QUANTIZED CONVOLUTION SEMIGROUPS

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ABSTRACT. We describe our construction of a continuous tensor product system in the sense of Arveson for a general W^*-continuous completely positive semigroup of $B(H)$ (H separable). This product system is canonically isomorphic to the product system of the minimal dilation E_0-semigroup. We use this construction to show that contrary to previous speculation the minimal dilations of all quantized convolution semigroups are completely spatial. This class includes the heat flow on the CCR algebra studied recently by Arveson, which we show is cocycle conjugate to a CAR/CCR flow of index two. The analysis of these examples also involves Lévy processes and their stochastic area processes.

Additionally, we prove the following fact: given a product system F and a type I product system E, if there is a bijection $\psi : E \to F$, not necessarily measurable, that preserves fibers and multiplication and ψ is unitary fiberwise, then E and F are isomorphic.

1. INTRODUCTION

In this paper we are interested in completely positive semigroups of $B(H)$ for H separable, or CP semigroups for short, and their minimal dilations to E_0-semigroups (see [Bha96, Arv97b]). A one-parameter semigroup of unital normal completely positive maps $\{\phi_t : B(H) \to B(H)\}_{t\geq 0}$ is a CP semigroup if it satisfies the continuity condition

$$t \mapsto \langle \phi_t(X)\xi, \eta\rangle \text{ is continuous for each } X \in B(H), \xi, \eta \in H.$$

The minimal dilation of a CP semigroup with *bounded* generator is well understood. W. Arveson [Arv97a, Arv99] has shown that such an E_0-semigroup is always cocycle conjugate to a CAR/CCR flow, and its index can be described explictly in terms of the bounded generator. These results extended some previous work by B. V. R. Bhat [Bha98] on one-dimensional Evans-Hudson flows, and concurrent work by R. Powers [Pow99] on the minimal dilation of CP semigroups on matrix algebras.

In this paper we review our construction of the product system of a CP semigroup. This is applied effectively to the analysis of the CP semigroups which we call quantized convolution semigroups: they are semigroups of $B(L^2(\mathbb{R}))$ obtained from a modified Weyl-Moyal quantization of convolution semigroups of Borel probability measures on $\mathbb{R}^2$. In general quantized convolution semigroups have *unbounded* generators. A prime example which falls in this class is the heat flow on the CCR algebra, considered earlier by Arveson [Arv02] and which remained unidentified. We show its minimal dilation is cocycle conjugate to a CAR/CCR flow of index 2, and more generally we prove that, contrary to previous speculation, all quantized convolution semigroups have type I minimal dilations.

We also prove that on the class of type I product systems, isomorphisms can be replaced by weak isomorphisms, which differ from the former only in that no measurability requirement is imposed.

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It is appropriate to point out that B.V.R. Bhat and M. Skeide have studied dilations of CP semigroups on general C*-algebras, considering a generalization of product systems in [BS00]. We should also mention that after our work was completed, we were informed by P. S. Muhly and B. Solel of their independent study of dilations through product systems in [MS02].

I would like to take this opportunity to thank my thesis supervisor, Professor William Arveson, for suggesting the heat flow as a subject of investigation. I also thank Michael Anshelevich for his comments on a preliminary version of this paper.

2. Product systems of CP semigroups

We proceed to review succintly our construction given in full detail in [Mar02b]. Given a CP semigroup $(\phi_t)_{t\geq 0}$ of $B(H_0)$ with minimal dilation α, we obtain a product system $\mathbb{E}_\phi$ that is canonically isomorphic to $\mathcal{E}_\alpha$.

Fiber structure of $\mathbb{E}_\phi$. For a given $t > 0$ fixed, the fiber $\mathbb{E}_\phi(t)$ is defined to be an inductive limit of Hilbert spaces. A basic object for this definition is metric operator space of a normal completely positive map of $B(H)$.

Definition 2.1 (Arveson [Arv97a])**.** Given a normal CP map $P : B(H) \to B(H)$, its metric operator space $\mathcal{E}_P$ is a Hilbert space whose elements are all operators $T \in B(H)$ for which there is a constant $c > 0$ such that the map

$$X \in B(H) \mapsto cP(X) - TxT^* \tag{2.1}$$

is completely positive. If $T \in \mathcal{E}_P$, then $\langle T, T\rangle_{\mathcal{E}_P}$ is defined as the smallest constant c satisfying (2.1). When dealing with a CP semigroup ϕ, we will denote by $\mathcal{E}_\phi(t)$ the metric operator space corresponding to ϕ.

Definition 2.2. Let $a \leq b \in \mathbb{R}$. A subset $\mathcal{P} \subseteq [a, b]$ will be called a **partition of [a,b]** if $\mathcal{P}$ is finite and $a, b \in \mathcal{P}$. We will denote by $\mathrm{Part}[a, b]$ the set of all partitions of $[a, b]$.

