Regular $t$-graphs, antipodal distance regular graphs of diameter 3 and related combinatorial structures

by

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In memory of my parents
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contents</td>
<td>i</td>
</tr>
<tr>
<td>1  Introduction</td>
<td>3</td>
</tr>
<tr>
<td>2  Preliminaries</td>
<td>5</td>
</tr>
<tr>
<td>2.1 Coherent configurations and association schemes</td>
<td>5</td>
</tr>
<tr>
<td>2.2 Antipodal distance regular graphs</td>
<td>8</td>
</tr>
<tr>
<td>2.3 Symmetric transversal designs and related structures</td>
<td>14</td>
</tr>
<tr>
<td>2.4 Computer algebra tools</td>
<td>15</td>
</tr>
<tr>
<td>3  Regular 3-graphs</td>
<td>17</td>
</tr>
<tr>
<td>3.1 Two-graphs and regular two-graphs</td>
<td>17</td>
</tr>
<tr>
<td>3.2 Two ways to generalize two-graphs</td>
<td>20</td>
</tr>
<tr>
<td>3.3 Regular 3-graphs in D. G. Higman’s sense</td>
<td>20</td>
</tr>
<tr>
<td>3.4 Elements of Higman’s theory</td>
<td>22</td>
</tr>
<tr>
<td>3.5 Feasibility conditions</td>
<td>28</td>
</tr>
<tr>
<td>3.6 The symplectic example</td>
<td>30</td>
</tr>
<tr>
<td>4  Regular 3-graphs and antipodal covers</td>
<td>31</td>
</tr>
<tr>
<td>4.1 $(n, 3, c_2)$-covers with cyclic matchings</td>
<td>31</td>
</tr>
<tr>
<td>4.2 Regular 3-graphs and $(n, 3, c_2)$-covers with cyclic matchings</td>
<td>32</td>
</tr>
<tr>
<td>4.3 Cyclic covers and $(n, 3, c_2)$-covers with cyclic matchings</td>
<td>37</td>
</tr>
<tr>
<td>4.4 The list of feasible parameter sets revisited</td>
<td>38</td>
</tr>
<tr>
<td>5  From symmetric transversal designs to antipodal covers</td>
<td>41</td>
</tr>
<tr>
<td>5.1 Generalized Hadamard matrices, STD’s, and antipodal covers</td>
<td>41</td>
</tr>
<tr>
<td>5.2 Elements of Klin-Pech theory</td>
<td>41</td>
</tr>
<tr>
<td>5.3 Class-regular STD’s via generalized Hadamard matrices</td>
<td>43</td>
</tr>
<tr>
<td>5.4 Flag algebras of incidence structures</td>
<td>44</td>
</tr>
<tr>
<td>5.5 Flag algebras of STD’s: first observations</td>
<td>46</td>
</tr>
</tbody>
</table>
CONTENTS

6 Computer algebra experimentation 47
  6.1 Regular 3-graphs from cyclic \((n, 3, c_2)-\)covers . . . . . . . . . . . . . . 47
    6.1.1 Investigation of the symplectic example . . . . . . . . . . . . . 49
  6.2 Regular 3-graphs from class regular STD’s . . . . . . . . . . . . . . . 49
    6.2.1 A regular 3-graph with parameters \((9, 1, 3)\) . . . . . . . . 49
    6.2.2 A regular 3-graph with parameters \((36, 10, 12)\) . . . . . 50
    6.2.3 Two regular 3-graphs with parameters \((81, 25, 27)\) . . . . . 50
    6.2.4 A regular 3-graph with parameters \((144, 46, 48)\) . . . . . 51
    6.2.5 28 regular 3-graphs with parameters \((324, 106, 108)\) . . . . 52
  6.3 A sporadic example on 135 vertices . . . . . . . . . . . . . . . . . . . 53
  6.4 Two rank 6 coherent algebras . . . . . . . . . . . . . . . . . . . . . . . 54

7 Computer free interpretations 55
  7.1 Two DRG’s on 27 vertices . . . . . . . . . . . . . . . . . . . . . . . . . 55
  7.2 A sporadic example on 108 vertices revisited . . . . . . . . . . . . . . 56
    7.2.1 The general story . . . . . . . . . . . . . . . . . . . . . . . . . 56
    7.2.2 The transversal design \(STD_2(3)\) . . . . . . . . . . . . . . . 56
    7.2.3 A model of the \(STD_2(3)\) via \(A_5\) . . . . . . . . . . . . . . 57
    7.2.4 Inspection of the action of \(\text{Aut}(STD_2(3))\) on flags . . . . 60
  7.3 Other structures . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 61
    7.3.1 Exceptional example on 972 vertices . . . . . . . . . . . . . . . 61

8 Concluding remarks 63

Bibliography 67

A Code 73
  A.1 Feasible parameter sets of regular 3-graphs . . . . . . . . . . . . 73
  A.2 Construction of the two \((9, 3, 3)\)-covers . . . . . . . . . . . . 74
  A.3 COCO output for the action of \(\text{Aut}(STD_2(3))\) on its flags . . . . 75
  A.4 COCO output for the action of \(\mathbb{Z}_3.A_6\) on the flags of \(STD_2(3)\) . . 82

List of Figures 97
List of Tables 98
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Chapter 1

Introduction

This thesis stemmed from a short draft [23] by D. G. Higman, called “A note on regular 3-graphs”, written in 1994. We begin with a short account of the history leading to the study of regular \( t \)-graphs. Regular two-graphs were introduced by G. Higman to study 2-transitive representations of certain simple groups, were studied by D. E. Taylor in his D. Phil. thesis (1971), and subsequently (1973) surveyed by J. J. Seidel in [37]. A few years later a generalization of two-graphs to higher dimensional objects was introduced in the framework of design theory (see [19], [35], [10]), and referred to as \( t \)-graphs.

In 1990, D. G. Higman introduced a generalization of two-graphs in a different direction and surprisingly called these new objects \( t \)-graphs, also. In an unpublished note [23], D. G. Higman uses the machinery of association schemes to characterize regular 3-graphs. This note by Higman is written with obscure notation, and moreover contains misprints and miscalculations which lead to false results.

In this thesis, we reformulate, extend and clarify this approach. Furthermore, we establish new observations with the machinery developed by Higman, and introduce the connection between regular 3-graphs and antipodal distance regular graphs of diameter 3 that satisfy a special property. Having established this connection, we construct new examples of regular 3-graphs via antipodal distance regular covers of complete graphs. The latter are obtained from constructions using other combinatorial objects, namely, a special kind of block design (termed a class regular symmetric transversal design), and generalized Hadamard matrices; these constructions were suggested by M. Klin and Ch. Pech in [26].

In our journey to construct new regular 3-graphs from antipodal distance regular covers of complete graphs via the above process, we observed new connections between the objects involved. This investigation is not yet complete; it is presented in this thesis in its current form, and we believe that a more complete
account will be ready soon.

Preliminaries for the objects of interest in this thesis (except $t$-graphs) are presented in chapter 2. In chapter 3 we provide some background about two-graphs and $t$-graphs, and introduce our own detailed version of Higman’s theory of regular 3-graphs. Our new results about regular 3-graphs and antipodal covers are detailed in chapter 4. In chapter 5 we outline our investigation of the connections between symmetric transversal designs and antipodal covers. The computer is an important tool in our research, namely, we use certain computer algebra systems. Most of our computer aided investigation is described in chapter 6. In chapter 7 we present a few computer-free interpretations of some of the results that appeared throughout the thesis, and specifically in chapter 6. In the concluding chapter 8, a summarizing discussion appears, including a short section about promising future development and further directions of research.
Chapter 2

Preliminaries

In this chapter some objects of interest are defined, together with some related, well-known results.

2.1 Coherent configurations and association schemes

We refer to [1], [7] for more details. A color graph is a pair \((\Omega, R)\), where \(\Omega\) is a set of vertices and \(R\) a partition of \(\Omega^2\) into a set of binary relations on \(\Omega\). According to D. G. Higman (see [18]), a coherent configuration is a color graph \(m = (\Omega, R)\), \(R = \{R_i | i \in I\}\) such that the following conditions are satisfied:

1. The identity relation \(Id_\Omega = \{(x,x) | x \in \Omega\}\) is a union of some relations \(R_i\), \(i \in I', I' \subseteq I\).
2. For each \(i \in I\) there exists \(i' \in I\) such that \(R^t_i = R_{i'}\), where \(R^t_i = \{(y,x) | (x,y) \in R_i\}\).
3. For any \(i, j, k \in I\) the number \(p_{ij}^k\) of elements \(z \in \Omega\) such that \((x,z) \in R_i\) and \((z,y) \in R_j\) is constant provided that \((x,y) \in R_k\).

The numbers \(p_{ij}^k\) are called the intersection numbers of \(m\). We refer to \(R_i\) as basic relations, graphs \(\Gamma_i = (\Omega, R_i)\) as basic graphs, and adjacency matrices \(A_i = A(\Gamma_i)\) as basic matrices of \(m\). The notion of a fiber of a coherent configuration, and that of its type, appears in [20]. The number \(|I|\) is called the rank of \(m\), and the number \(|\Omega|\) is the order of \(m\).

If \((G, \Omega)\) is a permutation group, \(2 - orb(G, \Omega)\) denotes the set of all 2-orbits of \((G, \Omega)\). These are the orbits of the induced action of \(G\) on \(\Omega^2\). It is easy to see that \((\Omega, 2 - orb(G, \Omega))\) is a coherent configuration. Such coherent configurations will be called Schurian (cf. [14]).
The particular case of a coherent configuration $\mathbf{m}$ in which the identity relation $Id_\Omega$ is one of the basic relations of $\Omega$ is called a homogeneous coherent configuration, or an association scheme. Typically, the basic relation $Id_\Omega$ is denoted by $R_0$. All remaining basic relations of an association scheme are called classes. As in [1], an association scheme is not presumed to be commutative.

The adjacency algebra and the intersection algebra

The notion of a coherent configuration may be reformulated in terms of matrices. Consider the set of basic matrices $\{A_i \mid i \in \mathcal{I}\}$ of the coherent configuration $\mathbf{m} = (\Omega, \mathcal{R})$. Then the conditions (1) – (3) can be reformulated as follows:

$$
(1') \sum_{i \in \mathcal{I}} A_i = I,
$$

$$
(2') A_i^t = A_{i'},
$$

$$
(3') A_i A_j = \sum_{k \in \mathcal{I}} p_{ij}^k A_k.
$$

Notice that the fact that $\mathcal{R}$ is a partition of $\Omega$ translates into

$$
\sum_{i \in \mathcal{I}} A_i = J,
$$

which means that the $A_i$ are linearly independent over $\mathbb{C}$, and (1') – (3') imply that they generate an $|\mathcal{I}|$-dimensional $\mathbb{C}$-algebra with an identity, which is also closed under transposition. This is called the adjacency algebra of $\mathbf{m}$.

Let $B_i$ be the matrix whose $(j, k)$-th entry is $p_{ij}^k$. The $B_i$’s are called the intersection matrices of $\mathbf{m}$, or the tensor of structure constants of $\mathbf{m}$, and the $\mathbb{C}$-algebra generated by them is called the intersection algebra of $\mathbf{m}$. This is an $|\mathcal{I}|$-dimensional algebra which is isomorphic to the adjacency algebra of $\mathbf{m}$.

Commutative association schemes

Let $\mathbf{m} = (\Omega, \mathcal{R})$ be a commutative association scheme, meaning that $p_{ij}^k = p_{ji}^k$, or equivalently that $A_i A_j = A_j A_i$ for all $i, j$. In this case, the adjacency algebra of $\mathbf{m}$ has a basis of idempotents $E_0, \ldots, E_{|\mathcal{I}|}$ such that:

$$
(1) \ E_i E_j = \delta_{ij} E_i,
$$

$$
(2) \ \text{The columns of } E_i \ \text{are eigenvectors of each } A_j,
$$

$$
(3) \ \sum_{i \in \mathcal{I}} E_i = I,
$$

$$
(4) \ E_i^* = E_i \ \text{(here } ^* \text{ means the usual conjugate transpose operation)}.
$$
2.1. COHERENT CONFIGURATIONS AND ASSOCIATION SCHEMES

For every \( i \in I \) there are \(|I|\) scalars \( p_i(j) \) such that

\[
A_i = \sum_{j \in I} p_i(j)E_j,
\]

and there are scalars \( q_i(j) \) such that

\[
E_i = \frac{1}{|\Omega|} \sum_{j \in I} q_i(j)A_j.
\]

The numbers \( p_i(j) \) are called the eigenvalues of \( m \). These are algebraic integers, since they are the eigenvalues of 0,1-matrices \( A_i \). As mentioned before, the columns of \( E_i \) are eigenvectors of each matrix in the adjacency algebra, and thus the column space of \( E_i \) is an eigenspace of the adjacency algebra, with dimension equal to \( \text{rank}(E_i) \). Since \( E_i \) is an idempotent, all its eigenvalues are equal to 1, and \( \text{rank}(E_i) = \text{tr}(E_i) \). The numbers \( m_i = \text{tr}(E_i) \) are called the multiplicities of \( m \). The matrices

\[
P = (P)_{i,j} = p_j(i)
\]

and

\[
Q = (Q)_{i,j} = q_j(i)
\]

are called the first eigenmatrix and the second eigenmatrix respectively of \( m \) (cf. [1]). Sometimes (cf. [21]), the first eigenmatrix \( P \) together with the list of multiplicities is called the character-multiplicity table of \( m \).

The coherent closure

A coherent algebra \( W \) (see [21]) is a set of square matrices of order \( n \) over the field \( \mathbb{C} \) which forms a matrix algebra, is closed with respect to the operations of Schur-Hadamard (pointwise) multiplication and transposition, and contains both the identity matrix \( I \) and the all-one matrix \( J \). The set of basic matrices \( \{A_i \mid i \in I\} \) of a coherent configuration \( m \) serves as a standard basis of the corresponding coherent algebra \( W \); in this case we write \( W = \langle A_0, A_1, \ldots, A_{r-1} \rangle \). We may abuse notation, referring to \( W \) and the corresponding coherent configuration \( m \) as one and the same object.

The intersection of coherent algebras is again a coherent algebra. This implies the existence of the coherent closure of \( B \), denoted by \( \langle\langle B \rangle\rangle \), i.e., the smallest coherent algebra containing a prescribed set \( B \) of matrices of order \( n \). An efficient polynomial time algorithm for the computation of the coherent closure, often referred to as WL-stabilization, is described in [47].

We call a graph \( \Gamma = (V, E) \) a coherent graph if \( E \) is a basic relation of the coherent closure \( \langle\langle E \rangle\rangle \) (here we think of \( E \) as the adjacency matrix of \( \Gamma \)). For example, each distance regular graph (cf. section 2.2) is coherent.
Isomorphisms, automorphisms, mergings

An isomorphism of color graphs \((\Omega, R)\) and \((\Omega', R')\) is a bijection \(\phi\) from \(\Omega\) to \(\Omega'\) which induces a bijection of colors (relations) in \(R\) onto colors in \(R'\). A weak (or color) automorphism of \(\Gamma = (\Omega, R)\) is an isomorphism of \(\Gamma\) with itself. If the induced permutation of colors is the identity permutation, then this is a (strong) automorphism.

We denote by \(\text{CAut}(\Gamma)\) and \(\text{Aut}(\Gamma)\) the groups of all weak and strong automorphisms of \(\Gamma\), respectively. Clearly, \(\text{Aut}(\Gamma) \subseteq \text{CAut}(\Gamma)\). In the case where \(\Gamma\) is a Schurian coherent configuration, the group \(\text{CAut}(\Gamma)\) coincides with the normalizer of \(\text{Aut}(\Gamma)\) in the symmetric group \(S(\Omega)\).

An algebraic isomorphism between coherent configurations \((\Omega, R)\) and \((\Omega', R')\) is a bijection \(\phi : R \rightarrow R'\) such that for all \(i, j, k \in I\), \(p^k_{ij} = p^k_{\phi(i)\phi(j)}\). An algebraic isomorphism of a coherent configuration \(m = (\Omega, R)\) with itself is called an algebraic automorphism of \(m\). The group of algebraic automorphisms of \(m\) is denoted by \(\text{AAut}(m)\).

Clearly, \(\text{CAut}(m)/\text{Aut}(m) \hookrightarrow \text{AAut}(m)\). If the quotient group \(\text{CAut}(m)/\text{Aut}(m)\) is a proper subgroup of \(\text{AAut}(m)\) then the algebraic automorphisms of \(m\) which are not induced by \(\phi \in \text{CAut}(m)\) are called proper algebraic automorphisms. See [25] for more details.

If \(W'\) is a coherent subalgebra of a coherent algebra \(W\), then the corresponding coherent configuration \(m'\) is called a fusion (or merging configuration) of \(m\). In the case where \(m = (\Omega, 2 - \text{orb}(G, \Omega))\) for a suitable permutation group \(G\), overgroups of \(G\) in \(S(\Omega)\) lead to fusions of \(m\). Thus the most interesting fusions are the non-Schurian fusions, i.e., those which do not emerge from a suitable overgroup of \((G, \Omega)\).

For each subgroup \(K \leq \text{AAut}(m)\), its orbits on the set of relations define a merging coherent configuration, called the algebraic merging defined by \(K\). Again, those algebraic mergings which are non-Schurian are of a special interest, being in a sense “less predictable” combinatorial objects.

### 2.2 Antipodal distance regular graphs

Let \(\Gamma = (V, E)\) be an undirected connected graph, and \(v, u \in V\) be two vertices. The distance \(d(v, u)\) between \(v\) and \(u\) is the length of the shortest path between them, the diameter of \(\Gamma\) is the maximal distance between two vertices of \(\Gamma\). The graph \(\Gamma\) is called distance regular if for any two vertices \(u, v \in V\), the number of vertices at distance \(i\) from \(u\) and distance \(j\) from \(v\) depends only on \(i, j\) and \(k := d(u, v)\). Let \(\Gamma\) be a distance regular graph of diameter \(d\). Let \(v\) and \(u\) be two vertices at distance \(k\) from each other. The intersection number \(p^k_{ij}\) is the number of vertices at distance \(i\) from \(u\) and \(j\) from \(v\). Denote by \(c_i, a_i\) and \(b_i\) the number of neighbors of \(v\) at distance \(i - 1\), \(i\) and \(i + 1\), respectively, from \(u\).
2.2. ANTIPODAL DISTANCE REGULAR GRAPHS

(These are the intersection numbers \( p_i^1, p_i^{i-1,1}, p_i^{i+1,1} \)). It turns out that these numbers determine all the intersection numbers of \( \Gamma \). Since \( a_i + b_i + c_i = b_0 \) (the valency of \( \Gamma \)), the intersection array \( \{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\} \) determines all the intersection numbers of \( \Gamma \). \( \Gamma \) is called antipodal if the vertices at distance \( d \) from a given vertex are all at distance \( d \) from each other. Hence “being at distance \( d \)” is an equivalence relation on the vertices of \( \Gamma \), and the equivalence classes of this relation are called antipodal fibers (or simply fibers) of \( \Gamma \).

Let \( \pi \) be the partition of \( V \) into fibers. Then the antipodal quotient \( \Gamma/\pi \) is the graph with classes of \( \pi \) as vertices and with an edge between two vertices if and only if there is an edge between the corresponding classes in \( \Gamma \). \( \Gamma \) is called the covering graph of \( \Gamma/\pi \), and the map sending each vertex of \( \Gamma \) to its fiber is called the covering map. Notice that when \( \Gamma \) is connected all fibers must have the same size — this is the index of the covering. When the index is \( r \) we say that \( \Gamma \) is an \( r \)-fold cover of \( \Gamma/\pi \). The distance between two fibers of \( \Gamma \) is the distance between the corresponding vertices in the antipodal quotient \( \Gamma/\pi \).

We recall some results which can be found in \[7\], and are also mentioned in \[16\]:

**Proposition 2.1.** Let \( \Gamma = (V, E) \) be an antipodal distance regular graph of diameter \( d > 2 \) then:

(i) If there is an edge between two fibers of \( \Gamma \), then there is a perfect matching between them.

(ii) If the distance between two fibers of \( \Gamma \) is \( i \), then each vertex in the first fiber is at distance \( i \) from one vertex in the second fiber, and at distance \( d - i \) from all other vertices in the second fiber.

