak{H} \oplus \cdots$. And of course the ranges of the $\phi(V_n)$ sum up to $\alpha_t(I)\mathfrak{H}$.

In particular, the representation gives rise to an E_0-semigroup exactly when $[\phi(E_t)\mathfrak{H}] = \mathfrak{H}$.

Definition. We say that a representation $\phi : E \to B(\mathfrak{H})$ is **essential** if

$$[\phi(E_t)\mathfrak{H}] = \mathfrak{H}.$$

We shall say that ϕ is **singular** if

$$\cap_{t>0}\phi(E_t)\mathfrak{H} = \{0\}.$$

We want to show that every product system has essential representations.

6.2 Continuous Fock space

Definition. Given a product system E, we definte

$$L^2(E) = \{f : (0,\infty) \to E \text{ measurable sections such that } \int_0^\infty \langle f(t), f(t)\rangle dt < \infty\}$$

On this (separable) Hilbert space there is always a representation $\ell : E \to B(L^2(E)$ which we call the regular representation: if $x \in E(t)$, we set

$$(\ell_x f)(s) = \begin{cases} x \cdot f(s-t), & s > t \\ 0, & 0 < s \leq t \end{cases}$$

The regular representation is always singular, so we must work harder to prove our desired result.

6.3 Spectral C^*-algebras

We now proceed to introduce an analogue of group C^*-algebras for product systems. Consider the L^1 convolution algebra of sections:

$$L^1(E) = \{f : (0,\infty) \to E \text{ measurable sections such that } \int_0^\infty \|f(t)\| dt < \infty\}$$

where the convolution operation si defined by:

$$(f * g)(t) = \int_0^t f(s)g(t-s)ds$$

And define the representation $\ell : L^1(E) \to B(L^2(E))$ by

$$\ell(f)\xi = \int_0^\infty \ell_{f(t)}\xi \, dt$$

Definition. Given a product system E, we define

$$C^*(E) = \overline{\text{span}}\{\ell(f)\ell(g)^* : f, g \in L^1(E)\}$$

This C^*-algebra is a hereditary subalgebra of the C^*-algebra generated by $\ell(L^1(E))$, hence Morita equivalent to it.

Theorem (Arveson)**.** *For every nondegenerate representation π of $C^*(E)$ on a Hilbert space $\mathfrak{H}$, there exists a representation $\phi : E \to B(\mathfrak{H})$ such that for $f, g \in L^1(E)$*

$$\pi(\ell(f)\ell(g)^*) = \phi(f)\phi(g)^*$$

($\phi : L^1(E) \to B(\mathfrak{H})$ denotes the integrated form of ϕ).

Definition. We say that a state of $C^*(E)$ is essential if its GNS representation gives rise to an essential representation of E.

Theorem (Arveson)**.** *For every product system E, its spectral C^*-algebra $C^*(E)$ has essential states. It follows that every product system is isormorphic to the concrete product system arising from an E_0-semigroup.*

6.4 Properties of Spectral C^*-algebras

Theorem (Arveson)**.** *For every product system E, the spectral C^*-algebra is nuclear.*

Theorem (Arveson)**.** *If E is a product system of type I or II, then its spectral C^*-algebra is simple.*

Definition. A separable C^*-algebra A is **purely infinite** if for every positive $a \neq 0$ there exists a sequence $b_n \in A$ such that $b_n^* a b_n$ is an approximate unit for A.

Definition. A separable C^*-algebra is **K-contractible** if $KK(A, A) = 0$. In particular, $K_0(A) = K_1(A) = 0$.

Theorem (Zacharias)**.** *If E is a product system of type I or II, then its spectral C^*-algebra is purely infinite and K-contractible. It follows by the classification results of Kirchberg and Phillips that $C^*(E) \simeq \mathcal{O}_2 \otimes \mathcal{K}$.*

Theorem (Hirshberg). *For any product system E, the spectral C^*-algebra $C^*(E)$ is K-contractible.*

Theorem (Zacharias). *A large class of unitless product systems satisfying a certain technical condition have purely infinite spectral C^* algebra, hence are isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$.*

Remark. It still is unknown whether or not the spectral C^*-algebra of a product system is always purely infinite (or even simple).