For one partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, we define the Hilbert space

$$\mathbb{H}_\phi^{\mathcal{P}} = \mathcal{E}_\phi(\Delta t_1) \otimes \mathcal{E}_\phi(\Delta t_2) \otimes \cdots \otimes \mathcal{E}_\phi(\Delta t_n)$$

where $\Delta t_k = t_k - t_{k-1}$. Given two partitions $\mathcal{P} \subseteq \mathcal{Q} \in \mathrm{Part}[0, t]$, the isometric connecting map is denoted by $V_{\mathcal{QP}} : \mathbb{H}_\phi^{\mathcal{P}} \to \mathbb{H}_\phi^{\mathcal{Q}}$ and it is given by the adjoint of operator multiplication, as follows. Let $\mathcal{Q} = \{0 = s_0 < \cdots < s_n = t\}$, and $\mathcal{P} = \{s_0, s_{k_1}, \ldots, s_{k_m}, s_{k_{m+1}} = s_n\}$. Then

$$V_{\mathcal{QP}}^*(X_1 \otimes X_2 \otimes \ldots X_n) = X_1 X_2 \ldots X_{s_{k_1}} \otimes X_{s_{k_1}+1} \ldots X_{s_{k_2}} \otimes \cdots \otimes X_{s_{k_m}+1} \ldots X_n$$

These maps satisfy the basic inductive condition: $V_{\mathcal{QP}} = V_{\mathcal{QR}} V_{\mathcal{RP}}$ for $\mathcal{P} \subseteq \mathcal{R} \subseteq \mathcal{Q}$.

Finally, we define the desired inductive limit Hilbert space:

$$\mathbb{E}_\phi(t) = \lim_{\mathcal{P} \in \mathrm{Part}[0,t]} \mathbb{H}_\phi^{\mathcal{P}}$$

with canonical isometries $V_{\mathcal{P}} : \mathbb{H}_\phi^{\mathcal{P}} \to \mathbb{E}_\phi(t)$. Here we include for future reference that for each $t > 0$,

$$\mathbb{E}_\phi(t) = \overline{\bigcup_{\mathcal{P} \in \mathrm{Part}[0,t]} \mathrm{Range}\, V_{\mathcal{P}}}.$$

Multiplication structure of $\mathbb{E}_\phi$. Given $t, s > 0$, the multiplication map is defined as the unique bounded bilinear map $\mathbb{E}_\phi(t) \times \mathbb{E}_\phi(s) \to \mathbb{E}_\phi(t+s)$ that satisfies

$$V_{\mathcal{P}}(X) \ \cdot \ V_{\mathcal{Q}}(Y) = V_{\mathcal{P}\oplus\mathcal{Q}}(X \otimes Y)$$

for all $\mathcal{P} \in \mathrm{Part}[0,t]$, $\mathcal{Q} \in \mathrm{Part}[0,s]$, $X \in \mathbb{H}_\phi^{\mathcal{P}}$, $Y \in \mathbb{H}_\phi^{\mathcal{Q}}$, and where $\mathcal{P} \oplus \mathcal{Q} = \mathcal{P} \cup (t + \mathcal{Q})$ is the new partition obtained by joining $\mathcal{P}$ and the translation of $\mathcal{Q}$ by t.

Notice that this multiplication operation corresponds very naturally to "tensoring", as should be hoped for a product system multiplication.

Borel structure of $\mathbb{E}_\phi$.

Definition 2.3. A **measurable Hilbert bundle** is a standard Borel space E with a surjective measurable map $p : E \to (0, \infty)$, such that for each $t > 0$, the fiber $E_t = p^{-1}(\{t\})$ has a Hilbert space structure, and there is a Borel isomorphism $E \to (0, \infty) \times \ell^2$ that is unitary fiberwise.

We define the measurable Hilbert bundle structure of $\mathbb{E}_\phi$ in terms of a universal property on a class of measurable Hilbert bundles.

Definition 2.4. For a partition $\mathcal{P} = \{x_0 < \cdots < x_n\} \in \mathrm{Part}[0,1]$, define the field of Hilbert spaces

$$\mathcal{E}_\phi^{\mathcal{P}} = \{(t, X) : X \in \mathbb{H}_\phi^{t\mathcal{P}}\}$$

where $t\mathcal{P} = \{tx_0, \ldots tx_n\}$ for $t > 0$. We also define the map between Hilbert bundles $V^{\mathcal{P}} : \mathcal{E}_\phi^{\mathcal{P}} \to \mathbb{E}_\phi$, given fiberwise by $V^{\mathcal{P}}(A) = V_{t\mathcal{P}}(A)$ if $A \in \mathcal{E}_\phi^{\mathcal{P}}(t) = \mathbb{H}_\phi^{t\mathcal{P}}$.