**Proposition 2.2.** Let \( \Gamma = (V, E) \) be a connected antipodal distance regular graph of diameter \( d > 2 \), intersection array \( \{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\} \) and index \( r \). Let \( \pi \) be the partition of \( V \) into antipodal classes (the fibers), then:

(i) The antipodal quotient \( \Gamma/\pi \) is a distance regular graph of diameter \( m = \lfloor d/2 \rfloor \) and intersection array \( \{b_0, \ldots, b_{m-1}; 1, c_2, \ldots, c_m\} \) if \( d \) is odd, and intersection array \( \{b_0, \ldots, b_{m-1}; 1, c_2, \ldots, rc_m\} \) if \( d \) is even.

(ii) Every eigenvalue of \( \Gamma/\pi \) is also an eigenvalue of \( \Gamma \) with the same multiplicity.
CHAPTER 2. PRELIMINARIES

Metric association schemes

Let $\Gamma = (V, E)$ be a distance regular graph of diameter $d$. Define $d$ relations $R_0, R_1, \ldots, R_d$ on $V$ such that

$$(u, v) \in R_i \iff d(u, v) = i.$$ 

Then $m = (V, \{R_0, R_1, \ldots, R_d\})$ is an association scheme; $m$ is called metric.

There is an axiomatic definition of a metric association scheme as follows: A symmetric association scheme $m = (X, \{R_i\}_{0 \leq i \leq d})$ is called metric with respect to the ordering $R_0, R_1, \ldots, R_d$ if $p^k_{ij} \neq 0$ implies $k \leq i + j$ and moreover $p^k_{ij} \neq 0$ for all $i, j, k$. Similarly, an association scheme is called cometric with respect to an ordering of the minimal idempotents if these conditions hold for the $q^k_{ij}$. It is easy to observe that the ordering is determined by the relation $R_1$ (or in the cometric case, by $E_1$).

Metric association schemes are in one-to-one correspondence with distance regular graphs. To see this, let $m = (X, \{R_i\}_{0 \leq i \leq d})$ be metric with respect to the ordering $R_0, R_1, \ldots, R_d$. Then $\Gamma_m = (X, R_1)$ is a distance regular graph, and $(x, y) \in R_j$ if and only if $d_{\Gamma_m}(x, y) = j$.

Another related notion is that of a $P$-polynomial (and the dual notion of $Q$-polynomial) association scheme: A symmetric association scheme $m = (X, \{R_i\}_{0 \leq i \leq d})$ is called a $P$-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$ if there exist complex coefficient polynomials $p_i(x)$ of degree $i$ such that $A_i = p_i(A_1)$ for all $0 \leq i \leq d$, where $A_i$ is the adjacency matrix of the relation $R_i$. Again, in a dual fashion we call an association scheme $Q$-polynomial, with respect to an ordering $E_0, E_1, \ldots, E_d$ of the minimal idempotents, if there exist polynomials $q_i(x)$ of degree $i$ such that $E_i = q_i(E_1)$ (here multiplication is the Schur-Hadamard product) for all $0 \leq i \leq d$. It turns out that the polynomials $p_0(x), p_1(x), \ldots, p_d(x)$ and the polynomials $q_0(x), q_1(x), \ldots, q_d(x)$ are closely connected to the first and second eigenmatrices $P$ and $Q$ respectively, and satisfy the relations: $P_{ij} = p_j(\theta_i)$ and $Q_{ij} = q_j(\theta'_i)$ where $\theta_i = P_{i1}$ and $\theta'_i = Q_{i1}$. To sum up this subsection we formulate the following proposition, the proof of which can be found in [1] or in [7]:

**Proposition 2.3.** Let $m = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme with ordering $R_0, R_1, \ldots, R_d$ of its relations and $E_0, E_1, \ldots, E_d$ of its minimal idempotents. Let $P$ and $Q$ be the eigenmatrices of $m$. The following are equivalent:

(i) $m$ is a metric association scheme (that is, $(X, R_1)$ is a distance regular graph).

(ii) There are rational coefficient polynomials $p_i(x)$ of degree $i$ such that $P_{ij} = p_j(\theta_i)$, where $\theta_i = P_{i1}$.

(iii) $m$ is $P$-polynomial.
Dually, the following are equivalent:

(i) $m$ is a cometric association scheme.

(ii) There are real coefficient polynomials $q_i(x)$ of degree $i$ such that $Q_{ij} = q_j(\theta'_i)$, where $\theta'_i = Q_{i1}$.

(iii) $m$ is $Q$-polynomial.

\[\square\]

Antipodal covers of complete graphs

We focus our interest on antipodal distance regular graphs (ADRG for short) of diameter 3. As implied by Proposition 2.2, these graphs are covers of a distance regular graph of diameter 1, i.e. a complete graph $K_n$, where $n = |\Gamma/\pi|$ is the number of fibers of $\Gamma$.

Proposition 2.4. The intersection numbers of an antipodal distance regular graph $\Gamma$ of diameter 3 are completely determined by the parameters $(n, r, c_2)$.

Proof. Clearly $\Gamma$ is regular of valency $n - 1$, and so $b_0 = n - 1$. Let $v \in V$ be a vertex of $\Gamma$, and denote by $\Gamma_i(v)$ the set of vertices in $\Gamma$ at distance $i$ from $v$. Since $\Gamma$ is antipodal, the vertices in $\Gamma_3(v)$ are all at distance 3 from each other, and using Proposition 2.2, we obtain $b_2 = 1$ and $c_3 = n - 1$. Now, by counting in two ways the edges between $\Gamma_2(v)$ and $\Gamma_3(v)$ we find that there are $(r - 1)(n - 1)$ vertices in $\Gamma_2(v)$, and by counting in two ways the edges between $\Gamma_1(v)$ and $\Gamma_2(v)$ we have $(n - 1)b_1 = (r - 1)(n - 1)c_2$. Thus, we obtain the intersection array of $\Gamma$ in terms of $(n, r, c_2)$:

$\{n - 1, (r - 1)c_2, 1; 1, c_2, n - 1\}$

\[\square\]

We recall a result of C. D. Godsil and A. D. Hensel in [16] for later use:

Lemma 2.5 (Lemma 3.1 in [16]). An $r$-fold cover of $K_n$ is antipodal distance regular if and only if there exists a constant $c_2$ such that any two non-adjacent vertices from different fibers of the cover have exactly $c_2$ common neighbors. \[\square\]

Since all the parameters of an ADRG $\Gamma$ of diameter 3 are determined by $n, r$ and $c_2$, we call $\Gamma$ an $(n, r, c_2)$-cover.

For later use we notice that any two non-neighbors from different fibers (these are vertices at distance 2 from each other) have exactly $c_2$ common neighbors, and any two neighbors (they must be from different fibers) have exactly $a_1 = n - 2 - (r - 1)c_2$ common neighbors.
CHAPTER 2. PRELIMINARIES

Elements of Godsil-Hensel theory

In this section we present more results of Godsil and Hensel from [16], some of these are formulated in the language used by Klin and Pech in [26], which differs from the original language of Godsil and Hensel.

Let $\Gamma = (V, E)$ be an $(n, r, c_2)$-cover. Let $X$ be the set of fibers of $\Gamma$. We give an algebraic description of a cover of a graph. First, we associate a set \{x_1, \ldots, x_r\} with each $x \in X$, then we record the matching between two fibers as a permutation in $S_r$.

**Definition 2.1.** For every pair $x, y \in X$ define $f(x, y) \in S_r$ such that

$$i f(x, y) = j \iff \{x_i, y_j\} \in E.$$

Clearly, with the above definition, $f(x, y)$ can be any element of $S_r$ depending on the choice of the labeling of the fibers $x$ and $y$. Nevertheless, given $f(x, y)$ for some pair of fibers $x, y \in X$ we may permute the labels of vertices of $x$ and obtain a labeling of $\Gamma$ in which $f(x, y) = 1$. In particular, we may choose a labeling in the following manner:

1. choose any fiber $x$, and label its vertices with \{x_1, \ldots, x_r\} arbitrarily,
2. for every fiber $y$ different from $x$ label its vertices by \{y_1, \ldots, y_r\} such that:
   $$\{x_i, y_i\} \in E \text{ for } 1 \leq i \leq r.$$

More generally we may:

1. choose a spanning tree of $K_n$,
2. for every pair $x, y$ of adjacent vertices in the spanning tree define $f(x, y) = 1$.

Such a labeling, i.e. one which has value 1 on a spanning tree of the underlying complete graph is called normalized. It turns out (see Section 7 of [16]) that there is a close connection between the group generated by the values of a normalized labeling $f$ and the subgroup of $\text{Aut}(\Gamma)$ which stabilizes each fiber of $\Gamma$. We give the definitions and then recall some results.

**Definition 2.2.** \(\langle f \rangle\) is the subgroup of $S_r$ generated by the set \{f(x, y) | x, y \in X\}.

**Definition 2.3.** The voltage group of $\Gamma$ is the subgroup $T \leq \text{Aut}(\Gamma)$ which stabilizes each fiber of $\Gamma$.

Properties of the voltage group $T$:

- $|T| \leq r$,
- $T$ acts semi-regularly on the fibers of $\Gamma$. 

If $T$ acts regularly on the fibers of $\Gamma$ then $\Gamma$ is called regular. $\Gamma$ is called abelian or cyclic when $T$ is abelian or cyclic respectively.

An alternative formulation in the language of matrices over the group ring $\mathbb{Z}[T]$, used by Klin and Pech in [26], is as follows: Let $T$ be a finite group. For a subset $M \subseteq T$ define the simple quantity $M \in \mathbb{Z}[T]$:

$$M = \sum_{m \in M} 1 \cdot m.$$  

Let $A = (a_{i,j}) \in \mathbb{Z}[T]^{n \times n}$ such that $a_{i,j} \in (\{g \mid g \in T\} \cup \{0\})$, all elements on the diagonal are equal to 0, and such that $A$ is self-adjoint. Then to $A$ we can associate two graphs:

1. The underlying graph $\Delta_A$ with vertex set $V(\Delta_A) = \{1, 2, \ldots, n\}$ and edge set $E(\Delta_A) = \{\{i, j\} \mid a_{i,j} \neq 0\}$,
2. The derived graph $\Gamma^A$ with vertex set $V(\Gamma^A) = \{1, 2, \ldots, n\} \times T$ and edge set $E(\Gamma^A) = \{\{(i, g), (j, h)\} \mid a_{i,j} \neq 0, \text{ and } g \cdot a_{i,j} = h\}$.

Such matrices, when defining connected covers with voltage group $T$, are called covering matrices. When $\Delta_A$ is a complete graph $K_n$, and $\Gamma^A$ is an $(n, r, c_2)$-cover of $\Delta_A$ then the matrix $A$ is called the Godsil-Hensel matrix of the cover (cf. [26]).

We recall another result from [16] for later use:

**Lemma 2.6** (Lemma 7.2 in [16]). Let $(f)$ be a group generated by a normalized labeling $f$, and $T$ the voltage group of $\Gamma$. Then

$$T \cong C_{S_r}((f)).$$  

**Cyclic $(n, r, c_2)$-covers**

When the voltage group of $\Gamma$ is cyclic, Godsil and Hensel obtained a restriction on the parameters of $\Gamma$:

**Theorem 2.7** (Theorem 9.2 in [16]). Let $\Gamma$ be a cyclic $(n, r, c_2)$-cover with $r > 2$. Then

$$r \mid n.$$  

2.3 Symmetric transversal designs and related structures

In this section we develop the preliminaries regarding the second theme of this thesis: connections between certain symmetric designs and DRG’s. Notation is mostly as in [3] and [4].

**t-designs**

Let $D = (V,B,I)$ be an incidence structure. $D$ is called a $t$-design if there exist parameters $\lambda,k$ such that:

- $|b| = k$ for all $b \in B$,
- every $t$-subset $Q \subset V$ is contained in the same number $\lambda$ of blocks from $B$.

In this case $D$ is called an $S_\lambda(t,k,v)$ where $v = |V|$.

**Resolvable designs and divisible designs**

Let $D = (V,B,I)$ be an incidence structure and let $B = B_1 \cup \cdots \cup B_m$ be a partition of the block set. Then the induced substructures $D_i = (V,B_i,I)$, $i = 1,\ldots,m$ are said to form a partition of $D$. Each $D_i$ is called a part of $D$. If a part of $D$ is an $S_r(1,k,v)$ then this part is called an $r$-factor, when $r = 1$ it is called a parallel class of $D$. If every part in a partition $B = B_1 \cup \cdots \cup B_m$ of $D$ is an $r$-factor then the partition is called an $r$-factorization and $D$ is called $r$-resolvable. For $r = 1$, $D$ is called resolvable, and a 1-factorization is called a resolution or parallelism. An incidence structure $D = (V,B,I)$ is called divisible if there is a partition of $V$ into point classes (sometimes called groups) such that any two points in the same class are incident with the same number $\lambda_1$ of common blocks. A divisible incidence structure is called a divisible design if it is balanced (this means that any two points in distinct classes are incident with the same number $\lambda_2$ of blocks). Our interest is in divisible designs with fixed point class size $g$ and fixed block size $k$, with the property that $\lambda_1 = 0$ and $\lambda_2 = \lambda$; such designs will be called divisible with parameter $\lambda$, and are denoted $GD_\lambda(k,g,kg)$.

**Transversal designs and nets, symmetric transversal designs**

A divisible design with parameter $\mu$ (i.e. a $GD_\mu(k,g,kg)$) is called a transversal design (or TD for short) if every block intersects every point class (uniquely, by definition of a point class). The dual structure of a transversal design is called a $\mu$-net, or for $\mu = 1$, simply a net. A TD with parameters $k > 2, \mu, g$ is denoted $TD_\mu(k,g)$. The dual of a $TD_\mu(k,g)$ is called a $(g,k,\mu)$-net. It is resolvable, has $k$ parallel classes, each parallel class contains exactly $g$ blocks of $g\mu$ points each, there are $g^2\mu$ points altogether and $kg$ blocks, and any two non-parallel blocks intersect precisely $\mu$ times. A $TD_\mu(k,g)$ is called symmetric if its dual is also a
2.4. COMPUTER ALGEBRA TOOLS

Class-regular STD’s and generalized Hadamard matrices

An $STD_{\mu}(g)$ is called class-regular if it admits a group of automorphisms $T$ of order $g$ that acts transitively (and hence regularly) on every point parallel class and on every block parallel class. This group $T$ is called a group of bitranslations.

A generalized Hadamard matrix $gH(T, \mu) = (h_{ij})$ over a group $T$ of order $g$ is a $g\mu \times g\mu$ matrix with entries from $T$, with the property that for every $i, j$, $1 \leq i < j \leq g\mu$, the multi-set

$$\left\{ t_{is} \cdot t_{js}^{-1} \mid 1 \leq s \leq g\mu \right\}$$

contains every element of $T$ exactly $\mu$ times. This can be formulated in the language of matrices over group rings (cf. section 2.2), as follows:

**Definition 2.4.** Let $T$ be a finite group and let $A = (a_{i,j}) \in \mathbb{Z}[T]^{n \times n}$ such that $a_{i,j} \in \{ g \mid g \in T \}$ for all $i$ and $j$. Then $A$ is called a generalized Hadamard matrix if for $\mu = \frac{n}{|T|}$, we have that

$$AA^* = A^*A = nI + \mu I(J - I).$$

Every generalized Hadamard matrix $gH(T, \mu)$ over a group $T$ of order $g$ determines a class-regular $STD_{\mu}(g)$ with a group of bitranslations isomorphic to $T$, and conversely, every class-regular $STD_{\mu}(g)$ with a group of bitranslations $T$ gives rise to a generalized Hadamard matrix $gH(T, \mu)$.

2.4 Computer algebra tools

Computer algebra facilities are playing a significant role in this thesis; in this section we give a brief description of most of the computer algebra tools used by us.

**GAP**

Short for “Groups, Algorithms, Programming” (see [15]). It is a computer algebra system. GAP supports a natural programming language making it a versatile system, in the sense that one can easily write code for performing very specific calculations, write whole packages or modify existing packages. Two GAP packages that are of great use in our area of research are GRAPE and DESIGN. The GRAPE package (see [41]) is a package for computing with graphs and groups, and is primarily designed for constructing and analysing graphs related to groups, finite geometries, and designs. The GRAPE package is written mostly in the GAP
language, but there are also certain functions in GRAPE which make use of B. D. McKay’s nauty program (see [33]), specifically, the automorphism group and isomorphism testing functions. The DESIGN package (see [40]) is for constructing, classifying, partitioning and studying block designs. Here again, some functions rely on the GRAPE package which in turn relies on nauty.

**COCO and COCO-II**

COCO is a package containing some programs to manipulate with coherent configurations. In particular, it is used to find all association schemes which are invariant with respect to a given permutation group, and their automorphism groups. COCO was developed in the early 1990’s in the former USSR by Faradžev and Klin (see [13]). 5 main programs are typically used, usually in the following order:

- **ind**: a program to compute the induced action of a permutation group on a combinatorial structure;
- **cgr**: a program to compute the centralizer algebra of a permutation group;
- **inm**: a program to compute the intersection numbers of a coherent configurations;
- **sub**: a program to find fusions (coherent subalgebras) of a coherent configuration (coherent algebra);
- **aut**: a program to compute the automorphism groups of a coherent configuration and its fusions.

COCO-II is a GAP package under construction. The main goal of its creators is to make the capabilities of COCO available in the GAP platform. Moreover, to implement the WL-stabilization algorithm, and develop new capabilities that are based on theoretical results obtained since COCO was created.

**Some theoretical background**

The Weisfeiler-Leman algorithm for the computation of the coherent closure of a given set of matrices is a polynomial time algorithm, mostly referred to as WL-stabilization (see [47]). In some cases, for example, when the order or rank of a coherent closure turns out to be large, there are certain computational tricks which can be used in conjunction with theoretic bounds to compute the coherent closure in just a few iterations.
3.1 Two-graphs and regular two-graphs

Two-graphs have roots originating in different areas in combinatorics, geometry and group theory, thus leading to different manifestations in the literature, such as: switching classes of graphs, sets of equidistant points in elliptic geometry, sets of equiangular lines in Euclidean geometry, binary maps of triples with vanishing coboundary, and double coverings of complete graphs (see the celebrated survey [37]). Our focus will be on the last two interpretations and the connection between them. We start with the definitions:

Definition 3.1. Let $X$ be a set of $n$ elements called vertices. $X^{(3)}$ is the set of all 3-subsets of $X$. A subset $\Delta \subseteq X^{(3)}$ is a two-graph if every 4-subset of $X$ contains an even ($\in \{0, 2, 4\}$) number of members of $\Delta$.

Sometimes we call $(X, \Delta)$ a two-graph, and $\Delta$ the set of odd triples.

Construction 1. Let $\Gamma = (V, E)$ be a simple graph. The set of triples $\{u, v, w\}$ of vertices, such that the induced subgraph $\Gamma|_{\{u,v,w\}}$ has an odd number of edges, forms a two-graph.

Definition 3.2. Let $X \subseteq V$ be a subset of vertices of $\Gamma = (V, E)$. Switching with respect to $X$ means interchanging the adjacencies between $X$ and its complement $V \setminus X$.

\begin{center}
\begin{tikzpicture}
    \node (1) at (0,0) {$1$};
    \node (2) at (1,0) {$2$};
    \node (3) at (1,-1) {$3$};
    \node (4) at (0,-1) {$4$};

    \draw (1) -- (2);
    \draw (3) -- (4);
    \draw (1) -- (3);
    \draw (2) -- (4);

    \node (b) at (2,0) {$b$};

    \begin{scope}[xshift=2.5cm]
        \node (1) at (0,0) {$1$};
        \node (2) at (1,0) {$2$};
        \node (3) at (1,-1) {$3$};
        \node (4) at (0,-1) {$4$};

        \draw (1) -- (2);
        \draw (3) -- (4);
        \draw (1) -- (3);
        \draw (2) -- (4);
    \end{scope}
    \node (c) at (3.5,0) {$c$};
\end{tikzpicture}
\end{center}
Example 1. The graph $c$ is obtained from $b$ by switching with respect to $X = \{2\}$.

