Regular 3-graphs

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June 24, 2011
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Two-graphs

Definition

Let $X$ be a set of $n$ elements called vertices. $X^{3}$ is the set of all 3-subsets of $X$. A subset $\Delta \subseteq X^{3}$ is a two-graph if every 4-subset of $X$ contains an even ($\in \{0, 2, 4\}$) number of members of $\Delta$. Sometimes we call $(X, \Delta)$ a two-graph, and $\Delta$ the set of odd triples.
A two-graph \((X, \Delta)\) is a function:

\[ f : X^3 \rightarrow U_2 \]

such that \(f(x) = -1 \iff x \in \Delta\), satisfying:

\[ f(\{x, y, z\}) \cdot f(\{x, y, t\}) \cdot f(\{x, z, t\}) \cdot f(\{y, z, t\}) = 1 \]

for any \(\{x, y, z, t\} \in X^4\).

Cohomological definition
Taylor and Levingston \(^1\) showed that there is a one-to-one correspondence between two-graphs and antipodal 2-fold covers of complete graphs.

Such a cover, when it is also distance regular is called a *Taylor graph*, these are distance regular graphs with intersection array

\[ \{k, \mu, 1; 1, \mu, k\} . \]

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Consider the combinatorial definition of a regular two-graph:

**Definition**

A two-graph $(X, \Delta)$ is called *regular* if every 2-subset $\{x, y\} \in X^{[2]}$ is contained in the same number of triples from $\Delta$.

In the correspondence established by Taylor and Levingston, Taylor graphs correspond to regular two-graphs. This will be a particular case of our more general result to come.
Using the cohomological language, there are two natural ways to generalize two-graphs:

- $t$-cocycles into $U_2$.
- $3$-cocycles into $U_t$.

The first direction was examined a little bit by Mielants, Cameron. Also, D. Higman has a generalization of E. Schult’s graph extension theorem to $t$-graphs in this sense. We develop the second direction.
Two-graphs and regular two-graphs

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Cohomological definition

A few preliminary definitions and notations:

- A $p$-cochain into $U_t$ is a function $f : X^p \rightarrow U_t$ such that:
  - $f(x) = 1$ for every $x \in X^p$ in which $x_i = x_j$ for some $i \neq j$.
  - $f(x) = \sigma(f(y))$ whenever $y$ is the result of interchanging $x_i$ and $x_j$ for some $i \neq j$.

- 2-cochains are called weights (on $X$).

- The coboundary operator $\delta$ is a function from the set of $p$-cochains to the set of $(p + 1)$-cochains defined by:

$$\delta f(x_1, \ldots x_{p+1}) = \prod_{i=1}^{p+1} \sigma^i(f(\hat{x}_i))$$

where $\sigma$ is the inverse operation in $U_t$, and

$\hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{p+1})$. 
A $p$-cochain with *vanishing coboundary* (this means that $\delta f = 1$) is called a *$p$-cocycle*.

Every $p$-cocycle is the coboundary of a $(p - 1)$-cochain.

In particular, 3-cocycles are the coboundaries of weights.
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**Definition**

A *$t$-graph* is called *regular* if for every pair $x, y \in X$, the number of $z \in X \setminus \{x, y\}$ such that $f(x, y, z) = \alpha$ is constant. This number is denoted $m(\alpha)$. 
Remark

It is routine to check that in case $t = 2$, the above definitions are exactly those of two-graphs and regular two-graphs correspondingly.
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For regular 3-graphs, we have:

- $m(\zeta) = m(\zeta^2)$.
- Denote $a = m(1)$, $b = m(\zeta) = m(\zeta^2)$, then: $n = a + 2b + 2$ (recall that $n = |X|$).
Interpretation as a $t$-fold cover of $K_n$

Let $\Phi$ be a $t$-graph. Let $w : X^2 \rightarrow U_t$ be a weight on $X$ with values in $U_t$ such that $\Phi = \delta w$. Define the graph $\Gamma_w = (V, E)$, where:

- $V = X \times \{1, 2, \ldots, t\}$. For convenience, we denote the vertex $(x, i)$ by $x_i$.
- $\{x_i, y_j\} \in E \iff w(x, y) = \zeta^{j-i}$
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Clearly, $|V| = t \cdot n$, and the resulting graph is a labeled $t$-fold cover of the complete graph $K_n$. It turns out that this antipodal cover has the property that if $w(x, y) = \zeta^i$ then the set of edges between $x_1, x_2, \ldots, x_t$ and $y_1, y_2, \ldots, y_t$ form a perfect matching which is given by the $i^{th}$ power of the permutation matrix of $(1, 2, \ldots, t)$.
Two-graphs and regular two-graphs

t-graphs and regular t-graphs

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New constructions

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4. New constructions
In case $t = 3$, the corresponding graph $\Gamma_w$ is a 3-fold cover of $K_n$ with exactly 3 types of matchings between fibers:

\[
\begin{align*}
\text{Matchings between fibers of } \Gamma_w \\
\end{align*}
\]

1. $w(x, y) = 1$
2. $w(x, y) = \zeta$
3. $w(x, y) = \zeta^2$
Regular 3-graphs are antipodal distance regular covers

Consider the (combinatorial) definition of a regular 3-graph $\Phi$. Now consider the graph $\Gamma_w$ defined by some weight $w$ in the switching class of $\Phi$ ($\delta w = \Phi$). First we present our clarification of D. G. Higman’s initial result $^2$:

**Proposition**

$\Gamma_w$ is an antipodal distance regular cover of $K_n$ with parameters $(n, 3, b)$.

**Remark**

Not every ADRG gives a regular 3-graph as was the case with regular two-graphs. For example, the line graph of the Petersen graph is a $(5, 3, 1)$-cover which does not correspond to a 3-graph.

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Elements from the Godsil-Hensel paper

The main ingredients from the paper of Godsil and Hensel ³:

**Lemma (Lemma 3.1)**

An r-fold cover of $K_n$ is antipodal distance regular if and only if there exists a constant $c_2$ such that any two non-adjacent vertices from different fibers of the cover have exactly $c_2$ common neighbors.

**Definition**

The *voltage group* of $\Gamma$ is the subgroup $T \leq \text{Aut}(\Gamma)$ which stabilizes each fiber of $\Gamma$. When $T$ is cyclic of order $r$ (the size of a fiber), the graph $\Gamma$ is called a *cyclic cover*.

ADRG’s with cyclic matchings

Definition

Let $\Gamma = (V, E)$ be a labeled $(n, t, c_2)$-cover, i.e. such that $V = X \times \{1, 2, \ldots, t\}$. Let $\sigma = (1, 2, \ldots, t) \in S_t$. A cyclic matching in $\Gamma$ is a matching that corresponds to $\sigma^i$, for some $1 \leq i \leq t$.

Definition

A (not labeled) $(n, t, c_2)$-cover $\Gamma = (V, E)$ is said to be with cyclic matchings if there exists a labeling of $V$ such that:

- all the matchings of $\Gamma$ are cyclic, and
- every cyclic matching appears.
Remark

Note that in an \((n, 3, c_2)\)-cover with cyclic matchings, an edge between two fibers \(x, y \in X\) of \(\Gamma\) determines the matching between them. Explicitly, if \(\{x_i, y_j\} \in E\) then the only cyclic matching that satisfies this must also satisfy
\[
\{x_i-1 \pmod{3}, y_{j-1} \pmod{3}\}, \{x_{i+1} \pmod{3}, y_{j+1} \pmod{3}\} \in E.
\]
Using the above notions and analysing the induced subgraphs of $\Gamma_w$ on 3 fibers we proved:

**Proposition (K)**

Regular 3-graphs with parameters $(n, a, b)$ are equivalent to cyclic $(n, 3, b)$-covers (in the sense of Godsil and Hensel).
In his note, D. Higman considers the rank 4 metric association scheme (but not in the correct order!) of $\Gamma_w$. He then constructs a rank 6 fission of this scheme:

**Construction (Higman)**

Define the relations:

- $R_0 = \text{Id}_V$
- $R_1 = \{(x_i, x_{i+1} \pmod{3}) \mid i = 1, 2, 3, x \in X\}$
- $R_2 = \{(x_i, x_{i+2} \pmod{3}) \mid i = 1, 2, 3, x \in X\}$
- $R_3 = E$
- $R_4 = \{(x_i, y_j) \mid i = 1, 2, 3, \{x_{i+1} \pmod{3}, y_j\} \in E\}$
- $R_5 = \{(x_i, y_j) \mid i = 1, 2, 3, \{x_{i+2} \pmod{3}, y_j\} \in E\}$
Remark

- The relations $R_1$ and $R_2$ are a splitting of the distance 3 class (the antipodal fibers), and $R_1 = R_2^t$.
- The relations $R_4$ and $R_5$ are a splitting of the distance 2 class (the non-edges between fibers), and $R_4 = R_5^t$.

He then proves:

**Proposition (Higman)**

$A_6(\Gamma) := (V, \{R_i\}_{i=0}^5)$ is a commutative association scheme.

And this is a sufficient condition:

**Proposition (Higman)**

*Every rank 6 association scheme with parameters as in the above construction arises from a regular 3-graph.*
Combining the above results, we formulate the following new characterization of regular 3-graphs:

**Corollary**

Let $\Gamma$ be an $(n, 3, c_2)$-cover. The following are equivalent:

1. $\Gamma$ defines a regular 3-graph.
2. $\Gamma$ has cyclic matchings.
3. $\Gamma$ is a cyclic cover.
4. $A_6(\Gamma)$ is an association scheme.
Finding feasible parameter sets of regular 3-graphs

Three sources:

(i) the character-multiplicity table of $\text{A}_6(\Gamma)$,
(ii) eigenvalues and multiplicities of the weight,
(iii) Godsil-Hensel results about cyclic ADRG’s.
The following proposition is a correction and extension of the necessary conditions obtained by Higman in his note.

**Proposition**

**Necessary conditions for the set** \((n, a, b)\) **of parameters of a regular 3-graph are:**

(i) \(3|n\),

(ii) \(n = a + 2b + 2\),

(iii) The roots \(\alpha\) and \(\beta\) of the equation \(x^2 - (a - b)x - (n - 1) = 0\) are integers,

(iv) \(n - 1 + \alpha^2\) divides \(2n(n - 1)\),

(v) \(\alpha - \beta\) divides \(n\alpha\).
A corrected version of Higman’s feasibility conditions yields a list of 137 parameter sets with $n \leq 1000$. With our additional condition, this list boils down to a list of 64 parameter sets. We have constructions for 6 parameter sets, most of them are new examples of regular 3-graphs.
Higman \(^4\) presents several group theoretic examples of regular 3-cocycles. These examples are mainly extensions of examples by D. E. Taylor \(^5\). One of them provides an infinite family of regular \(t\)-graphs, called by Higman “the symplectic example”. With our new viewpoint of regular 3-graphs as cyclic covers, this family turns out to be the famous Thas-Somma construction. These were the only examples of regular 3-graphs known to Higman.


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The Klin-Pech construction

In their recent paper \(^6\), Klin and Pech provide a construction of cyclic \((n^2, 3, \frac{n^2}{3})\)-covers from a generalized Hadamard matrices of order \(n\). In addition, we used classifications of generalized Hadamard matrices with suitable parameters (due to V. Tonchev et al.) to construct all the corresponding non-isomorphic cyclic covers, which provide different regular 3-graphs. A summary of our new constructions of regular 3-graphs:

- 1 new example with parameters \((36, 10, 12)\),
- 1 new example with parameters \((45, 19, 12)\) (exceptional),
- 1 new example with parameters \((81, 25, 27)\),
- 1 new example with parameters \((144, 46, 48)\),
- 28 new examples with parameters \((324, 106, 108)\).

An ambitious goal is to understand (and then possibly generalize) the construction of Klin and Pech without the use of generalized Hadamard matrices. gH-matrices are equivalent to certain designs, thus we are trying to construct the ADRG in terms of these designs. For this purpose we use the language of flag algebras of designs, as it was developed by Pech in his MSc thesis (1998).
Thanks for your attention!