There is a canonical measurable Hilbert bundle structure on $\mathcal{E}_\phi^{\mathcal{P}}$, for every $\mathcal{P} \in \mathrm{Part}[0,1]$. Firstly, we observe that when $\mathcal{P} = \{0, 1\}$ (the trivial partition), we have simply

$$\mathcal{E}_\phi^{\mathcal{P}} = \{(t, X) : X \in \mathcal{E}_\phi(t)\}$$

and this is a subset $\mathcal{E}_\phi^{\mathcal{P}} \subseteq (0, \infty) \times B(H_0)$. This suggests a Borel structure for this special case. The space $(0, \infty) \times B(H_0)$ has a natural Polish space Borel product structure, obtained from the usual structure on $(0, \infty)$ and the weak* structure on $B(H_0)$. Indeed, this choice of Borel structure coupled with the given fiberwise Hilbert structure determines a Hilbert bundle structure for $\mathcal{E}_\phi$.

The general case follows by observing that for every partition $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ there is a CP semigroup $\phi^{\mathcal{P}}$ of $B(H_0^{\otimes^n})$ given by

$$\phi_t^{\mathcal{P}}(A_1 \otimes \cdots \otimes A_n) = \phi_{t\Delta x_1}(A_1) \otimes \cdots \otimes \phi_{t\Delta x_n}(A_n).$$

This leads to a canonical identification:

$$\mathcal{E}_\phi^{\mathcal{P}} \simeq \mathcal{E}_{\phi^{\mathcal{P}}}.$$

Therefore, we can define the measurable Hilbert bundle structure of $\mathcal{E}_\phi^{\mathcal{P}}$ to be that of $\mathcal{E}_{\phi^{\mathcal{P}}}$, obtained from the trivial partition case considered above.

Definition 2.5. A standard Borel structure on $\mathbb{E}_\phi$ is **admissible** if all maps $V^{\mathcal{P}} : \mathcal{E}_\phi^{\mathcal{P}} \to \mathbb{E}_\phi$ are measurable, $\forall \mathcal{P} \in \mathrm{Part}[0,1]$, and it has the following property. If F is a measurable Hilbert bundle, and for every $\mathcal{P} \in \mathrm{Part}[0,1]$ there is a Borel map $\psi^{\mathcal{P}} : \mathcal{E}_\phi^{\mathcal{P}} \to F$ satisfying

(1) $\psi^{\mathcal{P}}$ preserves fibers
(2) $\forall t > 0$, $\psi_t^{\mathcal{P}} : \mathcal{E}_\phi^{\mathcal{P}}(t) \to F_t$ is an isometry
(3) $\mathcal{P} \subseteq \mathcal{Q} \in \mathrm{Part}[0,1] \quad \Rightarrow \quad \psi^{\mathcal{P}} = \psi^{\mathcal{Q}} V^{\mathcal{Q}\mathcal{P}}$
(4) For all $t > 0$,

$$\bigcup_{\mathcal{P} \in \mathrm{Part}[0,1]} \mathrm{Range}(\psi_t^{\mathcal{P}}) \quad \text{is dense in } F_t$$

then there is a unique Borel isomorphism $\Psi : \mathbb{E}_\phi \to F$, that is unitary fiberwise and $\Psi \circ V^{\mathcal{P}} = \psi^{\mathcal{P}}$, for all $\mathcal{P} \in \mathrm{Part}[0,1]$.

Theorem 2.6 ([Mar02b]). *There exists a unique admissible Borel structure on $\mathbb{E}_\phi$. Furthermore, let α be an E_0-semigroup with product system F, and suppose α is a minimal dilation of ϕ. Then there is a canonical product system isomorphism $\psi : \mathbb{E}_\phi \to F$.*

As it turns out, the proof of existence of an admissible Borel structure is intertwined with the construction of the canonical isomorphism. The uniqueness part follows immediately from the universal property involved in the definition of the admissible Borel structure.

An important consequence of this result is that any cocycle conjugacy invariant of a minimal dilation of ϕ can be studied directly through $\mathbb{E}_\phi$. We proceed to indicate how this can be done in practice.

3. Quantized convolution semigroups

Basic definition. In [Arv02], Arveson introduced a CP semigroup whose minimal dilation has no invariant normal states. He called that CP semigroup the heat flow on the CCR algebra, and it is defined as follows. Let $H_0 = L^2(\mathbb{R})$, and let

$$P = \frac{1}{i}\frac{d}{dx}, \qquad Q = \mathrm{Mult}_x$$

be the usual quantum mechanics momentum and position operators. We consider

$$D_P(X) = i[P, X], \qquad D_Q(X) = i[Q, X]$$

their corresponding unbounded derivations, which can be defined on a suitable dense subalgebra of operators with smooth kernels. The heat flow on $B(H_0)$ is the CP semigroup with *unbounded* generator

$$L = D_P^2 + D_Q^2.$$

Before settling the question of the classification of the heat flow's minimal dilation, we consider a larger class of CP semigroups. Some of the elements of this construction can be found in [Arv02]. Let W_z, for $z \in \mathbb{C}$, be the Weyl unitary operators associated to P and Q, given by

$$W_{a+bi} = e^{i(aQ+bP)} = e^{i\frac{ab}{2}} e^{iaQ} e^{ibP}.$$

Recall that $W_z W_{z'} = e^{i\omega(z,z')} W_{z+z'}$, where ω is the symplectic form on $\mathbb{C}$ given by $\omega(z, z') = \frac{1}{2}\,\mathrm{Im}(z\overline{z'})$.