Definition 3.3. The Seidel adjacency matrix $A = (a_{i,j})$ of a graph $\Gamma = (V, E)$ is a $\{0, -1, 1\}$-matrix having:

$$a_{i,j} = \begin{cases} 
0 & i = j \\
-1 & \{i, j\} \in E \\
1 & \{i, j\} \notin E
\end{cases}$$

In this notation, if the graph $\Gamma'$ is obtained from $\Gamma$ by switching with respect to $X \subseteq V$, then its adjacency matrix $A'$ is obtained from $A$ via a similarity transformation by a diagonal matrix having $\{-1, 1\}$ on its diagonal. Explicitly:

$$A' = DAD,$$

here $D_{i,i} = -1 \iff i \in X$.

Some facts about switching:

- Switching is an equivalence relation.
- Switching equivalent graphs have the same Seidel spectrum.
- Switching does not change the parity of the number of edges among any 3 vertices.

Lemma 3.1. For any graph with 4 vertices, the number of induced subgraphs on 3 vertices having an odd number of edges is even ($\in \{0, 2, 4\}$).

As a corollary from the above facts, Construction 1 indeed provides two-graphs. Actually, this is not just a particular construction, any two-graph can be obtained in this way:

Theorem 3.2. There is a 1-1 correspondence between two-graphs and switching classes of graphs.

Cohomological definition

A two-graph can be interpreted as a function from $X^{(3)}$ into $\mathbb{Z}_2$ (the indicating function of $\Delta$). A more convenient notation for later use is to interpret it multiplicatively, i.e., as a function into $U_2 = \{-1, 1\}$, the group of square roots of unity:

Definition 3.4. A two-graph $(X, \Delta)$ is a function:

$$f : X^{(3)} \rightarrow U_2$$
such that $f(x) = -1 \iff x \in \Delta$, satisfying:

$$
f(\{x, y, z\}) \cdot f(\{x, y, t\}) \cdot f(\{x, z, t\}) \cdot f(\{y, z, t\}) = 1$$

for any $\{x, y, z, t\} \in X^{(4)}$.

- A $p$-cochain into $U_2$ is a function $f : X^{(p)} \rightarrow U_2$.
- The coboundary operator $\delta$ is a function from the set of $p$-cochains to the set of $(p + 1)$-cochains defined by:

$$
\delta f(\{x_1, \ldots, x_{p+1}\}) = \prod_{i=1}^{p+1} \sigma^i(f(\hat{x}_i))
$$

where $\sigma$ is the inverse operation in $U_2$, and $\hat{x}_i = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{p+1}\}$.
- A $p$-cochain with vanishing coboundary (this means that $\delta f = 1$) is called a $p$-cocycle.
- Every $p$-cocycle is the coboundary of a $(p - 1)$-cochain.

In the above terminology:

- Two-graphs are 3-cocycles (into $U_2$).
- Every 3-cocycle is the coboundary of a 2-cochain (which is a graph).
- Two 2-cochains have the same coboundary if and only if their quotient is a 1-cochain (which is a function from $X$ to $U_2$).
- Thus the cohomology classes of 2-cochains are the switching classes of the corresponding graphs.

**Regular two-graphs**

**Definition 3.5.** A two-graph $(X, \Delta)$ is called regular if every 2-subset $\{x, y\} \in X^{(2)}$ is contained in the same number of triples from $\Delta$.

**Theorem 3.3.** A two-graph is regular if and only if it has two distinct (Seidel) eigenvalues $\rho_1 > 0 > \rho_2$, such that $\rho_1 \rho_2 = 1 - |X|$.

As was described in the introduction, two-graphs were introduced by G. Higman to study 2-transitive representations of certain sporadic groups, in his description he used antipodal 2-fold covers of complete graphs. Such a cover, when it is also distance regular is called a Taylor graph, these are distance regular graphs with intersection array

$$
\{k, \mu, 1; 1, \mu, k\}.
$$
In [45], Taylor and Levingston show that there is a one-to-one correspondence between two-graphs and antipodal 2-fold covers of complete graphs. In particular, Taylor graphs correspond to regular two-graphs.

This will be a particular case of our more general result in Chapter 4.

An alternative definition of a regular two-graph is: a two-graph \((X, \Delta)\) which is also a 2-design. This definition stems from the problem of extension of graphs (see [10], [29] and E. Shult’s paper [38]).

### 3.2 Two ways to generalize two-graphs

Considering the cohomological definition of two-graphs, two very natural generalizations arise:

- \(t\)-cocycles into \(U_2\).
- \(3\)-cocycles into \(U_t\) (here \(U_t\) is the group of \(t\)-th roots of unity).

Historically, the first of these was indeed the first to be considered. The first appearance of the term \(t\)-graph as a \(t\)-cocycle over \(U_2\) is in D. G. Higman’s generalization of E. Shult’s graph extension theorem (see [19]). Other sources on this (design theoretical) generalization can be found in Mielants [35] or in [10].

In this case, a regular \(t\)-graph is a \(t\)-cocycle into \(U_2\) which is also a \(t\)-design. Here just few examples are known: regular 3-graphs on 8 and 12 points and a regular 5-graph on 12 points (see [34]). Our interest in this thesis is the second way to generalize two-graphs, i.e. 3-cocycles into \(U_t\). This direction was examined by D. G. Higman, and the main source of this is [22]. In the next section we present this notion, focusing on the first step away from two-graphs, that is, \(t\)-graphs with \(t = 3\).

### 3.3 Regular 3-graphs in D. G. Higman’s sense

Following [22], we give the definitions leading up to the definition of a \(t\)-graph.

**Cohomology**

Let \(X\) be a finite set with \(|X| = n\). Let \((U, \cdot)\) be a commutative group, and define a structure of a monoid on the set \(U \cup \{0\}\) in a natural way by defining \(0 \cdot a = a \cdot 0 = 0\) for all \(a \in U \cup \{0\}\). Let \(M \subseteq U \cup \{0\}\) be such that \(M \cap U\) is a subgroup of \(U\). Write \(\sigma(u) = u^{-1}\) for \(u \in U\) and \(\sigma(0) = 0\). The support of a function \(f : X^p \rightarrow M\) is \(\text{supp}(f) = \{x \in X^p | f(x) \neq 0\}\). \(f\) is called a \(p\)-cochain if:

(i) \(\text{diag}X^p \subseteq \text{supp}(f)\) (here \(\text{diag}X^p = \{(x, \ldots, x) \in X^p | x \in X\}\)),

\(\text{supp}(f)\) denotes the support of the function \(f\), which is the set of all points in \(X^p\) where \(f\) is not equal to 0. The \(\text{diag}X^p\) is the diagonal of \(X^p\), i.e., the set of all points of the form \((x, \ldots, x)\) for \(x \in X\).
(ii) \( f(x) = 1 \) for all \( x \in \text{supp}(f) \) such that \( x_i = x_j \) for some \( i \neq j \),

(iii) if \( y \) results from \( x \) by interchanging \( x_i \) and \( x_j \) for some \( i \neq j \) then \( f(y) = \sigma(f(x)) \).

The set of all \( p \)-cochains together with pointwise multiplication forms a monoid denoted by \( C_p(X, M) \) (the dot in the subscript denotes the operation in the monoid \( U \cup \{0\} \)). Define the coboundary operator:

\[
\delta : C_p(X, M) \rightarrow C_{p+1}(X, M),
\]

by

\[
\delta f(x) = \prod_{i=0}^{p} (\sigma^i f)(\hat{x}_i),
\]

where \( \hat{x}_i \in X^p \) is obtained from \( x \in X^{p+1} \) by deleting the \( i^{th} \) coordinate. (convention: \( C_0(X, M) = M \) and \( \delta a(x) = a \) for \( a \in M \) and \( x \in X \)). For \( e \in X \) and \( p \geq 1 \) we have the monoid homomorphism

\[
\Delta_e : C^p(X, M) \rightarrow C^{p-1}(X, M)
\]

defined by

\[
\Delta_e f(x) = f(e, x)
\]

for \( x \in X^{p-1} \). The \( p \)-cochains \( f \) with full support (i.e. \( \text{supp}(f) = X^p \)) form a subgroup \( C^p(X, M \cap U) \) of \( C^p(X, M) \). Define the set of \( p \)-coboundaries:

\[
B_p(X, M) = \{ \delta f \mid f \in C^{p-1}(X, M) \},
\]

and the set of \( p \)-cocycles:

\[
Z_p(X, M) = \{ f \in C^p(X, M) \mid \delta f = 1 \}.
\]

A few basic observations:

- \( B^p(X, M \cap U) = B^p(X, M) \cap C^p(X, M \cap U) \).
- \( Z^p(X, M) = Z^p(X, M \cap U) \).
- \( \delta^2 f(x) = 0 \) or \( 1 \) for all \( x = (x_0, x_1, ..., x_{p+1}) \in X^{p+2} \) and \( \delta^2 f(x) = 1 \) if and only if \( (x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{p+1}) \in \text{supp}(f) \) for all \( i \neq j \).
- For \( p \geq 2 \), \( f(x) = [\delta \Delta_e f][\Delta_e \delta f](x) \) if \( x = (x_1, ..., x_p) \in \text{supp}(f) \) and \( (e, x_1, ..., \hat{x}_i, ..., x_p) \in \text{supp}(f) \) for all \( i \).
- \( Z^p(X, M) = B^p(X, M \cap U) \) for \( p \geq 2 \).
Weights and switching

- A weight on $X$ with values in $M$ is a 2-cochain $w \in C^2(X, M)$.

- The cohomology class of $w$ is $\{ w \cdot \delta u \mid u \in C^1(X, M \cap U) \}$ (here $w \cdot \delta u$ is the pointwise multiplication of functions). As a generalization of the notion of switching of graphs, we call it the switching class of $w$. Thus $B_1^1(X, M \cap U)$ is the switching class of the trivial weight $w = 1$.

- We can view a weight $w$ as an $|X| \times |X|$ matrix with entries in $M$. Then $w$ is $\sigma$-unitary (i.e. $(\sigma(w))^T = w$) and has unit diagonal. The weight $w \cdot \delta u$ as a matrix is obtained from $w$ by similarity transformations by the diagonal matrix with diagonal entries $u(x)$.

$t$-graphs and regular $t$-graphs

Let $U_t = \langle \zeta \rangle$ be the group of $t$-th roots of unity. $\zeta$ is a primitive $t$-th root of unity. With all the notions from above we define:

Definition 3.6. A $t$-graph $\Phi$ is a 3-cocycle into $U_t$. In the above notation: $\Phi \in Z^3(X, U_t)$.

Every 3-cocycle into $U_t$ is the common coboundary of switching equivalent weights into $U_t$, thus $t$-graphs correspond to switching classes of weights into $U_t$.

Definition 3.7. A $t$-graph is called regular if for every pair $x, y \in X$, the number of $z \in X$ such that $f(x, y, z) = \alpha$ is constant. This number is denoted $m(\alpha)$.

Remark 3.8. It is routine to check that in the case $t = 2$, the above definitions give two-graphs and regular two-graphs.

3.4 Elements of Higman’s theory

t-graphs as $t$-fold covers of $K_n$

Let $\Phi$ be a $t$-graph. Let $w : X^2 \rightarrow U_t$ be a weight on $X$ with values in $U_t$ such that $\Phi = \delta w$. Define a graph $\Gamma_w = (V, E)$, where:

- $V = X \times \{1, 2, \ldots, t\}$. For convenience, we denote the vertex $(x, i)$ by $x_i$.
- $\{x_i, y_j\} \in E \iff w(x, y) = \zeta^{j-i}$

The resulting graph is a $t$-fold cover of the complete graph $K_n$ ($n = |X|$), and if $w(x, y) = \zeta^i$ then the set of edges between $x_1, x_2, \ldots, x_t$ and $y_1, y_2, \ldots, y_t$ form a perfect matching which is given by the $i^{th}$ power of the permutation matrix of $(1, 2, \ldots, t)$. For example, when $i = 1$, see Figure 3.1.
Permuting $x_1, x_2, ..., x_t$ according to some power of $(1, 2, ..., t)$ amounts to a change of $w$ in its switching class.

From this point, we focus our attention on regular 3-graphs; $\zeta$ is a cube root of unity, and $U_3 = \langle \zeta \rangle$.

**Regular 3-graphs as 3-fold covers of $K_n$**

Let $w : X^2 \to U_3$ be a weight on $X$. The coboundary $\delta w$ of $w$ is a 3-graph $\Phi \in Z^3(X, U_3)$ on $X$. Assume that $\Phi$ is regular, this means that for every pair $x, y$ of distinct elements of $X$ and $\alpha \in U_3$, the number $m(\alpha)$ of $z \in X \setminus \{x, y\}$ such that $\Phi(x, y, z) = \alpha$ is independent of the choice of $x$ and $y$. Then by the definition of a 3-cochain we have:

$$m(\zeta) = m(\zeta^2).$$

Denote

$$a := m(1),$$
$$b := m(\zeta) = m(\zeta^2).$$

By fixing two elements $x, y \in X$, we may count all the other elements of $X$:

$$|X \setminus \{x, y\}| = \left|\{z \in X \mid \Phi(x, y, z) = 1\}\right| +$$
$$+ \left|\{z \in X \mid \Phi(x, y, z) = \zeta\}\right| + \left|\{z \in X \mid \Phi(x, y, z) = \zeta^2\}\right|.$$

Thus, we have:

$$n = a + 2b + 2.$$

The corresponding graph $\Gamma_w$ is a 3-fold cover of $K_n$ with exactly 3 types of matchings between fibers:
Remark 3.9. This description arose earlier in a different language in the context of Godsil-Hensel theory (Definition 2.1). In the case of regular 3-graphs, the dictionary between the two languages is given by the isomorphism $\zeta \mapsto (1,2,3)$ of $U_3 \cong C_3$.

Clearly, every such graph defines a weight on $X$ and switching corresponds to cyclic permutation of vertices in the fibers. In particular, non-isomorphic such covers define non-switching equivalent weights, or in other words different regular 3-graphs.

Regular 3-graphs and association schemes

The following are corrected constructions and results of D. G. Higman in [23]. Let $\Gamma := \Gamma_w = (V,E)$ be the graph defined by a weight $w$, such that $\Phi = \delta w$ is regular (cf. section 3.4).

Construction 2. Define symmetric binary relations $\{S_i\}_{i=0}^3$ partitioning $V^2$ as follows:

\begin{align*}
S_0 &= \text{Id}_V, \\
S_1 &= \{(x_i,x_j) \mid i \neq j, x \in X\}, \\
S_2 &= E, \\
S_3 &= V^2 \setminus \{S_0 \cup S_1 \cup S_2\}.
\end{align*}

Proposition 3.4 (Higman). $A_4(\Gamma) := (V,\{S_i\}_{i=0}^3)$ is a symmetric (and thus commutative) association scheme.

Proof. We compute the intersection matrices of $A_4(\Gamma)$. For example, we compute $p_{22}^2$; let $(x_i,y_j) \in S_2$. We may assume that

\[ w(x_i,y_j) = \zeta, \]

otherwise, we take a different $w'$ in the switching class of $w$, such that

\[ w'(x_i,y_j) = \zeta. \]
3.4. ELEMENTS OF HIGMAN’S THEORY

Thus \( j = i + 1 \) (mod 3). We count the number of \( z_k \in V \) such that \((x_i, z_k) \in S_2 \) and \((z_k, y_j) \in S_2 \); there are 3 types of \( z \in X \) which contain such a \( z_k \):

- \( k = i \implies w(x, z) = 1 \)
  \( w(z, y) = \zeta \implies \delta w(x, y, z) = \zeta \cdot 1 \cdot \zeta^2 = 1 \)

- \( k = i + 1 \) (mod 3) \implies w(x, z) = \zeta \)
  \( w(z, y) = 1 \implies \delta w(x, y, z) = \zeta \cdot \zeta^2 \cdot 1 = 1 \)

- \( k = i - 1 \) (mod 3) \implies w(x, z) = \zeta^2 \)
  \( w(z, y) = \zeta^2 \implies \delta w(x, y, z) = \zeta \cdot \zeta \cdot \zeta = 1 \)

Thus \( \delta w(x, y, z) = 1 \iff z \) is one of the above 3 types, hence \( p^2_{22} = a \).

In the same manner we obtain the intersection matrices \( \{B_i\}_{i=0}^3 \):

\[
B_0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 2 & 1
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & n-1 & 0 & a \\
0 & n-1 & 2b & a + b
\end{bmatrix},
B_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 2 & 1 \\
0 & n-1 & 2b & a + b \\
2(n-1) & n-1 & 2(a+b) & a + 3b
\end{bmatrix}.
\]

The association scheme \( A_4(\Gamma) \) admits a coherent rank 6 refinement:

**Construction 3.** Define:

\[
R_0 = \text{Id}_V,
R_1 = \left\{(x_i, x_{i+1} \mod 3) \mid i = 1, 2, 3, x \in X\right\},
R_2 = \left\{(x_i, x_{i+2} \mod 3) \mid i = 1, 2, 3, x \in X\right\},
R_3 = E,
R_4 = \left\{(x_i, y_j) \mid i = 1, 2, 3, \{x_{i+1} \mod 3, y_j\} \in E\right\},
R_5 = \left\{(x_i, y_j) \mid i = 1, 2, 3, \{x_{i+2} \mod 3, y_j\} \in E\right\}.
\]

**Remark 3.10.** Notice that the relations \( R_1, R_2, R_4, R_5 \) are anti-symmetric and that \( R_1 = R_2^t \) and \( R_4 = R_5^t \).

**Proposition 3.5** (Higman). \( A_6(\Gamma) := (V, \{R_i\}_{i=0}^5) \) is a commutative association scheme.
Proof. We compute the intersection matrices of $A_6(\Gamma)$. For example, we compute $p^4_{14}$; let $(x_i, y_j) \in R_4$. As before, we may assume that
\[ w(x_i, y_j) = \zeta, \]
otherwise we can take a switching equivalent $w'$, thus
\[ j = i + 1 \pmod{3}. \]

We count the number of $z_k \in V$ such that $(x_i, z_k) \in R_4$ and $(z_k, y_j) \in R_4$; there are 3 types of $z \in X$ which contain such a $z_k$:

- $k = i \implies w(x, z) = \zeta^2 \implies \delta w(x, y, z) = \zeta \cdot \zeta \cdot \zeta^2 = \zeta$
- $k = i + 1 \pmod{3} \implies w(x, z) = 1 \implies \delta w(x, y, z) = \zeta \cdot 1 \cdot 1 = \zeta$
- $k = i - 1 \pmod{3} \implies w(x, z) = \zeta \implies \delta w(x, y, z) = \zeta \cdot \zeta^2 \cdot \zeta = \zeta$

Thus $\delta w(x, y, z) = \zeta \iff z$ is one of the above 3 types, hence $p^4_{14} = b$.

In the same manner we obtain the intersection matrices $\{B_i\}_{i=0}^5$:

\[
B_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, 
B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, 
B_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
B_3 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}, 
B_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}, 
B_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
The following proposition implies that this is a sufficient condition:

**Proposition 3.6 (Higman).** Every rank 6 association scheme with parameters as in Construction 3 arises from a regular 3-graph.

**Proof.** Let \((V, \{R_i\}_{i=0}^5)\) be an association scheme with parameters (and notation) as in 3. Define \(T = R_0 \cup R_1 \cup R_2\), then \(T\) is an equivalence relation on \(V\) with equivalence classes of size 3. Denote \(X = V/T\). We give a labeling of the elements of \(V\) as a 3-fold cover of \(K_{|X|}\), and then we verify that in this cover we only have matchings of the 3 types shown in section 3.4. Let \(a \in X\) be any fiber, and label its elements by \(a_1, a_2, a_3\) so that \((a_1, a_2) \in R_1\). Then 
\[
(a_i, a_{i+1} \pmod{3}) \in R_1 
\]
for \(i = 1, 2, 3\). We now label the elements of each fiber \(x \neq a\) in \(X\) by \(x_1, x_2, x_3\) so that 
\[
(a_i, x_i) \in R_3
\]
for \(i = 1, 2, 3\). To prove that \((V, R_3)\) is a 3-fold cover of \(K_{|X|}\) with matchings of the 3 permitted types, we prove two things:

1. \((x_i, x_{i+1} \pmod{3}) \in R_1\) for all \(x \in X\) and \(i = 1, 2, 3\),
2. there is no matching such that \((x_i, y_j) \in R_3\) and \((x_j, y_i) \in R_3\), where \(i \neq j\).