6.5 Tsirelson's examples

Theorem (Tsirelson '00). *There are uncountably many pairwise non-isomorphic product systems of type II_0.*

Theorem (Tsirelson '00). *There are uncountably many pairwise non-isomorphic type III product systems.*

A Appendix

A.1 Automorphism groups are all cocycle conjugate

Suppose $\alpha_t(X) = V_t X V_t^*$ and $\beta_t(X) = W_t X W_t^*$. Set $U_t = W_t V_t^*$. Then:

1. $U_{t+s} = W_{t+s} V_{t+s}^* = W_t W_s V_s^* V_t^* = (W_t V_t^*)(V_t W_s V_s^* V_t^*) = U_t \alpha_t(U_s)$.
2. $U_t \alpha_t(X) U_t^* = W_t X W_t^* = \beta_t(X)$.

A.2 Propagator Solution

How do we solve the equation defining the cocycle arising from a bounded perturbation, and why is it what we want?

$$\frac{dU_t}{dt} = iU_t \alpha_t(B), \qquad U_0 = I$$

Heuristically, we wish we could say simply that $U(t) = \exp(\int_0^t \alpha_s(B)\, ds)$. Even thought this expression can be made to make sense, it does not satisfy the properties we need, namely

$$\exp(\int_0^{t+s} \alpha_u(B)\, du) \neq \exp(\int_0^t \alpha_u(B)\, du) \exp(\int_t^s \alpha_u(B)\, du)$$

Nevertheless there is a formula which makes better sense and has all the properties we need:

$$U_t = I + i \int_0^t \alpha_{u_1}(B) du_1 + i^2 \int_0^t \int_0^{u_1} \alpha_{u_1}(B) \alpha_{u_2}(B)\, du_2 du_1 + \dots$$

This converges uniformly, it is strongly cts, and satisfies:

1. The propagator equation
2. $U_t \alpha_t(U_s)$
3. The generator of $U_t \alpha_t(X) U_t^* = \delta(X) + i[B, X]$.

A.3 Standard Borel Spaces

Definition. A Polish space is a topological space which is metrizable and under this metric it is separable and complete.

Theorem. *Countable direct products and direct sums of Polish spaces are Polish spaces. Open subsets and G_δ subsets (i.e countable intersection of open sets) of Polish spaces are Polish spaces.*

Definition. A Borel space is called **standard** if it is Borel isomorphic to a Borel subset of a Polish space.

Theorem. *An uncountable standard Borel space is Borel isomorphic to the unit interval* $[0, 1]$ *with its usual structure.*

Definition. A Borel space X is countably separated if there exists a countable family $E_i \subseteq X$ of Borel sets such that for every $x \in X$ there exist i, j such that $x \in E_i$ and $x \notin E_j$.

This property is frequently associated with equivalence relations which are "tame".

A.4 C^* definitions

Definition. A C^*-algebra A is nuclear if for every C^*-algebra B, the algebraic tensor product $A \odot B$ has a unique C^*-crossnorm. For example, $C(X)$, $K(\mathfrak{H})$, inductive limits of such, ideals, quotients and cross products by amenable group C^*-algebras. But not subalgebras, hence the theory of exact C^*-algebras.

Definition. Two C^*-algebras A and B are Morita equivalent if there exists an A-B-imprimitivity bimodule. This implies that A and B have equivalent representation categories, and $A \otimes K(\mathfrak{H}) \simeq B \otimes K(\mathfrak{H})$.

Remark. If A is full hereditary subalgebra of B then they are Morita equivalent with imprimitivity bimodule $\overline{AB}$ (submodule of B).