Given μ an infinitely divisible (Borel) probability measure on $\mathbb{R}^2$, there exists a unique one-parameter family $\{\mu_t, : t \geq 0\}$ of (Borel) probability measures on $\mathbb{R}^2$, satisfying $\mu_0 = \delta_0$, and $\mu_{t+s} = \mu_t * \mu_s$ for all $t, s > 0$ (see [RW94]). We shall call this the convolution semigroup of probability measures associated to μ.

Associated to an infinitely divisible measure on the plane, it is possible to define a canonical CP semigroup of $B(L^2(\mathbb{R}))$ as follows.

The Weyl-Moyal quantization of a Schwartz function f on $\mathbb{R}^2$ is the operator $\mathcal{Q}(f)$ in $B(L^2(\mathbb{R}^2))$ defined by

$$\mathcal{Q}(f) = \int_{\mathbb{R}^2} \widehat{f}(z) W_z \, dz, \tag{3.1}$$

where $\widehat{f}(a,b) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x,y) e^{-i(ax+by)}\, dxdy$. (See also [Fol89] for a slightly different normalization and more details about the Weyl-Moyal quantization, also sometimes called Weyl Correspondence).

Given a Schwartz function f, we may also consider its symplectic Fourier transform f^S, given by

$$f^S(z) = \int_{\mathbb{R}^2} e^{2i\omega(\xi,z)} f(\xi) d\xi.$$

In terms of the Fourier transform, we have $f^S(a+bi) = 2\pi \widehat{f}(-b,a)$. The symplectic version of the Weyl-Moyal quantization $\mathcal{Q}^S(f)$ is obtained by replacing the Fourier transform by the symplectic Fourier transform in (3.1):

$$\mathcal{Q}^S(f) = \int_{\mathbb{R}^2} f^S(z) W_z \, dz.$$

Definition 3.1. The *CP semigroup associated with an infinitely divisible measure* μ, denoted by ϕ_t^μ is defined as the unique family of normal maps satisfying

$$\phi_t^\mu(\mathcal{Q}^S(f)) = \mathcal{Q}^S(\mu_t * f), \qquad \text{for } f \text{ Schwartz function in } \mathbb{R}^2.$$

Remark 3.2. When one considers the convolution semigroup of probability measures $\{\rho_t : t > 0\}$ on the plane given by the density

$$d\rho_t(x+yi) = \frac{1}{4\pi t} e^{-\frac{x^2+y^2}{4t}} dxdy \tag{3.2}$$

the corresponding CP semigroup is precisely the heat flow.

Connection with Lévy processes. Let μ be a fixed infinitely divisible Borel probability measure on the plane. In order to classify ϕ^μ, we apply the program suggested by the previous section, namely characterize its product system. The first step is to describe $\mathcal{E}_\phi(t)$.

Lemma 3.3 ([Mar02b]). *For every $t > 0$,*

$$\mathcal{E}_\phi(t) = \{\int_{\mathbb{R}^2} f(x,y) W_{x+iy} dxdy : f \in L^2(\mathbb{R}^2, d\mu_t)\}$$

and $\| \int_{\mathbb{R}^2} f(x,y) W_{x+iy} dxdy \|_{\mathcal{E}_\phi} = \|f\|_{L^2(\mathbb{R}^2, d\mu_t)}$.

This result leads to an analogous identification for every $\mathcal{P} = \{x_0 < \cdots < x_n\} \in \mathrm{Part}[0,1]$:

$$\mathcal{E}_\phi^{\mathcal{P}}(t) \simeq L^2((\mathbb{R}^2)^n, d\mu_{\Delta x_1} \otimes \cdots \otimes d\mu_{\Delta x_n})$$

We denote this identification as follows:

$$\mathcal{E}_\phi^{\mathcal{P}}(t) \ni f \mapsto \widetilde{f} \in L^2((\mathbb{R}^2)^n, d\mu_{\Delta x_1} \otimes \cdots \otimes d\mu_{\Delta x_n})$$

In terms of this identification we have the following description for the connecting isometries of the inductive limit.

Lemma 3.4 ([Mar02b]). *Let $\mathcal{Q} = \{0 = s_0 < s_1 < \cdots < s_m = t\}$ be a partition refining $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, and set $I(j) = \{u : t_{j-1} < s_u \le t_j\}$. The isometry $V_{\mathcal{QP}} : \mathbb{H}_\phi^{\mathcal{P}} \to \mathbb{H}_\phi^{\mathcal{Q}}$ is given by*

$$(\widetilde{V_{\mathcal{QP}} h})(\xi_1, \ldots, \xi_m) = \widetilde{h}\left(\sum_{k \in I(1)} \xi_k \, , \ldots , \sum_{k \in I(n)} \xi_k \right) \exp\left[i \sum_{r=1}^{n} \sum_{k<l \in I(r)} \omega(\xi_l, \xi_k) \right] \tag{3.3}$$

On one hand this description might seem to obscure matters. On the other hand, one might observe that except for the phase factor on the right side, this connecting map suggests some sort of concatenation of paths of an appropriate stochastic process.