**Proof of (1):** Assume that \((x_i, x_{i+1}) \in R_2\). Let \(k\) be such that \((a_i, x_{i+1}) \in R_k\). Then we have:
\[
(x_i, a_i) \in R_3, \quad (x_i, x_{i+1}) \in R_2, \quad (x_{i+1}, a_i) \in R_{k'}.
\]
Therefore:
\[
p_{3k'}^3 \neq 0.
\]
Examining column 3 in the matrix \(B_2\), we deduce that \(k' = 4\), which means that \(k = 5\). Also, we have:
\[
(a_i, x_{i+1}) \in R_k, \quad (a_i, a_{i+1}) \in R_1, \quad (a_{i+1}, x_{i+1}) \in R_3.
\]
This implies that:
\[
p_{13}^k \neq 0.
\]
Examining row 3 in the matrix \(B_1\), we deduce that \(k = 4\), which is a contradiction.

**Proof of (2):** Assume that \((x_i, y_j) \in R_3\) and \((x_j, y_i) \in R_3\) for some \(i \neq j\). Let \(k\) be such that \((x_i, y_i) \in R_k\). W.l.o.g we may assume that \(j = i + 1 \pmod{3}\). Then we have:
\[
(x_i, y_i) \in R_k, \quad (x_i, x_{i+1}) \in R_1, \quad (x_{i+1}, y_i) \in R_3.
\]
Therefore:

\[ p_{13}^{k} \neq 0. \]

Examining row 3 in the matrix \( B_1 \) we see \( k = 4 \). Also, we have:

\( (y_i, x_i) \in R_{k'}, (y_i, y_{i+1}) \in R_1, (y_{i+1}, x_i) \in R_3. \)

Thus we obtain:

\[ p_{13}^{k'} \neq 0, \]

which implies that \( k' = 4 \) and \( k = 5 \), a contradiction.

It follows that all the matchings of the graph \( (V, R_3) \) are of the 3 types shown in section 3.4, and we can define a weight \( w \) on \( X \) by \( w(x, x) = 1 \) and \( w(x, y) = 1, \zeta \) or \( \zeta^2 \) for \( x \neq y \) according to as the matching is of the first, second or third type. It is straightforward to verify that \( \delta w \) is regular.

**Remark 3.11.** Since the relations \( R_0, \ldots, R_5 \) depend only on the graph \( \Gamma \), we may construct them for any antipodal 3-fold cover of \( K_n \), and check if \( A_6(\Gamma) \) is an association scheme (i.e. the intersection numbers exist). This can also be checked using WL-stabilization; explicitly, computing the coherent closure of \( A_0, \ldots, A_5 \) — the adjacency matrices of \( R_0, \ldots, R_5 \).

### 3.5 Feasibility conditions

In this part of Higman’s note there appears the incorrect result concluded from a miscalculation in the matrices \( B_3 \) and \( B_5 \) (compare these matrices from this thesis and from the note [23]). We formulate a corrected version of these results.

We derive feasibility conditions from 2 sources: the character-multiplicity tables of the above constructed association schemes \( A_4(\Gamma) \) and \( A_6(\Gamma) \), and the matrix \( B = w - I \).

The character-multiplicity table of \( A_4(\Gamma) \) is:

\[
\begin{bmatrix}
1 & 2 & n - 1 & 2(n - 1) \\
1 & 2 & -1 & -2 \\
1 & -1 & \alpha & -\alpha \\
1 & -1 & \beta & -\beta
\end{bmatrix}
\begin{bmatrix}
1 \\
n - 1 \\
z_1 \\
z_2
\end{bmatrix}.
\]

Here:

- \( \alpha \) and \( \beta \) are the roots of \( x^2 - (a - b)x - (n - 1) = 0, \)
- \( z_1 = \frac{2n\beta}{\beta - \alpha}, \)
- \( z_2 = 2n - z_1 = \frac{2n\alpha}{\alpha - \beta}. \)
If \( z_1 = z_2 = n \) then we have \( \alpha = -\beta \), and \( \alpha, \beta = \pm \sqrt{n-1} \).

Otherwise, \( z_2 - z_1 \) is a non-zero integer, and we have:

\[
z_2 - z_1 = 2n \left( \frac{\alpha}{\alpha - \beta} \right) = 2n \left( \frac{\alpha + \beta}{\alpha - \beta} \right).
\]

This means that \( \alpha - \beta = \sqrt{(a-b)^2 + 4(n-1)} \) is rational, i.e. \( (a-b)^2 + 4(n-1) \) is a square, which implies that \( \alpha \) and \( \beta \) are rational algebraic integers, and thus are integers.

The character-multiplicity table of \( A_6(\Gamma) \) is:

\[
\begin{bmatrix}
1 & 1 & 1 & n-1 & n-1 & n-1 \\
1 & \zeta & \zeta^2 & \alpha & r & r^* \\
1 & \zeta^2 & \zeta & \alpha & r^* & r \\
1 & 1 & 1 & -1 & -1 & -1 \\
1 & \zeta & \zeta^2 & \beta & t & t^* \\
1 & \zeta^2 & \zeta & \beta & t^* & t \\
\end{bmatrix}
\begin{bmatrix}
1 \\
z_1/2 \\
z_1/2 \\
n-1 \\
z_2/2 \\
z_2/2 \\
\end{bmatrix}
\]

Here:

- \( r \) and \( r^* \) are the roots of \( x^2 + \alpha x + \alpha^2 = 0 \),
- \( t \) and \( t^* \) are the roots of \( x^2 + \beta x + \beta^2 = 0 \).

Remark 3.12. In Higman’s note the following equations appeared:

\[
x^2 - \alpha x + \left( \frac{3(n-1)}{2} + \alpha^2 \right) = 0, \text{ and } x^2 - \beta x + \left( \frac{3(n-1)}{2} + \beta^2 \right) = 0,
\]

which led him to the false conclusion that \( n \) must be odd.

Finally, we consider the matrix \( B = w - I \) (here we think of \( w \) as a matrix, cf. section 3.3). The eigenvalues of \( B \) with their multiplicities do not depend on the choice of \( w \) in its switching class, this is why we call them the eigenvalues and multiplicities of \( \Phi = \delta w \). Notice that

\[
B^2 = (a-b)B + (n-1)I,
\]

thus the eigenvalues of \( B \) are \( \alpha \) and \( \beta \) with multiplicities:

\[
m_\alpha = \frac{n\beta}{\beta - \alpha}, \quad m_\beta = \frac{n\alpha}{\alpha - \beta}.
\]

Altogether, we obtain the restrictions:

Proposition 3.7 (Higman). Necessary conditions for the set \( (n, a, b) \) of parameters of a regular 3-graph are:

(i) \( n = a + 2b + 2 \),
(ii) The roots $\alpha$ and $\beta$ of the equation $x^2 - (a - b)x - (n - 1) = 0$ are integers,

(iii) $\alpha - \beta$ divides $n\alpha$.

Using the above necessary conditions, we obtain a list of 137 feasible parameter sets of regular 3-graphs with $n \leq 1000$. We will improve this result later.

Summary of changes to Higman’s note

Higman had the feasibility condition: $n = a + 2b + 2$ is odd, this was derived from the character-multiplicity table of $A_6(\Gamma)$ which was miscalculated because of an earlier miscalculation of the intersection matrices of $A_6(\Gamma)$ (see section 3.5). Other changes include a more coherent notation, and detailed proofs of the propositions.

Remark 3.13. All the theory developed above for regular 3-graphs can be naturally extended to regular $t$-graphs with $t \geq 4$, only if we have some more restrictions. See more in section 8.

3.6 The symplectic example

In [22], Higman presents several group theoretic examples of regular 3-cocycles. These examples are mainly extensions of examples by D. E. Taylor in [44]. We consider one of them. In this example, we consider weights with values in the additive group of the field $GF(q)$, thus we will use additive notation: $C^2_+ \text{ instead of } C^2_+ \cdot \delta$, $\delta_+ \text{ instead of } \delta \cdot$ etc (cf. section 3.3).

Let $V$ be a $2m$ dimensional vector space over $GF(q)$. Let $B$ be a non-degenerate alternating bilinear form on $V$. Then $B \in C^2_+(V,GF(q))$ is a weight on $V$ with values in $GF(q)$. In case $q$ is a prime, this 3-cocycle is a $q$-graph. To see that in this case $\Phi = \delta_+ B$ is a regular $q$-graph we consider the symplectic group $Sp(2m,q)$. It acts transitively on the non-zero vectors of $V$, thus the subgroup $H := VSp(2m,q)$ of the affine group on $V$ acts 2-transitively on the vectors of $V$. The coboundary $\Phi = \delta_+ B$ is invariant under translations and is therefore invariant under the action of $H$ on $V$. Thus this example provides an infinite family of regular $q$-graphs for every prime $q$. 

Chapter 4

Regular 3-graphs and antipodal covers

In this chapter, my results about the connection between regular 3-graphs and antipodal covers are presented. At the end of this chapter a table of feasible parameter sets with \( n \leq 1000 \), which was obtained by us is included with all previously known examples, and some new examples constructed by us.

4.1 \((n,3,c_2)\)-covers with cyclic matchings

Recall the interpretation of a matching between two fibers \( x, y \in X \) as an element \( f(x, y) \) in \( S_r \) (Definition 2.1). We formulate a property of \((n,r,c_2)\)-covers that is later (in section 4.3) proved to be equivalent to the property of being cyclic (as defined in Definition 2.3).

**Definition 4.1.** Let \( \Gamma = (V,E) \) be a labeled \((n,t,c_2)\)-cover, i.e. such that \( V = X \times \{1,2,\ldots,t\} \). Let \( \sigma = (1,2,\ldots,t) \in S_t \). A cyclic matching in \( \Gamma \) is a matching that corresponds to \( \sigma^i \), for some \( 1 \leq i \leq t \).

**Definition 4.2.** A (not labeled) \((n,t,c_2)\)-cover \( \Gamma = (V,E) \) is said to be with cyclic matchings if there exists a labeling of \( V \) such that:

- all the matchings of \( \Gamma \) are cyclic, and
- every cyclic matching appears.

**Remark 4.3.** Note that in an \((n,t,c_2)\)-cover with cyclic matchings, an edge between two fibers \( x, y \in X \) of \( \Gamma \) determines the matching between them. For \( t = 3 \), if \( \{x_i, y_j\} \in E \) then the only cyclic matching that satisfies this must also satisfy \( \{x_{i-1} \pmod{3}, y_{j-1} \pmod{3}\}, \{x_{i+1} \pmod{3}, y_{j+1} \pmod{3}\} \in E \).
4.2 Regular 3-graphs and \((n, 3, c_2)\)-covers with cyclic matchings

**Proposition 4.1.** Every regular 3-graph with parameters \((n, a, b)\) defines an \((n, 3, b)\)-cover with cyclic matchings.

**Proof.** Let \(\Phi = \delta w\) be a regular 3-graph, let \(\Gamma_w = (V, E)\) be the graph defined by \(w\). Clearly, \(\Gamma_w\) is a 3-fold cover of \(K_n\). To use Lemma 2.5, we need to show that \(c_2\) exists. Let \(x_i\) and \(y_j\) be two non-adjacent vertices from different fibers \(x\) and \(y\) of \(X\). We may assume that \(w(x, y) = \zeta\), otherwise, we take a different weight \(w'\) in the switching class corresponding to \(\Phi\). Thus

\[ j = i \text{ or } j = i - 1. \]

If \(j = i\) then the common neighbors of \(x_i\) and \(y_j\) are \(z_k \in z \in X\) for which

\[ w(x, z) = w(z, y)^{-1}, \]

hence these are exactly the fibers \(z \in X\) such that

\[ \delta w(x, y, z) = w(x, y) \cdot w(x, z)^{-1} \cdot w(y, z) = \zeta. \]

By the regularity of \(\delta w\), the number of such \(z \in X\) is independent of the choice of \(x\) and \(y\), and is equal to \(b = m(\zeta)\).

If \(j = i - 1\) then the common neighbors of \(x_i\) and \(y_j\) are \(z_k \in z \in X\) for which

\[ w(x, z) = w(z, y)^{-1} \cdot \zeta^2, \]

hence these are exactly the fibers \(z \in X\) such that

\[ \delta w(x, y, z) = w(x, y) \cdot w(x, z)^{-1} \cdot w(y, z) = \zeta^2. \]

Again, using the regularity of \(\delta w\), the number of such \(z \in X\) is independent of the choice of \(x\) and \(y\), and is equal to \(b = m(\zeta^2)\). We conclude that any two non-adjacent vertices from different fibers of \(\Gamma_w\) have the same number \(c_2 = b\) of common neighbors. Finally, we apply Lemma 2.5 and obtain that \(\Gamma_w\) is an antipodal distance regular cover of \(K_n\) with parameters \((n, 3, b)\). In addition, by definition, \(\Gamma_w\) has only cyclic matchings and all cyclic matchings (cf. Definition 4.2) appear, hence \(\Gamma_w\) is an \((n, 3, b)\)-cover with cyclic matchings.

**Remark 4.4.** Recall that when \(\Gamma\) is an \((n, 3, c_2)\)-cover, \(a_1\) is the number of common neighbors of two adjacent vertices, and note that when such a graph is defined by a regular 3-graph we have that \(a_1 = a\) (this is proved in a similar manner to the proof of \(c_2 = b\) above).
4.2. REGULAR 3-GRAPHS AND $(N, 3, C_2)$-COVERS WITH CYCLIC MATCHINGS

**Remark 4.5.** Recall Construction 2 of $A_4(\Gamma)$. Notice that $A_4(\Gamma)$ with the ordering $(S_0, S_2, S_3, S_1)$ is the metric association scheme of the ADRG $\Gamma$.

**Proposition 4.2.** Every $(n, 3, c_2)$-cover with cyclic matchings defines a regular 3-graph with parameters $(n, a_1, c_2)$.

*Proof.* Let $\Gamma = (V, E)$ be an $(n, 3, c_2)$-cover with cyclic matchings. We prove that $A_6(\Gamma)$ is an association scheme. It suffices to compute the intersection matrices $B_0, B_1, \ldots, B_5$. We deal with these matrices by dividing each matrix to 4 blocks of size $3 \times 3$ as follows. For $i = 0, 1, \ldots, 5$ denote:

- $B_i^1 :=$ upper left $3 \times 3$ block of $B_i$,
- $B_i^2 :=$ upper right $3 \times 3$ block of $B_i$,
- $B_i^3 :=$ lower left $3 \times 3$ block of $B_i$,
- $B_i^4 :=$ lower right $3 \times 3$ block of $B_i$.

We first notice that since $\Gamma$ is antipodal, the fibers are a partition of $V$, thus if $x, y$ are fibers of $\Gamma$ then either $x = y$ or $x \cap y = \emptyset$. From this we observe that:

- (i) if $(x_i, y_j) \in R_0, R_1$ or $R_2$ then $x = y$,
- (ii) if $(x_i, y_j) \in R_3, R_4$ or $R_5$ then $x \cap y = \emptyset$,

and thus the blocks $B_1^2, B_1^3, B_2^2, B_2^3, B_3^1, B_4^1, B_5^1$ are all zero blocks.

Next, we show that $B_0 = I$; let $(x_i, y_j) \in R_k$ and $(z_k, y_j) \in R_l$. Also, $(x_i, z_k) \in R_0$, thus $x_i = z_k$. We obtain that $(x_i, y_j) \in R_k$ and $(x_i, y_j) \in R_l$, then $(B_0)_{jk} = p_{0j}^k = \delta_{jk}$.

The computation of elements from $B_1^1, B_1^4, B_2^1, B_2^4, B_3^2, B_3^4, B_4^2, B_4^4, B_5^2, B_5^4$ are all of similar flavour, for example we compute $p_{34}^0$; let $(x_i, y_j) \in R_0$, we count the number of $z_k \in V$ such that $(x_i, z_k) \in R_3$ and $(z_k, y_j) \in R_4$. We have:

- $(x_i, y_j) \in R_0 \implies y_j = x_i$
- $(x_i, z_k) \in R_3 \implies \{x_i, z_k\} \in E$
- $(z_k, y_j) \in R_4 \implies \{z_{k+1} \pmod{3}, y_j\} \in E$

which implies $d(z_k, z_{k+1} \pmod{3}) \leq 2$ - a contradiction to the antipodality of $\Gamma$, therefore $p_{34}^0 = 0$.

The computation of elements from $B_3^1, B_4^1, B_5^1$ are all of similar flavour, for example we compute $p_{44}^1$; let $(x_i, y_j) \in R_4$, we count the number of $z_k \in V$ such that $(x_i, z_k) \in R_4$ and $(z_k, y_j) \in R_4$. We have:

- $(x_i, y_j) \in R_4 \implies \{x_{i+1} \pmod{3}, y_j\} \in E$
\[(x_i, z_k) \in R_4 \implies \{x_{i+1} \pmod{3}, z_k\} \in E\]
\[(z_k, y_j) \in R_4 \implies \{z_{k+1} \pmod{3}, y_j\} \in E\]
which imply the following configuration (the \((\text{mod} \ 3)\) in the subscript is omitted):

\[\begin{array}{c}
\text{\(x\)} \\
\text{\(x_{i-1}\)} & \text{\(y_{j-1}\)} & \text{\(y\)} \\
\text{\(x_i\)} & \text{\(y_j\)} & \text{\(y_{j+1}\)} \\
\text{\(x_{i+1}\)} & \text{\(y_{j+1}\)} & \\
\text{\(z_{k-1}\)} & \text{\(z_k\)} & \text{\(z_{k+1}\)} \\
\text{\(z\)}
\end{array}\]

Figure 4.1: \(C_9\) configuration

Thus we have:

\[p^4_{44} = \left|\{z_k \in V \mid d(x_{i+1}, z_k) = 1, d(z_k, y_j) = 2, d(z_k, y_{j-1}) = 1\}\right| =
\left|\{z_k \in V \mid d(x_{i+1}, z_k) = 1, d(z_k, y_j) = 2\}\right| -
\left|\{z_k \in V \mid d(x_{i+1}, z_k) = 1, d(z_k, y_{j+1}) = 1\}\right| .\]

Since \(\Gamma\) is distance-regular and \(d(x_{i+1}, y_j) = 1\) and \(d(x_{i+1}, y_{j+1}) = 2\) we have:

\[\left|\{z_k \in V \mid d(x_{i+1}, z_k) = 1, d(z_k, y_j) = 2\}\right| = b_1,\]
\[\left|\{z_k \in V \mid d(x_{i+1}, z_k) = 1, d(z_k, y_{j+1}) = 1\}\right| = c_2.\]

Thus:

\[p^4_{44} = b_1 - c_2.\]

Recall that since \(\Gamma\) is an \((n, 3, c_2)\)-cover we have \(b_1 = 2c_2\), thus:

\[p^4_{44} = c_2.\]

The computation of the last three blocks led to a deeper understanding of what is going on, and to a more elegant proof of Proposition 4.2. For this purpose we start by understanding the structure of \(\Gamma\mid_{\{x, y, z\}}\), the graph induced on three distinct fibers \(x, y, z \in X\) of \(\Gamma\).
Lemma 4.3. Let $\Gamma = (V, E)$ be an $(n, 3, C_2)$-cover with cyclic matchings, and let $X$ be the set of fibers of $\Gamma$. Then for any $x, y, z \in X$ we have:

- (1) $\Gamma\{|x,y,z\} \cong 3 \circ C_3$ or $\Gamma\{|x,y,z\} \cong C_9$,

- (2) For any two fibers $x, y \in X$ the number of $z \in X$ such that $\Gamma\{|x,y,z\} \cong 3 \circ C_3$ is $a_1$,

- (3) For any two fibers $x, y \in X$ the number of $z \in X$ such that $\Gamma\{|x,y,z\} \cong C_9$ is $2c_2$.

Proof. (1) Let $1 \leq i, j, k \leq 3$ be such that $\{x_i, y_j\}, \{x_i, z_k\} \in E$. There are two options:

- $\{z_k, y_j\} \in E$, Using Remark 4.3, we obtain that in this case
  $\Gamma\{|x,y,z\} \cong 3 \circ C_3$.