Definition 3.5. Let μ be an infinitely divisible Borel probability measure on $\mathbb{R}^2$. There exists an $\mathbb{R}^2$-valued process $\{\mathbf{X}_t : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbf{X}_0 = 0$ a.s., and stationary, independent increments, such that the distribution of $\mathbf{X}_1$ is μ. We will call such a process a *Lévy process associated to* μ. Recall that a Lévy process has a unique modification, up to indistinguishability, with *càdlàg* paths (=right continuous paths on $[0,\infty)$ with left limits on $(0,\infty)$), jointly measurable on $[0,\infty) \times \Omega$, which is also a Lévy process (see [Pro90]). For each $t > 0$, let $(\mathcal{F}_t)_{t>0}$ be the smallest right-continuous filtration of $\mathcal{F}$ generated by the process. A choice of such a modification and its corresponding filtration will be called a *standard Lévy process framework for* μ.

Associated to a Lévy process we also define A_t to be its real-valued "stochastic area" process (cf. [JW84]) defined by the Itô integral

$$A_t = \int_0^t \omega(\mathbf{X}_s, d\mathbf{X}_s) := \frac{1}{2} \int_0^t \mathbf{X}^{(1)}_{s-} d\mathbf{X}^{(2)}_s - \mathbf{X}^{(2)}_{s-} d\mathbf{X}^{(1)}_s. \tag{3.4}$$

where $\mathbf{X}_{t-}$ is the process with left-continuous paths defined by $\mathbf{X}_{t-} = \lim_{s \uparrow t} \mathbf{X}_s$. Notice A_t is a well-defined stochastic integral, since the coordinate processes $\mathbf{X}^{(1)}_t$, $\mathbf{X}^{(2)}_t$ of $\mathbf{X}_t$ are semimartingales (see [RW87, section VI.2]), and notice also that A_t is $\mathcal{F}_t$-adapted. Given a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, we define also the $\mathcal{F}_t$-adapted real-valued stochastic process

$$A^{\mathcal{P}}_t = A_t - \sum_{k=1}^{n} \omega(\mathbf{X}_{t_k}, \mathbf{X}_{t_{k-1}}).$$

Since $A^{\mathcal{P}}_t$ is càdlàg, by replacing it by an indistinguishable modification, we may assume that $A^{\mathcal{P}}(t, \cdot)$ is jointly measurable on $[0,\infty) \times \Omega$.

There is a canonical product system corresponding to a standard Lévy process framework for μ. Ultimately, it depends on the following "shift" map. In general, we may assume Ω is the space of paths of the stochastic process, and so for every $c > 0$ there is a map $\sigma_c : \Omega \to \Omega$ such that

$$\sigma_c(\gamma)(t) = \gamma(t + c) - \gamma(c)$$

Definition 3.6. Define the product system $\mathbb{L}^\mu$ to be the set

$$\mathbb{L}^\mu = \{(t, f) \in (0,\infty) \times L^2(\Omega, \mathcal{F}, \mathbb{P}) : f \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})\}.$$

The (standard) Borel structure of $\mathbb{L}^\mu$ is inherited via its inclusion as a Borel subset of $(0,\infty) \times L^2(\Omega, \mathcal{F}, \mathbb{P})$. The product structure $\odot$ on $\mathbb{L}^\mu$ is determined uniquely by the property that $(f \odot g)(\gamma) = f(\gamma) g(\sigma_t(\gamma))$ for $f \in \mathbb{L}^\mu_t, g \in L^\mu_s$.

Theorem 3.7 ([Mar02a])**.** *For every Borel infinitely divisible probability measure μ on the plane, the product system $\mathbb{L}^\mu$ is type I.*

Weak isomorphism. We now reach the main result.

Definition 3.8. Let E and F be product systems. A map $\Psi : E \to F$ will be called a *weak isomorphism (of product systems)* if it is a fiber-preserving bijection, unitary fiberwise and it preserves multiplication.

For every partition $\mathcal{P} = \{0 = t_0 < \cdots < t_n = t\}$ of $[0,t]$, define the map $\psi_{\mathcal{P}} : \mathbb{H}^{\mathcal{P}}_\phi \to L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ by setting

$$\psi_{\mathcal{P}}(h) = \widetilde{h}(\mathbf{X}_{t_1} - \mathbf{X}_{t_0}, \ldots, \mathbf{X}_{t_n} - \mathbf{X}_{t_{n-1}}) \, \exp(i A^{\mathcal{P}}_t)$$

Theorem 3.9. *The family of maps $\{\psi^{\mathcal{P}} : \mathcal{E}^{\mathcal{P}}_\phi \to \mathbb{L}^\mu \,|\, \mathcal{P} \in \mathrm{Part}[0,1]\}$ induces a weak isomorphism $E_{\phi^\mu} \simeq \mathbb{L}^\mu$.*

Once this map is identified, appropriately defined and guaranteed to exist, the proof becomes straightforward but technical. To complete the program, it would be necessary to prove that the induced weak isomorphism is in fact measurable and hence a *bona fide* isomorphism. This technical fact has been in fact accomplished in [Mar02b].