- $\{z_k, y_j\} \notin E$. Then $\{z_k, y_{j-1 (\mod 3)}\} \in E$ or $\{z_k, y_{j+1 (\mod 3)}\} \in E$, in each of these cases Remark 4.3 implies that $\Gamma\{|x,y,z\} \cong C_9$.

(2) Let $x, y \in X$ be two fibers of $\Gamma$. Let $z \in X$ be such that $\Gamma\{|x,y,z\} \cong 3 \circ C_3$, then there are $1 \leq i, j, k \leq 3$ such that $\{x_i, y_j\}, \{x_i, z_k\}, \{z_k, y_j\} \in E$.

![Figure 4.2: The case when $\Gamma\{|x,y,z\} \cong 3 \circ C_3$](image)

Therefore:

$$|\{z \in X \mid \Gamma\{|x,y,z\} \cong 3 \circ C_3}\}| =$$

$$= |\{z_k \in V \mid d(x_i, z_k) = 1, d(z_k, y_j) = 1, d(x_i, y_j) = 1\}| =$$

Since $\Gamma$ is distance-regular and $d(x_i, y_j) = 1$ we have:

$$= |\{z_k \in V \mid d(x_i, z_k) = 1, d(z_k, y_j) = 1\}| = a_1.$$

(3) Let $x, y \in X$ be two fibers of $\Gamma$. Let $z \in X$ be such that $\Gamma\{|x,y,z\} \cong C_9$, then there are $1 \leq i, j, k \leq 3$ such that $\{x_i, y_j\}, \{x_i, z_k\} \in E$ and $\{z_k, y_j\} \notin$
E. Since \( z \neq y \) we have \( d(z_k, y_j) = 2 \) then either \( d(z_k, y_{j+1} \text{ (mod 3)}) = 1 \) or \( d(z_k, y_{j-1} \text{ (mod 3)}) = 1 \).

Therefore:

\[
\left| \{ z \in X \mid \Gamma|_{\{x,y,z\}} \cong C_9 \} \right| = \\
= \left| \{ z_k \in V \mid d(x_i, z_k) = 1, d(z_k, y_{j+1} \text{ (mod 3)}) = 1, d(x_i, y_j) = 1 \} \right| + \\
+ \left| \{ z_k \in V \mid d(x_i, z_k) = 1, d(z_k, y_{j-1} \text{ (mod 3)}) = 1, d(x_i, y_j) = 1 \} \right| \\
= c_2 + c_2 = 2c_2.
\]

Now we are ready to give a nicer proof of Proposition 4.2:

**Proof.** Let \( w_\Gamma \) be the weight on \( X \) defined by \( \Gamma \). To show that \( \delta w_\Gamma \) is regular we need to show that the numbers \( a = m(1), b = m(\zeta) = m(\zeta^2) \) exist. Recall that:

\[
m(1) = |\{ z \in X \mid \delta w_\Gamma(x, y, z) = 1 \}|, \\
m(\zeta) = |\{ z \in X \mid \delta w_\Gamma(x, y, z) = \zeta \}|, \\
m(\zeta^2) = |\{ z \in X \mid \delta w_\Gamma(x, y, z) = \zeta^2 \}|.
\]

Let \( x, y \in X \). Then:

- If \( \delta w_\Gamma(x, y, z) = 1 \), then \( \Gamma|_{\{x,y,z\}} \cong 3 \circ C_3 \) and by Lemma 4.3, in this case the number of \( z \in X \) such that \( \Gamma|_{\{x,y,z\}} \cong 3 \circ C_3 \) is \( a_1 \), and so \( a = m(1) = a_1 \).
4.3 Cyclic covers and \((N, 3, C_2)\)-covers with cyclic matchings

- If \(\delta w_\Gamma(x, y, z) = \zeta\) or \(\delta w_\Gamma(x, y, z) = \zeta^2\), then \(\Gamma_{\{x, y, z\}} \cong C_9\). As we have seen in the proof of Lemma 4.3, for three fibers \(x, y, z \in X\) such that \(\Gamma_{\{x, y, z\}} \cong C_9\) there are two options: either \(d(z_k, y_{j+1} \pmod{3}) = 1\) or \(d(z_k, y_{j-1} \pmod{3}) = 1\), one of which corresponds to \(z \in X\) such that \(\delta w_\Gamma(x, y, z) = \zeta\), and the other corresponds to \(z \in X\) such that \(\delta w_\Gamma(x, y, z) = \zeta^2\). Therefore \(b = m(\zeta) = m(\zeta^2) = c_2\).

\[\blacksquare\]

4.3 Cyclic covers and \((n, 3, c_2)\)-covers with cyclic matchings

To complete the picture, we use Lemma 2.6 to prove:

**Proposition 4.4.** Every \((n, 3, c_2)\)-cover with cyclic matchings is a cyclic cover.

**Proof.** From the cyclic matchings property we have \(\langle f \rangle = C_3 = \langle (1, 2, 3) \rangle\). By Lemma 2.6 the voltage group

\[T \cong C_{S_3}(C_3) = C_3,\]

hence, this is a cyclic cover. \[\blacksquare\]

By Theorem 2.7 we obtain a further restriction on the feasible parameters of regular 3-graphs:

**Corollary 4.5.** If \((n, a, b)\) are the parameters of a regular 3-graph then 3 \mid n. \[\blacksquare\]

Combining Proposition 4.2, Proposition 4.4 and the sufficient condition in Proposition 3.6, we formulate the following characterization of regular 3-graphs in our new language:

**Corollary 4.6.** Let \(\Gamma\) be a \((n, 3, c_2)\)-cover. The following are equivalent:

1. \(\Gamma\) defines a regular 3-graph.
2. \(\Gamma\) has cyclic matchings.
3. \(\Gamma\) is a cyclic cover.
4. \(A_6(\Gamma)\) is an association scheme. \[\blacksquare\]
4.4 The list of feasible parameter sets revisited

Having obtained a further restriction on the set of feasible parameters of a regular 3-graph in Corollary 4.5, we upgrade Higman’s list of necessary conditions:

**Proposition 4.7.** Necessary conditions for the set \((n, a, b)\) of parameters of a regular 3-graph are:

(i) \(n = a + 2b + 2\),

(ii) \(3|n\),

(iii) The roots \(\alpha\) and \(\beta\) of the equation \(x^2 - (a - b)x - (n - 1) = 0\) are integers,

(iv) \(\alpha - \beta\) divides \(n\alpha\).

This narrows down the list of 137 sets of feasible parameters of regular 3-graphs with \(n \leq 1000\) obtained earlier to a list of 64. I used Mathematica (see section A.1) to generate this list. In Table 4.1 all feasible parameter sets are presented, and for the cases that we have constructed, some additional information is added.

<table>
<thead>
<tr>
<th>Parameters of the regular 3-graph ((n, a, b))</th>
<th>Intersection array of the corresponding ADRG ({b_0, b_1, b_2; c_1, c_2, c_3})</th>
<th># Previously known examples</th>
<th># New examples</th>
<th>Description</th>
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<td>The symplectic example. See subsection 6.2.1 and section 7.1.</td>
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<td>See subsection 6.2.2 and section 7.2</td>
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4.4. THE LIST OF FEASIBLE PARAMETER SETS REVISITED

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<th>Parameters of the regular 3-graph $(n,a,b)$</th>
<th>Intersection array of the corresponding ADRG ${b_0,b_1,b_2; c_1, c_2, c_3}$</th>
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<th># New examples</th>
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### Parameters of the regular 3-graph $(n, a, b)$

<table>
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<th>Intersection array of the corresponding ADRG ${b_0, b_1, b_2; c_1, c_2, c_3}$</th>
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Table 4.1: Feasible parameters of regular 3-graphs
Chapter 5

From symmetric transversal designs to antipodal covers

5.1 Generalized Hadamard matrices, STD’s, and antipodal covers

In this chapter, we begin by presenting elements of the Klin-Pech theory of constructing regular antipodal covers of complete graphs. The essence of this theory is the characterization of such ADRG’s in the language of Godsil-Hensel matrices (see section 2.2) and their connection with skew generalized Hadamard matrices. In the core of this theory lies a construction of skew generalized Hadamard matrices of large size from any general such matrix of smaller size. From another perspective, generalized Hadamard matrices are equivalent to class-regular symmetric transversal designs. This fact led us to the investigation of links between symmetric transversal designs and regular antipodal distance regular covers of complete graphs. We investigate these connections using the flag algebra of a design and the metric association scheme of a suitable ADRG. A crucial example in this investigation is the new ADRG with 108 vertices (a (36, 3, 12)-cover) discovered by Klin and Pech in [26]. We find some interesting features of this example, hoping that in the future they will be supported by a general theory that we are now developing.

5.2 Elements of Klin-Pech theory

Recall the definitions of a covering matrix and Godsil-Hensel matrix given in section 2.2. The following (Lemma 5.3 in [26]) is an alternative way to characterize Godsil-Hensel matrices in algebraic terms:
**Lemma 5.1.** Let $T$ be a finite group and let $A \in \mathbb{Z}[T]^{n \times n}$ be a covering matrix with $\Delta_A = K_n$. Then $\Gamma^A$ is distance regular if and only if there exist numbers $r$ and $c_2$ such that

$$(A + I)^2 = nI + (n - rc_2)A + c_2T(J - I).$$

From this formulation of the criterion, the connection of regular $(n, r, c_2)$-covers with generalized Hadamard matrices becomes evident. In particular, if $n - rc_2 = 0$ then $A + I$ must be a $gH(T, n)$. We obtain the following characterization of regular $(n, r, c_2)$-covers with $n - rc_2 = 0$.

**Corollary 5.2.** Let $T$ be a finite group with neutral element $e$, and let $A \in \mathbb{Z}[T]^{n \times n}$ be a covering matrix with $\Delta_A = K_n$. Then the graph $\Gamma^A$ is a regular antipodal distance regular cover of $K_n$ with $n - rc_2 = 0$ if and only if $(A + I)$ is a self-adjoint $gH(T, n)$ with diagonal entries all equal to $e$.

**Remark 5.1.**

(i) Self-adjoint generalized Hadamard matrices with identity diagonal are sometimes called *skew*. We will furthermore adopt this name.

(ii) If $T = \{\{+1, -1\}, \cdot\}$ is the cyclic group of order 2, then generalized Hadamard matrices over $T$ coincide with classical Hadamard matrices and skew generalized Hadamard matrices over $T$ coincide with symmetric Hadamard matrices with identity diagonal. Moreover, all distance regular double covers of $K_n$ are regular covers. Hence, we determine that distance regular antipodal double covers of $K_n$ with $\delta = -2$ are equivalent to symmetric Hadamard matrices with identity diagonal.

(iii) As described in section 3.1, distance regular double covers of $K_n$ are also known as Taylor graphs, and there is a one-to-one correspondence between Taylor graphs and regular two-graphs. Thus symmetric Hadamard matrices with identity diagonal give a rich source of regular two-graphs.

(iv) The parameters of the regular cover obtained from a skew $gH(T, n)$ are

$$ (n, r, c_2) = (n, |T|, n/|T|). $$

The following theorem is a result from [26], in which Klin and Pech describe a method for obtaining a skew $gH(T, n^2)$ from any $gH(T, n)$.

**Theorem 5.3.** Let $T$ be a finite abelian group and let $H = (h_{i,j})$ be any $gH(T, n)$. Let $\psi : \{1, 2, \ldots, n\}^2 \to \{1, 2, \ldots, n^2\}$ be any bijection. Define the matrix $R_H = (r_{(i,j)\psi, (k,l)\psi})$ of order $n^2$ over $T$ according to

$$ r_{(i,j)\psi, (k,l)\psi} := h_{k,j} \cdot h^{-1}_{i,l}. $$

Then $R_H$ is a skew $gH(T, n^2)$. 

Corollary 5.4. For a finite abelian group $T$, if there is a $gH(T,n)$, then for all $t \in \mathbb{N} \setminus \{0\}$ there is a skew $gH(T,n^2)$.

Remark 5.2. The condition in Theorem 5.3 that $T$ be abelian can be dropped if instead we assume that $H^T$ is also a generalized Hadamard matrix. This is equivalent to the assumption that the element wise conjugation $H^-$ of $H$ is a generalized Hadamard matrix.

Remark 5.3. In the construction of $R_H$ from $H$, the choice of the bijection $\psi$ is not of importance, because different choices will give equivalent skew generalized Hadamard matrices that give rise to isomorphic covers of $K_n$. So further on it will be convenient to assume that in the construction, $\psi$ is taken such that $(i,j)^\psi = (i-1)n+j$, where $n$ is the order of $H$.

An alternative description of the construction using block matrices is as follows:

Let $H$ be a generalized Hadamard matrix $gH(T,n)$ such that $H^-$ is also a generalized Hadamard matrix. Take $\varphi : \{1,2,\ldots,n\}^2 \rightarrow \{1,2,\ldots,n^2\}$, defined by $(k,l) \mapsto (l-1)n+k$. Moreover, take $\psi$ as defined previously. Now define the matrix $R'_H = (r_{(i,j)^\psi,(k,l)^\varphi})$ of order $n^2$. A close inspection reveals that the thus defined matrix is of block-shape $R'_H = H^T \otimes H^-$. More concretely

$$R'_H = \begin{pmatrix}
H^T \cdot h_{1,1}^{-1} & H^T \cdot h_{2,1}^{-1} & \cdots & H^T \cdot h_{n,1}^{-1} \\
H^T \cdot h_{1,2}^{-1} & H^T \cdot h_{2,2}^{-1} & \cdots & H^T \cdot h_{n,2}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
H^T \cdot h_{1,n}^{-1} & H^T \cdot h_{2,n}^{-1} & \cdots & H^T \cdot h_{n,n}^{-1}
\end{pmatrix}.$$  

If we permute the columns of $R'_H$ by the permutation $\varphi^{-1}\psi$, then we obtain $R_H$.

5.3 Class-regular STD’s via generalized Hadamard matrices

To present the connection between class-regular STD’s and generalized Hadamard matrices (cf. section 2.3) we start with the more general idea of a $(g,k;\lambda)$-difference matrix over a group $H$ (see [3]).

Definition 5.4. Let $H$ be a group of order $g$, let $\lambda$ be a positive integer, and $D = (d_{ij})$ for $i = 1, \ldots, k$, and $j = 1, \ldots, g\lambda$ be a matrix with entries from $H$ satisfying the condition that

$$(d_{il} - d_{jl})_{l=1,\ldots,g\lambda}$$

contains each element of $H$ precisely $\lambda$ times, whenever $i \neq j$ and $i,j \in \{1,\ldots,k\}$. Note: we are using additive notation, but $H$ is not assumed to be abelian.
Then $D$ is called a $(g, k; \lambda)$-difference matrix over the group $H$. If $k = g\lambda$ then $D$ is a generalized Hadamard matrix, denoted by $gH(g, k)$ or by $gH(H, k)$.

A theorem by D. Jungnickel (cf. [3, VIII.3, Thm 3.6]) establishes the equivalence of $(g, k; \lambda)$-difference matrices over a group $H$ (of order $g$) with class-regular STDs. The main consequence of the above discussion is that symmetric class-regular STD's are equivalent to generalized Hadamard matrices.

### 5.4 Flag algebras of incidence structures

We recall the definition of a 1-design (an $S_\lambda(1, k, v)$, see section 2.3), this time in some other words. Let $P$ be a set of $v$ points, $B$ a set of $b$ blocks, $I$ the incidence relation. A 1-design $(P, B, I)$ can be identified with the subset $D \subseteq P \times B$ such that:

- the number $r = |\{l \in B \mid (p, l) \in D\}|$ is independent of the choice of the point $p$.
- the number $k = |\{p \in P \mid (p, l) \in D\}|$ is independent of the choice of the block $l$.

The elements $(p, l)$ of $D$ are called flags, and $|D| = vb = kr$. A traditional way to represent this structure is the point adjacency matrix, a $(0, 1)$-matrix $A = (a_{p,q})$ indexed by the point set $P$ such that:

$$a_{p,q} = 1 \iff (p, l), (q, l) \in D \text{ for some } l \in B.$$ 

Considering flags, two other $(0, 1)$-matrices arise naturally:

- The collinearity matrix $L = (l_{f_1, f_2})$ indexed by flags, where:
  $$l_{f_1, f_2} = 1 \iff l_1 = l_2, p_1 \neq p_2.$$ 
- The concurrence matrix $N = (n_{f_1, f_2})$ indexed by flags, where:
  $$n_{f_1, f_2} = 1 \iff l_1 \neq l_2, p_1 = p_2.$$ 

Here $f_1 = (p_1, l_1)$ and $f_2 = (p_2, l_2)$. The matrices $L$ and $N$ satisfy:

$$L^2 = (k - 2)L + (k - 1)I, \quad N^2 = (r - 2)N + (r - 1)I.$$ 

Clearly, the flag structure determines the original design. Thus, in order to study the flag structure, we investigate the coherent closure of $\{L, N\}$, called the flag algebra of $D$ and denoted by:

$$W_F(D) = \langle\langle L, N\rangle\rangle.$$
Flag algebras have been mainly used in order to find necessary conditions for the existence of certain designs. Another direction of investigation is the problem of finding classes of designs that have a flag algebra of given rank. Some examples of the last direction include Steiner 2-designs which have a rank 7 flag algebra, and generalized $n$-gons which have a rank $2n$ flag algebra (cf. [30] or [24]). In general, the question of when the flag algebra of a design has restricted rank is difficult and remains open.

**Quasi-balanced designs**

This class of designs which was studied by Pech in his thesis [36] is of interest to us, since STD’s are a particular case of this class. We begin with the definition:

**Definition 5.5.** A design $D = (P, B)$ is called quasi-balanced if there exist $u, w, x, y \in \mathbb{N}$, $u < w$, $x < y$, such that:

- for any two points $p_1, p_2 \in P$, either $|p_1 \cap p_2| = u$ or $|p_1 \cap p_2| = w$.
- for any two blocks $b_1, b_2 \in B$, either $|b_1 \cap b_2| = x$ or $|b_1 \cap b_2| = y$.

$(u, w; x, y)$ are called the intersection numbers of $D$.

Notice that an $STD_{\mu}(g)$ is quasi-balanced with intersection numbers $(0, \mu; 0, \mu)$.

A crucial concept in the theory of flag algebras is that of distance between flags. We begin by introducing a flag walk of $D$.

**Definition 5.6.** Let $D = (P, B)$ be an incidence structure, and let $L$ and $N$ be the collinearity and concurrency matrices of $D$. A flag walk $Q$ of length $n$ is a sequence of flags $(f_0, \ldots, f_n)$ such that either $(L)_{f_i, f_{i+1}} = 1$ or $(N)_{f_i, f_{i+1}} = 1$ for all $0 \leq i < n$.

To each flag walk $Q$ of $D$ we associate the pattern of $Q$, this is the word $w(Q) = x_1x_2\ldots x_n$ over the alphabet $\Sigma = \{L, N\}$ such that $x_i = \begin{cases} L & \text{if } (L)_{f_{i-1}, f_i} = 1 \\ N & \text{if } (N)_{f_{i-1}, f_i} = 1 \end{cases}$.

**Definition 5.7.** Let $\Sigma = \{L, N\}$, and let $\Sigma^*$ be the free monoid generated by the alphabet $\Sigma$. Define the sequence of words:

- $w_0 = I$,
- $w_1 = L$,
- $w_2 = N$,
- $w_{2n+1} = \begin{cases} w_{2n-1}L & \text{if } 2|n \\ w_{2n-1}N & \text{else} \end{cases}$,
- $w_{2n+2} = \begin{cases} w_{2n}N & \text{if } 2|n \\ w_{2n}L & \text{else} \end{cases}$.
Here $I$ is the empty word.

1. A flag walk $Q$ is called a flag path if $w(Q) = w_i$ for some $i$.
2. A flag path is called an $L$-path if $i$ is odd, otherwise it is called an $N$-path.
3. For two flags $f$ and $g$, the $L$-distance $d_L(f, g)$ is the length of the shortest $L$-path from $f$ to $g$ if it exists, and $\infty$ otherwise. The $N$-distance is defined analogously.
4. The $L$-diameter and the $N$-diameter of a design $D$ is the longest $L$-path in $D$, and the longest $N$-path in $D$, respectively.
5. $D$ is called $L$-connected (respectively $N$-connected) if $\text{diam}_L(D) \neq \infty$ (respectively $\text{diam}_N(D) \neq \infty$).

An important result about $L/N$-connectedness:

**Lemma 5.5.** Let $D$ be a design with parameters $(v, b, r, k)$, then $D$ is $L$-connected if and only if it is $N$-connected.

More details can be found in the MSc thesis of Ch. Pech [36].

The theory of flag algebras developed in this section is not explicitly used in further parts of this thesis. In some places (for example in section 6.4 and subsection 7.2.4), it is used implicitly, and we hope that in the future this theory will play a central role in a general theory of the connections between STD’s and ADRG’s.

### 5.5 Flag algebras of STD’s: first observations

Our first observations appeared when studying the $STD_2(3)$ with 18 points and 108 flags (see detailed description in subsection 6.2.2 and section 7.2). In this case, the metric association scheme of the $(36, 3, 12)$-cover obtained by the Klin-Pech construction from the $STD_2(3)$ appears as the intersection of two rank 6 coherent subalgebras of the rank 11 flag algebra of $STD_2(3)$. Both rank 6 algebras can be described in terms of $L$ and $N$. After this observation, a similar investigation of other cases was done, and we expect to generalize these observations to a theoretical level in the near future, thus generalizing the results of Klin and Pech in [26].
In this chapter I present some of my computer aided investigations. Some new examples of regular 3-graphs will be presented, along with a few previously known examples that are of interest.

6.1 Regular 3-graphs from cyclic \((n, 3, c_2)\)-covers

In section 3.6 we presented an example by Higman in which he describes an infinite family of regular 3-graphs. With our new point of view on regular 3-graphs as cyclic covers, we revisit this example and find that this infinite family of cyclic covers is a subfamily of the Thas-Somma family. Using the equivalence of regular 3-graphs and covers with cyclic matchings we search for new examples of regular 3-graphs. In Table 6.1 we summarize the constructions of infinite classes of \((n, r, c_2)\)-covers known to us (for this table with more details cf. [26]).
Most of these families do not provide us examples of regular 3-graphs. In the following inspection of the above mentioned families we use the set of restrictions we obtained on the parameters of regular 3-graphs and the fact that \( r = 3 \).

**Mathon**  
In this case \( q = 3c + 1 \), thus \( q \equiv 1 \pmod{3} \) and so \( n = q + 1 \equiv 2 \pmod{3} \). Since \( 3 \nmid n \) these are not cyclic covers.

**Cameron**  
Since \( c_2 \) is a natural number, we have \( 3 \mid q^3 - 1 \). Then \( n = q^3 + 1 \equiv 2 \pmod{3} \), and thus \( 3 \nmid n \) and these covers are not cyclic.

**Bondy**  
Here we have a \((5, 3, 1)\)-cover (this is the line graph of the Petersen graph) and clearly \( 3 \nmid 5 \).

**Thas-Somma**  
This construction indeed provides an infinite family of regular 3-graphs. It will be described in subsection 6.1.1.

**Brouwer**  
Here \( s = 2 \), and these are \((2t + 1, 3, t - 1)\)-covers. Since we have \( n = a + 2b + 2 \), then \( 2t + 1 = a + 2t - 2 + 2 \), i.e. \( a = 1 \). The only feasible parameter set with \( a = 1 \) is \((9, 1, 3)\).

**Godsil-Hensel**  
Here \( p = 3 \) and \( i - k = 1 \). Then \( k = i - 1 \), and these are \((3^{2i}, 3, 3^{2i-1})\)-covers. This is a particular case of the Thas-Somma construction.
6.1.1 Investigation of the symplectic example

This construction appears as Example 6.1 in [22], and Higman calls this the symplectic example. We describe this family of regular 3-graphs in terms of \((n,3,c_2)\)-covers. We give the general description of the Thas-Somma construction: Let \(q = p^j\) be a prime power. Let \(V\) be a \(2j\)-dimensional vector space over \(GF(q)\), and let \(B\) be a non-degenerate symplectic form on \(V\). Let \(\Gamma\) be the graph with vertex set \(\{(\alpha,v) \mid \alpha \in GF(q), v \in V\}\) and with \((\alpha,v)\) adjacent to \((\beta,u)\) if and only if \(B(v,u) = \alpha - \beta\) and \(v \neq u\). Then \(\Gamma\) is a \((q^{2j},q,q^{2j-1})\)-cover. This well known family was first described by C. Somma in [42] as a generalization of a construction given by J. A. Thas in [46]. For \(q = 3\) this construction provides examples of regular 3-graphs with parameters \((3^{2j},3^{2j-1}-2,3^{2j-1})\). Note that not every \(j\) yields a cover with cyclic matchings, thus for example, there is no regular 3-graph with \(n = 27\).

The constructions of these regular 3-graphs that we describe next are carried out using the theory developed by M. Klin and Ch. Pech in [26].

6.2 Regular 3-graphs from class regular STD's

In this section we present some new examples of regular 3-graphs constructed by us with the use of class regular symmetric transversal designs, or equivalently, generalized Hadamard matrices. This construction is done with the aid of the Klin-Pech method presented in [26], and the classifications of generalized Hadamard matrices over \(U_3\) for some small orders (see [32] and [17]).

6.2.1 A regular 3-graph with parameters \((9,1,3)\)

There is a famous constructive proof that there are only 2 non-isomorphic \((9,3,3)\)-covers. This proof uses the fact that there is a unique strongly regular graph with parameters \((27,10,1,5)\) (the Schlaffli graph), that there are only 2 non-isomorphic spreads in the unique generalized quadrangle \(GQ(2,4)\), and that deletion of a spread in a generalized quadrangle yields a DRG. This construction will be presented with all details in section 7.1. GAP code for constructing both \((9,3,3)\)-covers is appended in section A.2.

There is just one generalized Hadamard matrix \(gH(U_3,3)\) up to monomial equivalence.

\[
H = \begin{pmatrix}
1 & 1 & 1 \\
1 & \zeta & \zeta^2 \\
1 & \zeta^2 & \zeta
\end{pmatrix}
\]
Using the Klin-Pech method a cyclic \((9,3,3)\)-cover is constructed. It is distance transitive, and its automorphism group of order 1296 is identified by GAP as:

\[
(((C3 \times C3) : C3) : Q8) : C3) : C2.
\]

This example is the member of the Thas-Somma construction, or Higman’s symplectic example.

### 6.2.2 A regular 3-graph with parameters \((36,10,12)\)

There is a only one \(gH(U_3,6)\) up to monomial equivalence. It is:

\[
H = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & \zeta & \zeta^2 & \zeta & \zeta \\
1 & \zeta & 1 & \zeta & \zeta^2 & \zeta \\
1 & \zeta^2 & \zeta & 1 & \zeta & \zeta^2 \\
1 & \zeta^2 & \zeta & 1 & \zeta & \zeta^2 \\
1 & \zeta & \zeta^2 & \zeta & \zeta^2 & \zeta \\
1 & \zeta & \zeta^2 & \zeta & \zeta^2 & \zeta \\
1 & \zeta & \zeta^2 & \zeta & \zeta^2 & \zeta
\end{pmatrix}.
\]

Using the Klin-Pech method, I constructed the corresponding cyclic \((36,3,12)\)-cover \(\Gamma\). It is not distance transitive, its automorphism group is transitive of rank 8 on the vertices of the cover, and is isomorphic to \((\mathbb{Z}_3.S_6)\).\(\mathbb{Z}_2\).

Then I constructed \(A_6(\Gamma)\), which turned out to be an association scheme, producing a new example of a regular 3-graph. It is a regular 3-graph with parameters \((36,10,12)\).

### 6.2.3 Two regular 3-graphs with parameters \((81,25,27)\)

There are just two generalized Hadamard matrices \(gH(U_3,9)\) up to monomial equivalence. This fact is established in [32], from which I extracted both matrices:

\[
H_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & \zeta & 1 & \zeta^2 & \zeta & \zeta & \zeta^2 & \zeta \\
1 & 1 & \zeta & \zeta & \zeta & \zeta^2 & \zeta & \zeta^2 & \zeta \\
1 & \zeta & 1 & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta & \zeta \\
1 & \zeta^2 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta \\
1 & \zeta^2 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta & \zeta \\
1 & \zeta^2 & \zeta & \zeta^2 & 1 & \zeta^2 & \zeta & \zeta & \zeta
\end{pmatrix}.
\]
6.2. REGULAR 3-GRAPHS FROM CLASS REGULAR STD’S

\[
H_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \zeta & \zeta^2 & \zeta & \zeta & \zeta & \zeta \\
1 & 1 & 1 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta & \zeta & \zeta^2 & 1 & \zeta^2 & 1 \\
1 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & 1 & \zeta & \zeta^2 & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & 1 & \zeta & \zeta^2 & \zeta & \zeta & \zeta
\end{pmatrix}.
\]

Using the method of Klin and Pech, I constructed two non-isomorphic cyclic \((81, 3, 27)\)-covers. The cover constructed from \(H_1\) is not distance transitive while the cover constructed from \(H_2\) is distance transitive. Surprisingly enough, it turned out that both have cyclic matchings, or in other words, for both of them \(A_6(\Gamma)\) is an association scheme. Since weights that are switching equivalent give isomorphic covers, we obtain two different regular 3-graphs with parameters \((81, 25, 27)\).

6.2.4 A regular 3-graph with parameters \((144, 46, 48)\)

There is only one \(\text{gH}(U_3, 12)\) up to monomial equivalence (see \([43]\)). I extracted this \(\text{gH}(U_3, 12)\) from \([26]\):

\[
H = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta \\
\zeta & 1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta \\
\zeta & \zeta^2 & 1 & \zeta & \zeta^2 & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta \\
1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta & \zeta \\
\zeta^2 & 1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta \\
\zeta^2 & 1 & \zeta & \zeta^2 & \zeta^2 & \zeta & \zeta & \zeta & \zeta
\end{pmatrix}.
\]

I constructed both the class-regular \(STD_4(3)\) and the cyclic \((144, 3, 48)\)-cover. As in the previous examples, \(A_6(\Gamma)\) is an association scheme, and thus we obtain a new regular 3-graph, with parameters \((144, 46, 48)\).

The cyclic cover has an automorphism group of order 1728 and structure description determined by \text{GAP} as

\[((C_2 \times C_6 \times A_4) : C_3) : C_2) : C_2,\]
it acts non-transitively on the 432 vertices with two orbits of size 216. The automorphism group of the $STD_4(3)$ is of order 864 and structure description:

$$(\left<(C_2 \times C_6 \times A_4) : C_3\right> : C_2,$$

and it acts transitively on both the points and the blocks of the design. Its action on the 432 flags of the $STD_4(3)$ is an index 2 subgroup of the automorphism group of the $(144, 3, 48)$-cover. It is intransitive on 432 flags with two orbits of size 216.

### 6.2.5 28 regular 3-graphs with parameters $(324, 106, 108)$

In the recently published paper [17], all class regular $STD_6(3)$’s are classified up to isomorphism. As a result, all generalized Hadamard matrices $gH(U_3, 18)$ are classified. There are 53 class regular $STD_6(3)$’s up to isomorphism, and 85 generalized Hadamard matrices $gH(U_3, 18)$’s up to monomial equivalence. I constructed the 85 corresponding cyclic $(324, 106, 108)$-covers via the Klin-Pech construction, and classified them up to isomorphism. There are 28 such covers which lead to 28 different regular 3-graphs, some basic information about these covers and the corresponding $STD_6(3)$’s is given in Table 6.3.
6.3. A SPORADIC EXAMPLE ON 135 VERTICES

This ADRG was discovered by M. Klin and Ch. Pech, and is presented in [26]. This (45, 3, 12)-cover is of special interest to us since it is the only example we have of a regular 3-graph with parameters of a form different than \((m^2, m^2 - 2, m^2)\).
It is not distance transitive, its automorphism group acts transitively of rank 10 on the 135 vertices of the cover.

### 6.4 Two rank 6 coherent algebras

As seems to be the case, when we use the Klin-Pech method to construct a regular \((n^2, g, n^2/g)\)-cover \(\Gamma_D\) from a class regular \(STD_\mu(g)\) \(D\) (cf. chapter 5), the action of the automorphism group of the design \(\text{Aut}(D)\) on the set \(F\) of \(g^3\mu^2\) flags of \(D\) is a subgroup of \(\text{Aut}(\Gamma_D)\). In terms of coherent algebras, this means that the metric association scheme of \(\Gamma_D\) appears as a rank 4 merging of the Schurian coherent configuration of the action \((\text{Aut}(D), F)\).

As will be described with more details in subsection 7.2.4, we identified two special rank 6 mergings of the color graph of the action of \(\text{Aut}(STD_2(3))\) on the 108 flags of \(STD_2(3)\). The special thing about these two algebras, is that their intersection yields the rank 4 metric association scheme of the corresponding \((36, 3, 12)\)-cover. In this case, they are algebraically equivalent, and they are non-Schurian. We observed the same situation in one of the \(STD_3(3)\)'s, specifically, in the one yielding the distance transitive \((n, 3, c_2)\)-cover. In this case, the two rank 6 algebras are not algebraically equivalent, they are both Schurian, as is the rank 4 metric association scheme of the corresponding \((81, 3, 27)\)-cover. One more example where this fact is observed is the exceptional example of a \(STD_6(3)\), which is described in subsection 7.3.1.
Chapter 7

Computer free interpretations

7.1 Two DRG’s on 27 vertices

The generalized quadrangle $GQ(2, 4)$

The generalized quadrangle $GQ(2, 4)$ has 27 points and 45 lines. Its point graph is the unique SRG with parameters $(27, 10, 1, 5)$. This SRG (or sometimes its complement) is commonly known as the Schl"afli graph and denoted by $Sch$. The graph $Sch$ has some exceptional features, including that of being 4-homogenous, which means that each isomorphism between any two of its induced subgraphs on $k$ vertices, for $k \leq 4$, can be extended to an automorphism of the whole graph. In fact, $Sch$ is strictly 4-homogenous (which means that it is not 5-homogenous). For more details see [28].

Spreads in generalized quadrangles

A spread in a generalized quadrangle is a subset of lines which partitions the point set. A result of A. E. Brouwer states that the deletion of a spread from the point graph of $GQ$ yields an antipodal DRG of diameter 3 (see [7]), i.e. an $(n, r, c_2)$-cover. In the case of $GQ(2, 4)$ it turns out that there just two classes of non-isomorphic spreads (see [8]).

Two spreads in $GQ(2, 4)$

We start with a group $K$ of order 324. It is a transitive group of degree 9, it is the normalizer in $S_9$ of a semi-regular group of order 3, denoted $\langle g_1^3 \rangle$. $K$ acts naturally on the set $\Omega$ of points of $GQ(2, 4)$. The corresponding association scheme $m = (\Omega, 2 - orb(K, \Omega))$ turns out to be the coherent closure of some 3
relations, 2 of which correspond to the orbits of different kinds of spreads in \(GQ(2, 4)\). For more details see [27].

**The related \((9, 3, 3)\)-covers**

With the aid of GAP, I constructed our model of \(GQ(2, 4)\), the spreads \(s_1\) and \(s_2\), and two non-isomorphic \((9, 3, 3)\)-covers. One of them is distance transitive, while the other is not. The distance transitive graph is a cyclic cover, while the other one is not. In the context of regular 3-graphs, the cyclic cover fulfills the sufficient condition that \(A_6(\Gamma)\) be an association scheme, thus producing the smallest example of a regular 3-graph. This is a regular 3-graph with parameters \((9, 1, 3)\).

**The corresponding \(STD_1(3)\)**

This well known TD has 9 points and 9 blocks and its automorphism group is of order 108, thus the induced action on the flags of \(STD_1(3)\) is an index 12 subgroup of the automorphism group of the cyclic cover of order 1296.

### 7.2 A sporadic example on 108 vertices revisited

#### 7.2.1 The general story

There is a construction of this cover in terms of the alternating group \(A_6\), using its exceptional Schur multiplier and other decorations of \(A_6\). Specifically, the automorphism group of this cover contains a subgroup which is an extension of the exceptional triple cover of \(S_6\). Historically, this goes back to the investigation of the primitive representation of the Hall-Janko group \(J_2\) on 280 points. This is a rank 4 action with subdegrees 1,36,108 and 135. The 2-closure of this action is \(\text{Aut}(J_2)\), its orbital graph \(\Gamma\) is an SRG with parameters \((280, 36, 8, 4)\). The stabilizer of a vertex in \(\text{Aut}(\Gamma) = \text{Aut}(J_2)\) has a faithful action on the suborbit of length 36, and this action coincides with the full automorphism group of a well known distance transitive graph which is the incidence graph of the \(STD_2(3)\). This led to the investigation of the actions on the suborbits of lengths 108 and 135, and to the further discovery by Klin and Pech of a \((36, 3, 12)\)-cover and a \((45, 3, 12)\)-cover using COCO and GAP. More details can be found in [26]. We wish to examine this example with our new tools.

**7.2.2 The transversal design \(STD_2(3)\)**

Recall that there is a unique \(gH(E_3, 6)\), and that \(STD_2(3)\) is the corresponding class-regular symmetric transversal design. This design is at the same time a
7.2. A SPORADIC EXAMPLE ON 108 VERTICES REVISITED

semibiplane \( SBP(18,6) \) with 18 points. We begin with a striking new model of it.

7.2.3 A model of the \( STD_2(3) \) via \( A_5 \)

Consider the following complementary graphs on 5 vertices:

- **The Pentagon**
  
  - \( C \)
  
  - \( S \)

Now consider the action of the alternating group \( A_5 \) of order 60 on these graphs, specifically, consider the orbits \( C^{A_5} \) and \( S^{A_5} \) of \( C \) and \( S \). Notice that in both cases the stabilizer \( S_{A_5}(C) \) is the automorphism group \( Aut(C) \) of \( C \) (and thus also of \( S \)), which is the dihedral group \( D_5 \) of order 10. This means that the orbits are of size 6. We depict them:

- Figure 7.1: The orbit \( C^{A_5} \)
  
  - \( c_1 \)
  
  - \( c_2 \)
  
  - \( c_3 \)
  
  - \( c_4 \)
  
  - \( c_5 \)
  
  - \( c_6 \)

- Figure 7.2: The orbit \( S^{A_5} \)
  
  - \( s_1 \)
  
  - \( s_2 \)
  
  - \( s_3 \)
  
  - \( s_4 \)
  
  - \( s_5 \)
  
  - \( s_6 \)
Also, consider the orbit of $A_5$ on the pair $\{C, S\}$. The stabilizer $S_{A_5}(\{C, S\})$ is still $D_5$ since a permutation sending $C$ to $S$ (for example, the permutation $(2, 4, 5, 3)$) is odd. Clearly, this orbit is 

$$\{C, S\}^{A_5} = \{\{c_i, s_i\}| 1 \leq i \leq 6\}.$$ 

The set $C^{A_5} \cup S^{A_5} \cup \{C, S\}^{A_5}$ is our set of 18 points of the $STD_2(3)$. Now we define the blocks. There are two types of blocks:

- 3 blocks of type 1,
- 15 blocks of type 2.

**Type 1 blocks**

These are the 3 blocks consisting of the sets $C^{A_5}$, $S^{A_5}$ and $\{C, S\}^{A_5}$.

**Type 2 blocks**

Consider the subgraphs of $K_5$ of the following type:

```
  •
 / \  \\
|   |  \\
\• \•  •
```

A graph of type $\gamma$

There are $15 = 5 \cdot 3$ such subgraphs, to each such (labeled) subgraph we associate the following 6 points:

```
  •  •  •  •  •  •
  \       / \       /
\• \•   \• \• \• \•
```

Figure 7.3: A block of type 2

Explicitly, the block associated with such a graph consists of

- the 2 points in $C^{A_5}$ containing this graph as a subgraph,
7.2. A SPORADIC EXAMPLE ON 108 VERTICES REVISITED

- the 2 points in $S^{A_5}$ containing this graph as a subgraph,
- the 2 points in $\{C, S\}^{A_5}$ for which neither element of the pair contains this graph as a subgraph.