Instead of doing so, we present a different proof. We point out that since $\mathbb{L}^{\mu}$ is always type I, the weak isomorphism implies the existence of an isomorphism. This is the main result of the next section. We point out some immediate consequences of the existence of the isomorphism.

Corollary 3.10. *Any quantized convolution semigroup has type I minimal dilation. The heat flow corresponds to a Brownian motion (Remark 3.2), hence it is cocyle conjugate to the CAR/CCR flow of index two (see* [Mey93, Mar02a]*).*

4. Weak isomorphisms of product systems

Are weak isomorphisms always measurable? The answer, in general, is no. Indeed, let E, F be isomorphic product systems, and let $\Phi : E \to F$ be a *bona fide* isomorphism. Let $\mathbb{T}$ be the group of complex numbers of norm one, and $f : (0, \infty) \to \mathbb{T}$ a nonmeasurable map such that $f(x + y) = f(x)f(y)$ (it is not hard to construct such a map). Then $\Psi : E \to F$ defined by $\Psi_t = f(t)\Phi_t$ is a weak isomorphism that is not measurable.

On the other hand, we proceed to show that, if either E or F is a type I product system, then the issue of measurability of a weak isomorphism $\Psi : E \to F$ can be circumvented. More precisely, in the presence of a weak isomorphism, E is type I if and only F is type I, and we show that in this case they have the same index. Since the index is a complete invariant of type I product systems, it follows that if E and F are weakly isomorphic, then they are in fact isomorphic.

It will be more convenient to describe this result in terms of Arveson's definition of a decomposable product system [Arv97c], which we present next for the convenience of the reader. It is worth mentioning that Arveson showed that a product system is decomposable if and only if it is type I. This difficult theorem is the main result, and motivation, for [Arv97c].

Definition 4.1. Let E be a product system. A nonzero vector $x \in E(t)$ is called *decomposable* if for every $0 < s < t$ there are $y \in E(s)$ and $z \in E(t - s)$ such that $x = yz$. The set of decomposable vectors in $E(t)$ will be written $D(t)$. We shall denote the projective space corresponding to $D(t)$ by $\Delta(t)$.

A product system, in general, may fail to have any decomposable vectors, e.g. the product system of R. Powers' nonspatial E_0-semigroup (see [Pow87]). If, however, for some $t_0 > 0$ there are enough decomposable vectors such that $D(t_0)$ spans $E(t_0)$, then in fact it is possible to show that

$$E(t) = \overline{\operatorname{span}}\, D(t), \quad \forall t > 0. \tag{4.1}$$

Definition 4.2. A product system E satisfying (4.1) will be called *decomposable.*

Definition 4.3. Let E be a product system. A section $w : (0, \infty) \to E$ will be called a *weak unit of* E if $w_{t+s} = w_t w_s$, $t, s > 0$ and $w_t \neq 0, \forall t > 0$.

Remark 4.4. Notice that a weak unit is not required to be a measurable map. In fact, (true) units correspond precisely to measurable weak units.

In the following, $\mathbb{T}$ denotes the set of complex numbers of norm one, and $\mathbb{C}^{\times}$ denotes the group of complex numbers without the origin, with multiplication.

Lemma 4.5. *Let E be a product system, and let w be a weak unit of E Then there is $f : (0, \infty) \to \mathbb{C}^{\times}$ such that $u_t = f(t)w_t$ is a (true) unit of E.*

Proof. Let $w_t^n = w_t/\|w_t\|$. By Proposition 10.3 of [Arv97c], there is $g : (0, \infty) \to \mathbb{T}$ such that $x_t = g(t)w_t^n$ is a measurable section of E. Now, we can write $x_{t+s} = c(t, s)x_t x_s$, where

$$c(t, s) = \frac{g(t + s)}{g(t)g(s)}$$

for all $t, s > 0$. Notice that it follows from its definition that, for every $t, s > 0$, $c(t, s) \in \mathbb{T}$. Furthermore, $c : (0, \infty)^2 \to \mathbb{T}$ is a measurable function, since $c(t, s) = \langle x_{t+s}, x_t x_s\rangle$, and the RHS is measurable in $t, s > 0$. It follows that $c(t, s)$ is a multiplier of $(0, \infty)$, cf. page 21 of [Arv89]. By the Corollary in page 24 of [Arv89], there is $h : (0, \infty) \to \mathbb{T}$ measurable such that

$$c(t, s) = \frac{h(t + s)}{h(t)h(s)}$$

Now, by defining $f^n(t) = h(t)^{-1}g(t)$, we clearly have that $u_t = f^n(t)w_t^n$ is a unit of E, and f^n takes values on $\mathbb{T}$. It is clear that $f(t) = f^n(t)\|w_t\|$ satisfies the desired conditions, and $u_t = f(t)w_t$. □

Definition 4.6. Let E be a product system. We will denote by $\mathcal{W}_E$ the set of weak units of the product system E, and $\mathcal{U}_E$ the set of units of E. We have canonically $\mathcal{U}_E \subseteq \mathcal{W}_E$.

Define an equivalence relation on $\mathcal{W}_E$ as follows: $x \sim y$ iff there is $f : (0, \infty) \to \mathbb{C}^\times$ (no measurability required) such that $x_t = f(t)y_t$. We will denote by $\mathcal{W}_E/\sim$ the set of equivalence classes of $\mathcal{W}_\mathbb{E}$ corresponding to this equivalence relation.

Recall that if E is a product system, then there is a conditionally positive-definite function $\gamma : \mathcal{U}_E \times \mathcal{U}_E \to \mathbb{C}$ determined uniquely by the identity

$$\langle u_t, v_t\rangle = e^{t\gamma(u,v)}, \qquad t > 0, \;\; u, v \in \mathcal{U}_E.$$

This function is called the *covariance function of* E.

Lemma 4.7. *Let E be a product system, let u_t be a unit of E and let $f : (0, \infty) \to \mathbb{C}^\times$ be a (not necessarily measurable) function such that $v_t = f(t)u_t$ is a unit. Then there is a unique $\alpha \in \mathbb{C}$ such that $f(t) = \exp(\alpha t)$, and moreover*

$$\alpha = \gamma(v, u) - \gamma(u, u),$$

where γ is the covariance function of E.

Proof. Notice that

$$f(t) = \frac{\langle v_t, u_t\rangle}{\|u_t\|^2} = \exp t[\gamma(v, u) - \gamma(u, u)]$$

Therefore continuity of f follows. Uniqueness of α follows from the fact that if $e^{ct} = 1$, for all $t > 0$, then $c = 0$. □

Recall that, for a set X, we denote by $\mathbb{C}_0X$ the set of finitely-supported complex-valued functions on X with zero mean. Additionally, the index of a product system E, when $\mathcal{U}_E \neq \varnothing$, is defined to be the dimension of the Hilbert space obtained from the vector space $\mathbb{C}_0\mathcal{U}_E$ endowed with the positive semi-definite form obtained from the covariance function of E.

Theorem 4.8. *Let E_1 and E_2 be product systems, and assume that E_1 is type I. Let $\Phi : E_1 \to E_2$ be a weak isomorphism. Then E_2 is type I, and* $\operatorname{index} E_1 = \operatorname{index} E_2$. *Therefore, $E_1 \simeq E_2$.*

Proof. Since E_1 is type I, it is decomposable (see [Arv97c, Theorem 11.1]). The decomposability of E_2 is immediate, and thus we turn our attention to the assertion about the index.

Let $e_1(t)$ be a unit of E_1. By Lemma 4.5, there is a unit e_2 of E_2, $e_2 \sim \Phi(e_1)$ (in the sense of Definition 4.6). By Theorem 8.2 of [Arv97c], there are $L_t^i : \Delta_i^2 \to \mathbb{C}$ continuous functions vanishing at the origin satysfying

$$e^{L_t^i(x,y)} = \frac{\langle x, y\rangle}{\langle x, e_i(t)\rangle \langle e_i(t), y\rangle}$$

for $x, y \in D_i(t)$, $t > 0$.

Observe that the map Φ gives rise to a bijection $\Phi : \mathcal{W}_{E_1} \to \mathcal{W}_{E_2}$, and this clearly descends to a bijection $\phi : \mathcal{W}_{E_1}/\sim \longrightarrow \mathcal{W}_{E_2}/\sim$. By Lemma 4.5, given any $\xi \in \mathcal{W}_{E_i}/\sim$, there is $u \in \mathcal{U}_{E_i}$ such that $u \in \xi$, $i = 1, 2$. Therefore, we actually have a bijection $\phi : \mathcal{U}_{E_1}/\sim \longrightarrow \mathcal{U}_{E_2}/\sim$.

Using ϕ we construct a bijection $\psi : \mathcal{U}_{E_1} \to \mathcal{U}_{E_2}$ as follows. For each $\xi \in \mathcal{U}_{E_1}/\sim$, choose a unit $u_\xi \in \xi$, with $\|u_\xi(t)\| = 1$, for $t > 0$. Set $\psi(u_\xi)$ to be any element of $\phi(\xi)$. Finally, given $v \in \xi$, set

$$\psi(v)_t = e^{t\gamma_1(v,u_\xi) - t\gamma_1(u_\xi,u_\xi)} \psi(u_\xi)_t = e^{t\gamma_1(v,u_\xi)} \psi(u_\xi)_t$$

Notice that we have $\psi(v) \sim \Phi(v)$. Indeed, $v \sim u_\xi$, and thus $\Phi(v) \sim \Phi(u_\xi)$. By definition $\psi(v) \sim \psi(u_\xi)$, and $\psi(u_\xi) \sim \Phi(u_\xi)$. Finally, Since ϕ is a bijection, Lemma 4.7 guarantees that this is 1-1. To see that ψ is a onto, fix V unit of E_2. Then $V \in \phi(\xi)$, for some $\xi \in \mathcal{U}_{E_1}/\sim$, since ϕ is onto. By Lemma 4.7 there is $\alpha \in \mathbb{C}$ such that $V_t = e^{\alpha t}\psi(u_\xi)_t$, and if we set $v_t = e^{\alpha t} u_\xi(t)$, we have $\gamma_1(v, u_\xi) = \alpha$, and $V = \psi(v)$. We conclude that ψ is a bijection.