The incidence in this structure is natural inclusion. We verify that the incidence structure defined is indeed an $STD_2(3)$.

The parallel classes

- The 6 parallel classes of points are:
  \[
  \{\{c_i, s_i, \{c_i, s_i\}\}|1 \leq i \leq 6}\.
  \]

- The 6 parallel classes of blocks are of two types, one parallel class is the set of type 1 blocks. Each of the other 5 parallel classes is determined by the 3 different graphs of type $\gamma$ corresponding to a fixed isolated vertex. Explicitly, for the vertex $x$, the corresponding parallel class of blocks is:

\[
\begin{align*}
\begin{array}{c}
  \bullet \bullet \\
  \bullet \bullet \\
  \bullet \bullet \\
  \bullet \bullet \\
  \bullet \bullet \\
  x
\end{array}
\end{align*}
\]

By definition we have:

\[
\begin{align*}
  v &= b = 18, \text{ and} \\
  k &= r = 6.
\end{align*}
\]

Clearly, $\lambda_1 = 0$. The most delicate part is to prove that $\lambda_2 = 2$. There are a few different cases of pairs of points from different parallel classes. These are:

1. $\{c_i, c_j\}$ for $i \neq j$,
2. $\{s_i, s_j\}$ for $i \neq j$,
3. $\{\{c_i, s_i\}, \{c_j, s_j\}\}$ for $i \neq j$,
4. $\{c_i, s_j\}$ for $i \neq j$,
5. $\{c_i, \{c_j, s_j\}\}$ for $i \neq j$,
6. $\{s_i, \{c_j, s_j\}\}$ for $i \neq j$. 
Proof. Pairs from case 1 are contained in the block $C^{A_5}$, and since they have exactly 2 common edges, they determine a unique graph of type $\gamma$, hence they are contained in a unique block of type 2. The same argument applies for case 2 pairs, and a similar argument applies for case 3 pairs. Pairs of type 4 have exactly 3 edges in common, of which 2 edges share a common vertex, thus there are exactly 2 graphs of type $\gamma$ that are subgraphs of $c_i$ and $s_j$, and these determine the 2 blocks containing this pair. For pairs of type 5 there are exactly 2 choices of graphs of type $\gamma$ such that this is a subgraph of $c_i$, and is not a subgraph of $c_j$ or $s_j$. The same argument applies to pairs of type 6.

To show that this $TD$ is an $STD$ we need to show that any pair of non-parallel blocks have 2 points in common. There are two types of such pairs:

1. A block of type 1 and a block of type 2.

2. Two blocks of type 2.

Proof. In the first case it is clear from the construction of type 2 blocks that there are exactly 2 points from each of the orbits $C^{A_5}$, $S^{A_5}$, $\{C, S\}^{A_5}$ which are exactly the blocks of type 1. The second case is a bit more sophisticated. There are 2 options, either the corresponding graphs of type $\gamma$ intersect in 1 edge, or they have empty intersection. In the case that they intersect in an edge, the graph of their union (a graph with 3 edges) is a subgraph of exactly 1 member of $C^{A_5}$ and 1 member of $S^{A_5}$ - these are the 2 points of intersection. In the case that their intersection is empty, the graph of their union (a graph with 4 edges) is a subgraph of exactly 1 member of either $C^{A_5}$ or $S^{A_5}$, and there is 1 pair of $\{C, S\}^{A_5}$ in which these 4 edges are partitioned into crossing pairs.

7.2.4 Inspection of the action of $\text{Aut}(STD_2(3))$ on flags

Understanding the structure of the 2-orbits of the action of $G = \text{Aut}(STD_2(3))$ on the set $F$ of flags of the $STD_2(3)$ is a crucial step in our investigation.

Using COCO, we constructed the color graph of the action $(G, F)$ and all its mergings (see section A.3); it is a rank 11 coherent algebra. We compiled a table of the flag relations, and have explanations for some parts of this table; specifically, we explained the 2 non-Schurian mergings of rank 6 of which the intersection is the rank 4 metric association scheme of the $(36, 3, 12)$-cover.

Also, we constructed the color graph of an index 2 subgroup of $G$, namely, $Z_3.A_6$ of order 1080, acting on $F$. This rank 18 coherent algebra contains as a rank 6 merging the association scheme $A_6(\Gamma)$ ($\Gamma$ is the corresponding cover). It also contains the two rank 6 algebras described above as mergings, and the rank 4 metric association scheme of the cover $\Gamma$ is the intersection of each of the two algebras with $A_6(\Gamma)$. Moreover, we identified two rank 9 merging of this action,
one of which contains one of the two rank 6 algebras and \( A_6(\Gamma) \), and the other contains the second rank 6 algebra and \( A_6(\Gamma) \) (see section A.4).

In the coming time, we wish to obtain a more formal understanding of both observations, and a computer free proof. This in turn, will hopefully provide crucial insight for a theoretical generalization.

7.3 Other structures

7.3.1 Exceptional example on 972 vertices

As was mentioned before, in [17] all \( gH(U_3, 18) \) are classified up to monomial equivalence. There are 28 of them, we constructed all of the corresponding STD’s and covers in GAP. One of these stands out with the largest group and smallest rank of its algebra. The automorphism group of the corresponding STD is transitive of order 38800 and structure description

\[
((C_3 \times (C_3.A_6)) : C_3) : C_2 : C_2.
\]

The corresponding association scheme is of rank 5.

The automorphism group of the cover is of order 933120, it is the only one of the 28 considered covers which has a transitive automorphism group, the rank of its corresponding association scheme is 16. We are now in the course of understanding it. We hope that in light of this very rich symmetry we will obtain some exceptional properties of this example.
Chapter 8

Concluding remarks

Summary of results

As described in the introduction, our journey began with Higman’s note [23], the main ideas of which are depicted in Figure 8.1. From these constructions Higman derived feasibility conditions for sets of parameters of regular 3-graphs.

Aside from correcting Higman’s error, our extra contribution is a clarification and extension of his original ideas, which provided an extra restriction on the parameters of a regular 3-graph. This is reflected in Figure 8.2.
Our approach establishes the graph-theoretic equivalent of regular 3-graphs. In this situation, the \((n, r, c_2)\)-cover turned out to be cyclic, thus allowing us to derive stronger feasibility conditions for regular 3-graph parameters. Having established our theoretical approach, we obtained a new and reduced list of feasible parameter sets of regular 3-graphs with \(n \leq 1000\), and constructed some examples of regular 3-graphs, most of which are new. All these new constructions are based on the use of the Klin-Pech approach described in [26], in conjunction with a careful analysis of known examples of generalized Hadamard matrices. In our eyes, the presented approach provides a deeper understanding of the links between class regular symmetric transversal designs and the corresponding cyclic \((n, r, c_2)\)-cover constructed from them.

**Extension to regular \(t\)-graphs with \(t \geq 4\)**

The theory outlined in chapter 4 can be extended to regular \(t\)-graphs with any \(t \geq 4\) only if we impose certain restrictions on the parameters of the regular \(t\)-graph. For example, when \(t\) is odd, the parameters of a regular \(t\)-graph are:

\[
\begin{align*}
  n, \\
  m(1), \\
  m(ζ) &= m(ζ^{t-1}), \\
  \vdots \\
  m(ζ^{t-1/2}) &= m(ζ^{t+1/2}).
\end{align*}
\]

A graph \(Γ_w\) defined by a regular \(t\)-graph will be distance regular only if most parameters of the regular \(t\)-graph are equal. Explicitly, in the case that \(t\) is odd we demand:

\[
m(ζ) = m(ζ^2) = \cdots = m(ζ^{t-1/2}).
\]

In this case, these will also be cyclic covers since for any \(t\) we have

\[
C_{S_t}(C_t) \cong C_t.
\]
Here we use the notation $C_t \leq S_t$ for the cyclic group $C_t = \langle (1, 2, \ldots, t) \rangle$.

Higman’s theory, which was presented in chapter 3, also extends to regular $t$-graphs with $t \geq 4$ in the case of equal parameters (as described above). The construction of $A_4(\Gamma_w)$ is exactly the same, and it has a rank $2t$ refinement which completely determines the weight $w$ (analogously to $A_6(\Gamma_w)$ in the case of regular 3-graphs). Thus, the extension of our theory to $t \geq 4$ is described schematically in Figure 8.3:

![Figure 8.3: Extension to $t \geq 4$](image)

**Direction of further research**

During the research described in this thesis some very appealing conjectures arose. Some of them, especially with regard to the connection between transversal designs and covers, are based on very recently published results, and thus we are eager to look into them. Another goal for the coming future is a detailed generalization of the approach developed here to regular $t$-graphs, as described in the previous section. Also, it seems appealing to investigate the other generalization of two-graphs. As the generalization investigated in this thesis studies regular 3-graphs in the language of $(n, 3, c_2)$-covers, the generalization in the other direction is in the language of 3-designs. In this direction, some interesting, new concepts such as association schemes on triples may appear. Finally, a very promising goal for the nearest future is to obtain the complete picture about the two rank 6 coherent subalgebras of the centralizer algebra of the automorphism group of a class regular symmetric transversal design acting on its flags, thus obtaining a possible generalization of the Klin-Pech method in [26].
Bibliography


67


[BIBLIOGRAPHY]


Appendices
Appendix A

Code

This section contains code I used for my computer aided research. The systems used are Mathematica, GAP and COCO.

A.1 Feasible parameter sets of regular 3-graphs

The following is Mathematica code for calculating the sets of feasible parameters \((n, a, b)\) of regular 3-graphs with \(n \leq 1000\); it is an implementation of the conditions in Proposition 4.7.

```
count = 0;
For[n = 1, n <= 1000, n = n + 1,
  For[a = 1, a < n, a++,
    b = (n - a - 2)/2;
    If[IntegerQ[b] && b > 0,
      alpha = ((a - b) + Sqrt[(a - b)^2 + 4*n - 4])/2;
      beta = ((a - b) - Sqrt[(a - b)^2 + 4*n - 4])/2;
      If[IntegerQ[alpha] && IntegerQ[beta] &&
        Divisible[n*alpha, alpha - beta] &&
        Divisible[n, 3],
        Print["[", n, ",", a, ",", b "]"];
        count++, Null],
    Null]
  ];
]
Print["The number of feasible parameter sets is: ", count];
```
A.2 Construction of the two \((9, 3, 3)\)-covers

The following is GAP code for constructing the two \((9, 3, 3)\)-covers; it is an implementation of the construction due to A. E. Brouwer described in section 7.1.

```gap
LoadPackage("grape");

# This function generates the symmetric closure
# of the binary relation \(r\)
Symmetrize := function(r)
    local i;
    for i in [1..Length(r)] do
        if not [r[i][2], r[i][1]] in r then
            Add(r, [r[i][2], r[i][1]]);
        fi;
    od;
    return r;
end;

# Blocks of GQ(2,4)
B1 := [[1, 10, 19], [1, 13, 22], [1, 16, 25], [2, 11, 20], [2, 14, 23], [2, 17, 26],
      [3, 12, 21], [3, 15, 24], [3, 18, 27], [4, 11, 22], [4, 14, 25], [4, 17, 19],
      [5, 12, 23], [5, 15, 26], [5, 18, 20], [6, 10, 24], [6, 13, 27], [6, 16, 21],
      [7, 12, 25], [7, 15, 19], [7, 18, 22], [8, 10, 26], [8, 13, 20], [8, 16, 23],
      [9, 11, 27], [9, 14, 21], [9, 17, 24]];  
S_even := [[1, 2, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12], [13, 14, 15], [16, 17, 18],
          [19, 20, 21], [22, 23, 24], [25, 26, 27]];  
S_odd := [[1, 5, 9], [4, 8, 3], [7, 6, 2], [10, 14, 18], [13, 17, 12], [16, 15, 11],
         [19, 23, 27], [22, 26, 21], [25, 24, 20]];  

g1 := Union(B1, S_odd);

g2 := Union(B1, S_even);

# This function returns the point graph of a list of blocks
MyPointGraph := function(b)
    local i, j, k, g;
    g := [];
    for i in [1..Length(b)] do
        for j in [1..Length(b[i])-1] do
            Add(g, [b[i][j], b[i][j+1]]);
        od;
    od;
    return g;
end;
```
A.3. COCO output for the action of $Aut(STD_2(3))$ on its flags

The following is a protocol of COCO output for computing the color graph of the automorphism group of the $STD_2(3)$ acting on its flags and all its mergings. Mergings number 5 and merging number 6 are the two rank 6 algebras of which the intersection is merging number 17, the metric association scheme of the cover.

```plaintext
### System CO-CO for construction of coherent configurations ###
output listing to file 108a.res
COCO>> ind autD6.gen * 108a 108a
* induced action of permutation group on structure *
input of group from autD6.gen
input of structure for inducing from <stdin>
output of induced group to file 108a.gen
output of mapping to file 108a.map
```
inducing on structure: 
\((1,\{1,4,7,10,13,16\})\)

length of orbit 1 is 108

induced transitive group on 108 points

COCO>> cgr 108a * 108a

* construction of 2-orbits of permutation group *

input of group from 108a.gen
input of options from <stdin>
output of colour graph to file 108a.cgr

orbit 1 has length 108

options(i,rank):

stabilizer of point 0 of orbit 1 has
11 suborbits on orbit 1

<table>
<thead>
<tr>
<th>number</th>
<th>length</th>
<th>representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>11</td>
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<tr>
<td>6</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>27</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>33</td>
</tr>
</tbody>
</table>

colour graph is saved

reflexive suborbits:0
symmetrical suborbits:1,2,3,4,5,6,7,10
pairs of antisymmetrical suborbits: (8,9)
A.3. COCO OUTPUT FOR THE ACTION OF $\text{AUT}(\text{STD}_2(3))$ ON ITS FLAGS

COCO>> inm 108a * 108a

* calculation of intersection numbers *

input of colour graph from file 108a.cgr
input of options from <stdin>
output of intersection numbers to file 108a.nrs

homogeneous colour graph of rank 11 on 108 vertices
there are 435 nonzero intersection numbers with $i$ of type 1,1

COCO>> sub 108a * 108a

* construction of subconfigurations *

input of intersection numbers from 108a.nrs
input of options from <stdin>
output of subconfigurations to file 108a.sub

disconnected classes: 1,2,3,6,7
imprimitive scheme of rank 11

options($s,p,sp$):

there exist 26 good symmetrical sets
there exist 0 pairs of good antisymmetrical sets
1. subscheme of rank 8 by merging (1,6)(2,7)(8,9)
2. subscheme of rank 7 by merging (1,7)(4,8,9)(5,10)
3. subscheme of rank 7 by merging (2,6)(4,8,9)(5,10)
4. subscheme of rank 6 by merging (1,7)(4,5,8,9,10)
5. subscheme of rank 6 by merging (1,5,8,9)(4,7,10)
6. subscheme of rank 6 by merging (2,4,10)(5,6,8,9)
7. subscheme of rank 6 by merging (2,6)(4,5,8,9,10)
8. subscheme of rank 6 by merging (1,7)(2,6)(4,8,9)(5,10)
9. subscheme of rank 5 by merging (2,4,5,6,8,9,10)
10. subscheme of rank 5 by merging (1,2,6,7)(4,8,9)(5,10)
11. subscheme of rank 5 by merging (1,4,5,7,8,9,10)
12. subscheme of rank 5 by merging (1,7)(2,6)(4,5,8,9,10)
13. subscheme of rank 4 by merging $(1,2,6,7)(4,5,8,9,10)$
14. subscheme of rank 4 by merging $(1,4,5,7,8,9,10)(2,6)$
15. subscheme of rank 4 by merging $(1,4,5,7,8,9,10)(3,6)$
16. subscheme of rank 4 by merging $(1,4,5,8,9,10)(2,3,6)$
17. subscheme of rank 4 by merging $(1,5,6,8,9)(2,4,7,10)$
18. subscheme of rank 4 by merging $(1,3)(2,4,5,6,8,9,10)$
19. subscheme of rank 4 by merging $(1,3,7)(4,5,6,8,9,10)$
20. subscheme of rank 4 by merging $(1,7)(2,4,5,6,8,9,10)$
21. subscheme of rank 3 with parameters $(108,5,4)$ by merging $(1,3,4,5,6,7,8,9,10)$
22. subscheme of rank 3 with parameters $(108,17,16)$ by merging $(1,3,7)(2,4,5,6,8,9,10)$
23. subscheme of rank 3 with parameters $(108,2,1)$ by merging $(1,2,4,5,6,7,8,9,10)$
24. subscheme of rank 3 with parameters $(108,5,4)$ by merging $(1,2,3,4,5,6,8,9,10)$
25. subscheme of rank 3 with parameters $(108,17,16)$ by merging $(1,4,5,7,8,9,10)(2,3,6)$

25 subschemes

COCO>> aut 108a 108a

* calculation of automorphism groups *

input of colour graph from file 108a.cgr
input of mergings from 108a.sub

homogeneous colour graph of rank 11 on 108 vertices

colour graph of rank 11
transitive automorphism group of order 2160
rank=11; subdegrees:1,10,5,2,20,10,10,5,20,20,5
base of length 2, 4 generators

colour graph of rank 8
transitive automorphism group of order 4320
rank=8; subdegrees:1,20,10,2,20,10,40,5
base of length 3, 5 generators

colour graph of rank 7
transitive automorphism group of order 33592320
rank=7; subdegrees:1,15,5,2,60,15,10
base of length 12, 14 generators

colour graph of rank 7
transitive automorphism group of order 33592320
rank=7; subdegrees:1,10,15,2,60,15,5
base of length 12, 14 generators

colour graph of rank 6
transitive automorphism group of order 24186470400
rank=6; subdegrees:1,15,5,2,75,10
base of length 15, 20 generators

colour graph of rank 6
transitive automorphism group of order 2160
rank=11; subdegrees:1,10,5,2,20,10,10,5,20,20,5
base of length 2, 4 generators

colour graph of rank 6
transitive automorphism group of order 2160
rank=11; subdegrees:1,10,5,2,20,10,10,5,20,20,5
base of length 2, 4 generators

colour graph of rank 6
transitive automorphism group of order 24186470400
rank=6; subdegrees:1,10,15,2,75,5
base of length 15, 20 generators

colour graph of rank 5
transitive automorphism group of order
7426385854913185593243803320320
rank=6; subdegrees:1,15,15,2,60,15
base of length 72, 75 generators

colour graph of rank 5
transitive automorphism group of order
4679882803280609280000000
rank=5; subdegrees:1,10,90,2,5
base of length 34, 45 generators
colour graph of rank 5
transitive automorphism group of order 14852771709826371186487606640640
rank=5; subdegrees:1,30,2,60,15
base of length 72, 76 generators

colour graph of rank 5
transitive automorphism group of order 4679882803280609280000000
rank=5; subdegrees:1,90,5,2,10
base of length 35, 47 generators

colour graph of rank 5
transitive automorphism group of order 5346997815537493627135538390630400
rank=5; subdegrees:1,15,15,2,75
base of length 72, 82 generators

colour graph of rank 4
transitive automorphism group of order 10693995631074987254271076781260800
rank=4; subdegrees:1,30,2,75
base of length 72, 86 generators

colour graph of rank 4
transitive automorphism group of order 515489797623215858436645552128000000
rank=4; subdegrees:1,90,15,2
base of length 72, 109 generators

colour graph of rank 4
transitive automorphism group of order 4589843716597329428480000000000000000000
rank=4; subdegrees:1,90,5,12
base of length 90, 110 generators

colour graph of rank 4
transitive automorphism group of order 4609709068124160000
rank=4; subdegrees:1,85,17,5
base of length 19, 33 generators
A.3. \textit{COCO output for the action of Aut(\textit{STD}_2(3)) on its flags}  

- Colour graph of rank 4  
  Transitive automorphism group of order 4320  
  Rank=8; subdegrees: 1, 20, 10, 2, 20, 10, 40, 5  
  Base of length 3, 5 generators

- Colour graph of rank 4  
  Transitive automorphism group of order 458984371659732942848000000000000000000000  
  Rank=4; subdegrees: 1, 12, 90, 5  
  Base of length 90, 111 generators

- Colour graph of rank 4  
  Transitive automorphism group of order 46097090681241600000  
  Rank=4; subdegrees: 1, 17, 5, 85  
  Base of length 20, 31 generators

- Colour graph of rank 4  
  Transitive automorphism group of order 5154897997623218688584366455521280000000  
  Rank=4; subdegrees: 1, 15, 90, 2  
  Base of length 72, 108 generators

- Colour graph of rank 3  
  Transitive automorphism group of order 988377165470577459200000000000000000000  
  Rank=3; subdegrees: 1, 102, 5  
  Base of length 90, 128 generators

- Colour graph of rank 3  
  Transitive automorphism group of order 3313544033643926650880000000000000000000000000000000000000  
  Rank=3; subdegrees: 1, 17, 90  
  Base of length 102, 189 generators

- Colour graph of rank 3  
  Transitive automorphism group of order 38427027945314505060762582515712000000000  
  Rank=3; subdegrees: 1, 105, 2  
  Base of length 72, 140 generators

- Colour graph of rank 3
transitive automorphism group of order
988377165470577459200000000000000000000
rank=3; subdegrees:1,102,5
base of length 90, 129 generators
colour graph of rank 3
transitive automorphism group of order
331354403364392665088000000000000000000
rank=3; subdegrees:1,90,17
base of length 102, 198 generators

COCO>> end
* end of CO-CO *

A.4 COCO output for the action of $Z_3.A_6$ on the flags of $STD_2(3)$

The following is a protocol of COCO output for computing the color graph of the group $Z_3.A_6$ acting on the flags of the $STD_2(3)$. The two rank 9 mergings are number 6 and number 8. Number 26 and number 31 are the two rank 6 algebras, and number 29 is $A_6(\Gamma)$.