The map ψ gives rise to a linear bijection $\Psi : \mathbb{C}_0\mathcal{U}_{E_2} \to \mathbb{C}_0\mathcal{U}_{E_1}$ defined by $\Psi(f) = f \circ \psi$. We will prove that this is a unitary map. First, to make our notation easier, we denote $L^i(u, v) = L_1^i(u_1, v_1)$ when u, v are units of E_i, $i = 1, 2$.

Claim 4.9. *For $i = 1, 2$ and for any $u, v \in \mathcal{U}_{E_i}$, we have*

$$L_t^i(u_t, v_t) = t\,[\gamma_i(u, v) - \gamma_i(u, e_i) - \gamma_i(e_i, v)]$$

Proof. Let

$$g(t) = L_t^i(u_t, v_t) - t\,[\gamma_i(u, v) - \gamma_i(u, e_i) - \gamma_i(e_i, v)]$$

Notice that $g(t)$ is continuous, $e^{g(t)} = 1$, for all $t > 0$, and $lim_{t\to 0+} g(t) = 0$. Therefore, $g \equiv 0$. □

Claim 4.10. *For $i = 1, 2$, given $f, g \in \mathbb{C}_0\mathcal{U}_{E_i}$, we have that*

$$\sum_{u,v\in\mathcal{U}_{E_i}} f(u)\overline{g(v)}\gamma_i(u, v) = \sum_{u,v\in\mathcal{U}_{E_i}} f(u)\overline{g(v)} L^i(u, v)$$

and given u, v units of E_1,

$$L^2(\psi(u), \psi(v)) = L^1(u, v)$$

Proof. To prove the first identity, notice that by the previous claim

$$L^i(u, v) = \gamma_i(u, v) - \gamma_i(u, e_i) - \gamma_i(e_i, v)$$

and

$$\sum_{u,v\in\mathcal{U}_{E_i}} f(u)\overline{g(v)}\gamma_i(u, e_i) = \sum_{u\in\mathcal{U}_{E_i}} f(u)\gamma_i(u, e_i) \sum_{v\in\mathcal{U}_{E_i}} \overline{g(v)} = 0$$

since $\sum g(v) = 0$. Analogously, the term involving $\gamma(e_i, v)$ vanishes. To prove the second identity, notice that for any $x, y \in D_1(t)$,

$$e^{L_t^2(\Phi(x),\Phi(y))} = \frac{\langle \Phi(x), \Phi(y)\rangle}{\langle \Phi(x), e_2(t)\rangle \langle e_2(t), \Phi(y)\rangle} = \frac{\langle x, y\rangle}{\langle x, e_1(t)\rangle \langle e_1(t), y\rangle} = e^{L_t^1(x,y)}$$

since $e_2 \sim \Phi(e_1)$. Now, the map $L_t^2 \circ \Phi$ is continuous and vanishes at zero. Therefore, by the uniqueness statement of Theorem 8.2 of [Arv97c], we conclude that

$$L_t^2(\Phi(x), \Phi(y)) = L_t^1(x, y)$$

for all $x, y \in D_1(t)$, $t > 0$. Applying this to u, v units of E_1, we have the required identity, since $\psi(u) \sim \Phi(u), \psi(v) \sim \Phi(v)$. $\square$

We are now ready to prove that Ψ is unitary. Let $f, g \in \mathbb{C}_0\mathcal{U}_{E_2}$. Then

$$\begin{aligned}
\langle \Psi(f), \Psi(g) \rangle &= \sum_{u,v \in \mathcal{U}_{E_1}} \Psi(f)(u)\overline{\Psi(g)(v)}\gamma_1(u, v) \\
&= \sum_{u,v \in \mathcal{U}_{E_1}} f(\psi(u))\overline{g(\psi(v))}L^1(u, v) = \sum_{u,v \in \mathcal{U}_{E_1}} f(\psi(u))\overline{g(\psi(v))}L^2(\psi(u), \psi(v)) \\
&= \sum_{u,v \in \mathcal{U}_{E_2}} f(u)\overline{g(v)}L^2(u, v) = \sum_{u,v \in \mathcal{U}_{E_2}} f(u)\overline{g(v)}\gamma_2(u, v) = \langle f, g \rangle
\end{aligned}$$

Thus we conclude that $\operatorname{index} E_1 = \operatorname{index} E_2$. Since the index is a complete isomorphism invariant of type I product systems, $E_1 \simeq E_2$. $\square$

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