*** system CO-CO for construction of coherent configurations ***
output listing to file 36bb.res
COCO>> ind z3xa6.gen * 36bb 36bb
* induced action of permutation group on structure *
input of group from z3xa6.gen
input of structure for inducing from <stdin>
output of induced group to file 36bb.gen
output of mapping to file 36bb.map

inducing on structure:
(18,0)
length of orbit 1 is 108
induced transitive group on 108 points

COCO>> inm 36bb * 36bb

* calculation of intersection numbers *

COCO>> cgr 36bb * 36bb

* construction of 2-orbits of permutation group *

input of group from 36bb.gen
input of options from <stdin>
output of colour graph to file 36bb.cgr

orbit 1 has length 108

options(i,rank):

stabilizer of point 0 of orbit 1 has
  18 suborbits on orbit 1

<table>
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<td>1</td>
<td>79</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
<td>93</td>
</tr>
</tbody>
</table>
colour graph is saved

reflexive suborbits:0
symmetrical suborbits:9,10,13,14,15
pairs of antisymmetrical suborbits:
(1,6)(2,4)(3,5)(7,12)(8,11)(16,17)

COCO>> inm 36bb * 36bb

* calculation of intersection numbers *

input of colour graph from file 36bb.cgr
input of options from <stdin>
output of intersection numbers to file 36bb.nrs

homogeneous colour graph of rank 18 on 108 vertices

there are 1098 nonzero intersection numbers with i of type 1,1

COCO>> sub 36bb * 36bb

* construction of subconfigurations *

input of intersection numbers from 36bb.nrs
input of options from <stdin>
output of subconfigurations to file 36bb.sub

disconnected classes: 7,8,9,11,12,15,16,17
imprimitive scheme of rank 18

options(s,p,sp):

there exist 35 good symmetrical sets
there exist 10 pairs of good antisymmetrical sets
1. subscheme of rank 12 by merging
   (9,15)(10,14)(1,2)(6,4)(7,11)(12,8)
2. subscheme of rank 11 by merging
   (3,5)(7,12)(8,11)(10,14)(16,17)(1,4)(6,2)
3. subscheme of rank 11 by merging
A.4. COCO OUTPUT FOR THE ACTION OF $\mathbb{Z}_3.A_6$ ON THE FLAGS OF $STD_2(85)$

(1,6)(2,4)(3,5)(9,15)(16,17)(7,8)(12,11)

4. subscheme of rank 10 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12,15)

5. subscheme of rank 10 by merging
(1,6,10)(2,4,14)(3,5,13)(8,11,9)

6. subscheme of rank 9 by merging
(1,6,2,4,10,14)(3,5,13)(7,12,15)

7. subscheme of rank 9 by merging
(9,10,13,14)(1,2,3,8)(6,4,5,11)

8. subscheme of rank 9 by merging
(1,6,2,4,10,14)(3,5,13)(8,11,9)

9. subscheme of rank 9 by merging
(10,13,14,15)(1,2,3,12)(6,4,5,7)

10. subscheme of rank 8 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12,15)(8,11,9)

11. subscheme of rank 8 by merging
(1,6,2,4)(3,5)(7,12,8,11)(9,15)(10,14)(16,17)

12. subscheme of rank 8 by merging
(1,6,2,4,3,5,10,13,14)(7,12,15)

13. subscheme of rank 8 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12)(8,11,9)(16,17)

14. subscheme of rank 8 by merging
(1,6,2,4,3,5,10,13,14)(8,11,9)

15. subscheme of rank 8 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12,15)(8,11)(16,17)

16. subscheme of rank 7 by merging
(1,6,2,4,10,14)(3,5,13)(7,12,15)(8,11,9)

17. subscheme of rank 7 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12,15)(8,11,9)(16,17)

18. subscheme of rank 7 by merging
(1,6,2,4,3,5,8,11,9,10,13,14)

19. subscheme of rank 7 by merging
(1,6,2,4,10,14)(3,5,13)(7,12)(8,11,9)(16,17)

20. subscheme of rank 7 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12,8,11,9,15)

21. subscheme of rank 7 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)

22. subscheme of rank 7 by merging
(1,6,2,4,10,14)(3,5,13)(7,12,15)(8,11)(16,17)

23. subscheme of rank 6 by merging
(1,6,2,4,3,5,10,13,14)(7,12,15)(8,11)(16,17)
24. subscheme of rank 6 by merging
(1,6,2,4,3,5,10,13,14)(7,12,15)(8,11,9)
25. subscheme of rank 6 by merging
(1,6,2,4,10,14)(3,5,13)(7,12,15)(8,11,9)(16,17)
26. subscheme of rank 6 by merging
(1,6,2,4,3,5,7,12)(8,11)(10,13,14,15)(16,17)
27. subscheme of rank 6 by merging
(1,6,2,4,3,5,10,13,14)(7,12)(8,11,9)(16,17)
28. subscheme of rank 6 by merging
(1,6,2,4,10,14)(3,5,13)(7,12,8,11,9,15)
29. subscheme of rank 6 by merging
(9,10,13,14,15)(1,2,3,12,8)(6,4,5,7,11)
30. subscheme of rank 6 by merging
(1,6,10)(2,4,14)(3,5,13)(7,12,8,11,9,15)(16,17)
31. subscheme of rank 6 by merging
(1,6,2,4,3,5,7,12)(9,10,13,14)(16,17)
32. subscheme of rank 5 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(8,11)(16,17)
33. subscheme of rank 5 by merging
(1,6,2,4,3,5,10,13,14)(7,12,8,11,9,15)
34. subscheme of rank 5 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(8,11,9)
35. subscheme of rank 5 by merging
(1,6,2,4,3,5,8,11,9,10,13,14)(7,12,16,17)
36. subscheme of rank 5 by merging
(1,6,2,4,3,5,10,13,14)(7,12,15)(8,11,9)(16,17)
37. subscheme of rank 5 by merging
(1,6,2,4,3,5,8,11,9,10,13,14)(7,12,15)
38. subscheme of rank 5 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(8,11,9)(16,17)
39. subscheme of rank 5 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(7,17)(12,16)
40. subscheme of rank 5 by merging
(1,6,2,4,10,14)(3,5,13)(7,12,8,11,9,15)(16,17)
41. subscheme of rank 4 by merging
(1,6,2,4,3,5,10,13,14)(7,12,8,11,9,15)(16,17)
42. subscheme of rank 4 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(8,11,9)(16,17)
43. subscheme of rank 4 by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(8,11,16,17)
44. subscheme of rank 4 by merging
(1,6,2,4,3,5,8,11,9,10,13,14)(7,12,15)(16,17)
45. subscheme of rank 4 by merging
(1,6,2,4,3,5,8,11,9,10,13,14)(7,12,16,17)
46. subscheme of rank 4 by merging
(1,6,2,4,3,5,7,12,8,11)(9,10,13,14,15)(16,17)
47. subscheme of rank 4 by merging
(1,6,2,4,3,5,8,11,10,13,14)(7,12,15,16,17)
48. subscheme of rank 4 by merging
(1,6,2,4,3,5,7,12,8,11,9,10,13,14,15)
49. subscheme of rank 4 by merging
(1,6,2,4,3,5,7,12,10,13,14)(8,11,9,16,17)
50. subscheme of rank 3 with parameters (108,2,1) by merging
(1,6,2,4,3,5,7,12,8,11,9,10,13,14,15)(16,17)
51. subscheme of rank 3 with parameters (108,5,4) by merging
(1,6,2,4,3,5,7,12,8,11,9,10,13,14,16,17)
52. subscheme of rank 3 with parameters (108,17,16) by merging
(1,6,2,4,3,5,8,11,9,10,13,14)(7,12,15,16,17)
53. subscheme of rank 3 with parameters (108,17,16) by merging
(1,6,2,4,3,5,7,12,10,13,14,15)(8,11,9,16,17)
54. subscheme of rank 3 with parameters (108,5,4) by merging
(1,6,2,4,3,5,7,12,8,11,10,13,14,15,16,17)

54 subschemes

COCO>> aut 36bb 36bb

* calculation of automorphism groups *

input of colour graph from file 36bb.cgr
input of mergings from 36bb.sub

homogeneous colour graph of rank 18 on 108 vertices

colour graph of rank 18
transitive automorphism group of order 1080
rank=18; subdegrees:1,10,10,5,10,5,10,5,5,5,10,5,1,1
base of length 2, 3 generators

colour graph of rank 12
transitive automorphism group of order 2160
rank=12; subdegrees:1,20,5,20,5,10,10,10,20,5,1,1
base of length 2, 4 generators
colour graph of rank 11
transitive automorphism group of order 2160
rank=11; subdegrees: 1, 20, 20, 10, 10, 10, 5, 20, 5, 5, 2
base of length 2, 4 generators
colour graph of rank 11
transitive automorphism group of order 2160
rank=11; subdegrees: 1, 20, 20, 10, 10, 10, 5, 10, 2
base of length 2, 4 generators
colour graph of rank 10
transitive automorphism group of order 262440
rank=10; subdegrees: 1, 30, 30, 15, 15, 5, 5, 1, 1
base of length 6, 7 generators
colour graph of rank 10
transitive automorphism group of order 262440
rank=10; subdegrees: 1, 30, 30, 15, 5, 15, 5, 1, 1
base of length 6, 7 generators
colour graph of rank 9
transitive automorphism group of order 524880
rank=9; subdegrees: 1, 60, 15, 15, 5, 5, 1, 1
base of length 6, 8 generators
colour graph of rank 9
transitive automorphism group of order 524880
rank=9; subdegrees: 1, 60, 15, 5, 15, 5, 5, 1, 1
base of length 2, 3 generators
colour graph of rank 9
transitive automorphism group of order 524880
rank=9; subdegrees: 1, 60, 15, 5, 15, 5, 5, 1, 1
base of length 6, 8 generators
colour graph of rank 9
transitive automorphism group of order 1080
rank=18; subdegrees: 1, 10, 10, 5, 10, 5, 5, 10, 5, 5, 5, 10, 5, 1, 1
base of length 2, 3 generators
colour graph of rank 9
transitive automorphism group of order 1080
rank=18; subdegrees: 1, 10, 10, 5, 10, 5, 5, 5, 10, 5, 5, 5, 10, 5, 1, 1
base of length 2, 3 generators
colour graph of rank 8
transitive automorphism group of order 54034068706919683560
rank=8; subdegrees:1,30,30,15,15,15,1,1
base of length 36, 37 generators

colour graph of rank 8
transitive automorphism group of order 4320
rank=8; subdegrees:1,40,10,20,10,20,5,2
base of length 2, 5 generators

colour graph of rank 8
transitive automorphism group of order 377913600
rank=8; subdegrees:1,75,15,5,5,5,1,1
base of length 6, 12 generators

colour graph of rank 8
transitive automorphism group of order 16796160
rank=8; subdegrees:1,30,30,15,10,15,5,2
base of length 12, 13 generators

colour graph of rank 8
transitive automorphism group of order 377913600
rank=8; subdegrees:1,75,5,15,5,5,1,1
base of length 6, 12 generators

colour graph of rank 8
transitive automorphism group of order 16796160
rank=8; subdegrees:1,30,30,15,15,10,5,2
base of length 12, 13 generators

colour graph of rank 7
transitive automorphism group of order 108068137413839367120
rank=7; subdegrees:1,60,15,15,15,1,1
base of length 36, 38 generators

colour graph of rank 7
transitive automorphism group of order 3713192927456592796621901660160
rank=7; subdegrees:1,30,30,15,15,15,2
APPENDIX A. CODE

base of length 72, 73 generators

colour graph of rank 7
transitive automorphism group of order
731231688012595200000000
rank=7; subdegrees:1,90,5,5,5,1,1
base of length 30, 40 generators

colour graph of rank 7
transitive automorphism group of order 33592320
rank=7; subdegrees:1,60,15,10,15,5,2
base of length 12, 14 generators

colour graph of rank 7
transitive automorphism group of order
108068137413839367120
rank=7; subdegrees:1,30,30,15,30,1,1
base of length 36, 38 generators

colour graph of rank 7
transitive automorphism group of order
731231688012595200000000
rank=7; subdegrees:1,90,5,5,5,1,1
base of length 30, 40 generators

colour graph of rank 7
transitive automorphism group of order 33592320
rank=7; subdegrees:1,60,15,10,15,5,2
base of length 12, 14 generators

colour graph of rank 6
transitive automorphism group of order 24186470400
rank=6; subdegrees:1,75,15,10,5,2
base of length 12, 22 generators

colour graph of rank 6
transitive automorphism group of order
77809058937964344326400
rank=6; subdegrees:1,75,15,15,1,1
base of length 36, 42 generators
colour graph of rank 6
transitive automorphism group of order
7426385854913185593243803320320
rank=6; subdegrees:1,60,15,15,15,2
base of length 72, 74 generators

colour graph of rank 6
transitive automorphism group of order
2160
rank=11; subdegrees:1,20,20,10,10,5,20,5,5,2
base of length 2, 4 generators

colour graph of rank 6
transitive automorphism group of order
24186470400
rank=6; subdegrees:1,75,10,15,5,2
base of length 12, 22 generators

colour graph of rank 6
transitive automorphism group of order
216136274827678734240
rank=6; subdegrees:1,60,15,30,1,1
base of length 36, 39 generators

colour graph of rank 6
transitive automorphism group of order
2160
rank=12; subdegrees:1,20,5,20,5,10,10,20,5,1,1
base of length 2, 4 generators

colour graph of rank 6
transitive automorphism group of order
7426385854913185593243803320320
rank=6; subdegrees:1,30,30,15,30,2
base of length 72, 74 generators

colour graph of rank 6
transitive automorphism group of order
2160
rank=11; subdegrees:1,20,20,10,10,5,20,5,5,2
base of length 2, 4 generators

colour graph of rank 5
transitive automorphism group of order
155618117875928688652800
rank=5; subdegrees:1,75,30,1,1
base of length 36, 46 generators

colour graph of rank 5
transitive automorphism group of order
467988280328069280000000
rank=5; subdegrees:1,90,10,5,2
base of length 30, 50 generators

colour graph of rank 5
transitive automorphism group of order
15055412006839440402447052308480000000
rank=5; subdegrees:1,90,15,1,1
base of length 36, 70 generators

colour graph of rank 5
transitive automorphism group of order
467988280328069280000000
rank=5; subdegrees:1,90,10,5,2
base of length 30, 50 generators

colour graph of rank 5
transitive automorphism group of order
5346997815537493627135538390630400
rank=5; subdegrees:1,75,15,15,2
base of length 72, 78 generators

colour graph of rank 5
transitive automorphism group of order
15055412006839440402447052308480000000
rank=5; subdegrees:1,90,15,1,1
base of length 36, 70 generators

colour graph of rank 5
transitive automorphism group of order
850921630807183327232000000000000000000
rank=5; subdegrees:1,90,6,6,5
base of length 90, 111 generators

colour graph of rank 5
transitive automorphism group of order
8509216308071833272320000000000000000000
rank=5; subdegrees:1,90,6,5,6
base of length 90, 110 generators

colour graph of rank 5
transitive automorphism group of order
14852771709826371186487606640640
rank=5; subdegrees:1,60,15,30,2
base of length 72, 75 generators

colour graph of rank 4
transitive automorphism group of order
10693995631074987254271076781260800
rank=4; subdegrees:1,75,30,2
base of length 72, 82 generators

colour graph of rank 4
transitive automorphism group of order
515489799762321868858436645552128000000
rank=4; subdegrees:1,90,15,2
base of length 72, 124 generators

colour graph of rank 4
transitive automorphism group of order
45898437165973294284800000000000000000
rank=4; subdegrees:1,90,12,5
base of length 90, 126 generators

colour graph of rank 4
transitive automorphism group of order
515489799762321868858436645552128000000
rank=4; subdegrees:1,90,15,2
base of length 72, 125 generators

colour graph of rank 4
transitive automorphism group of order
45898437165973294284800000000000000000
rank=4; subdegrees:1,90,12,5
base of length 90, 128 generators

colour graph of rank 4
transitive automorphism group of order 4320
rank=8; subdegrees:1,40,10,20,10,20,5,2
base of length 2, 5 generators

colour graph of rank 4
transitive automorphism group of order 4609709068124160000
rank=4; subdegrees:1,85,17,5
base of length 17, 35 generators

colour graph of rank 4
transitive automorphism group of order 6381414615226716018435129815859200000000
rank=4; subdegrees:1,105,1,1
base of length 36, 98 generators

colour graph of rank 4
transitive automorphism group of order 4609709068124160000
rank=4; subdegrees:1,85,17,5
base of length 17, 34 generators

colour graph of rank 3
transitive automorphism group of order 3842702794531450504076258251571200000000
rank=3; subdegrees:1,105,2
base of length 72, 161 generators

colour graph of rank 3
transitive automorphism group of order 9883771654705774592000000000000000000000
rank=3; subdegrees:1,102,5
base of length 90, 133 generators

colour graph of rank 3
transitive automorphism group of order 331354403364392665088000000000000000000000
rank=3; subdegrees:1,90,17
base of length 102, 179 generators

colour graph of rank 3
transitive automorphism group of order 331354403364392665088000000000000000000000
rank=3; subdegrees:1,90,17
base of length 102, 179 generators
A.4. COCO OUTPUT FOR THE ACTION OF $\mathbb{Z}_3.A_6$ ON THE FLAGS OF $STD_2(95)$

base of length 102, 177 generators

colour graph of rank 3
transitive automorphism group of order
988377165470577459200000000000000000000000000000000000
rank=3; subdegrees:1,102,5
base of length 90, 131 generators

COCO>> end

* end of CO-CO *
List of Figures

3.1 A perfect matching between fibers of $\Gamma_w$ .............................. 23
3.2 Matchings between fibers of $\Gamma_w$ ........................................... 24

4.1 $C_9$ configuration ................................................................. 34
4.2 The case when $\Gamma_{\{x,y,z\}} \cong 3 \circ C_3$ .............................. 35
4.3 Two cases when $\Gamma_{\{x,y,z\}} \cong C_9$ ....................................... 36

7.1 The orbit $C^{A_5}$ .................................................................... 57
7.2 The orbit $S^{A_5}$ ................................................................. 57
7.3 A block of type 2 ................................................................. 58

8.1 Higman’s draft ................................................................. 63
8.2 This thesis ................................................................. 64
8.3 Extension to $t \geq 4$ .......................................................... 65
List of Tables

4.1 Feasible parameters of regular 3-graphs . . . . . . . . . . . . . . . . . . 40
6.1 Infinite families of \((n, r, c_2)\)-covers . . . . . . . . . . . . . . . . . . . . . 48
6.3 Regular 3-graphs with parameters \((324, 106, 108)\) . . . . . . . . . . . . 53