COMPLEX TENSOR POWER IS A SYMMETRIC MONOIDAL FUNCTOR

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1. Introduction

In [2], we defined the "complex tensor power" of a finite-dimensional split unital complex vector space (i.e. a vector space V with a distinguished non-zero vector $\mathbb{1}$ and a splitting $V \cong \mathbb{C}\mathbb{1} \oplus U$). This "complex tensor power" $V^{\underline{\otimes} t}$ of V is an *Ind*-object in the category $\underline{Rep}(S_t)$, and comes with an action of $\mathfrak{gl}(V)$ on it. It can be shown that this object does not depend on the choice of splitting, but only on the pair $(V, \mathbb{1})$.

The "t-th tensor power" of V is defined for any $t \in \mathbb{C}$, and for $n = t \in \mathbb{Z}_+$, the functor $\underline{Rep}(S_{t=n}) \to Rep(S_n)$ takes this *Ind*-object of $\underline{Rep}(S_{t=n})$ to the usual tensor power $V^{\otimes n}$ in $Rep(S_n)$. Moreover, the action of $\mathfrak{gl}(V)$ on the former object corresponds to the action of $\mathfrak{gl}(V)$ on $V^{\otimes n}$.

In this note, we want to show that the functor $(\cdot)^{\underline{\otimes} t}$ is a symmetric monoidal functor, i.e. that given two unital finite-dimensional vector spaces V, V', we have a natural isomorphism

$$(V \otimes V')^{\underline{\otimes} t} \cong V^{\underline{\otimes} t} \otimes V'^{\underline{\otimes} t}$$

The notations will be the same as in [2].

2. Properties of the complex tensor power

Let $\nu \in \mathbb{C}$.

Consider the following category **Uni** of unital vector spaces. The objects of this category will be tuples $(V, \mathbb{1}, U)$, where V is a finite-dimensional vector space, $\mathbb{1} \in V \setminus \{0\}$, and U is a subspace of V such that $V = \mathbb{C}\mathbb{1} \oplus U$.

The morphisms in this category are given by

$$Mor_{\mathbf{Uni}}\left((V, \mathbb{1}, U), (V', \mathbb{1}', U')\right) := \{\phi \in \mathrm{Hom}_{\mathbb{C}}(V, V') : \phi(\mathbb{1}) = \mathbb{1}'\}$$

Remark 2.0.1. This category has a natural structure of a symmetric monoidal tensor category, with

$$(V, \mathbb{1}, U) \otimes (V', \mathbb{1}', U') := (V \otimes V', \mathbb{1} \otimes \mathbb{1}', U \otimes \mathbb{C}\mathbb{1}' \oplus \mathbb{C}\mathbb{1} \otimes U' \oplus U \otimes U')$$

We now construct a functor

$$(\cdot)^{\underline{\otimes} \nu} : \mathbf{Uni} \longrightarrow Ind - \underline{Rep}(S_\nu)$$

On objects of **Uni**, this is just $(V, \mathbb{1}, U) \mapsto V^{\underline{\otimes} \nu} := \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k}$. On morphisms, this functor is

$$\phi : (V, \mathbb{1}, U) \to (V', \mathbb{1}', U') \mapsto \Phi : \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k} \to \bigoplus_{k \geq 0} (U'^{\otimes k} \otimes \Delta_k)^{S_k}$$

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with the matrix coefficients $\Phi^{l,k} : (U^{\otimes k} \otimes \Delta_k)^{S_k} \to (U'^{\otimes l} \otimes \Delta_l)^{S_l}$ of Φ coming from maps $U^{\otimes k} \otimes \Delta_k \to U'^{\otimes l} \otimes \Delta_l$ which are defined by the formula

$$\sum_{\substack{\iota:\{1,...,l\}\to\{1,...,k\} \\ \text{strictly increasing}}} \phi_{U,U'}^{(Im(\iota))} \otimes \phi_{U,\mathbb{1}'}^{(\{1,...,k\}\setminus Im(\iota))} \otimes \left(\Delta_k \overset{res_\iota}{\to} \Delta_l\right)$$

Here

- The map $\phi_{U,U'} : U \to U'$ is the composition
$$U \hookrightarrow V \overset{\phi}{\to} V' \twoheadrightarrow U'$$
The notation $\phi_{U,U'}^{(Im(\iota))}$ means that we apply the map $\phi_{U,U'}$ only to the factors $\iota(1), \iota(2)$, etc. of $U^{\otimes k}$.
- The map $\phi_{U,\mathbb{1}'} : U \to \mathbb{C}$ is defined so that the composition
$$U \hookrightarrow V \overset{\phi}{\to} V' \twoheadrightarrow \mathbb{C}\mathbb{1}'$$
is the map $u \mapsto \phi U, \mathbb{1}'\mathbb{1}'$. The notation $\phi_{U,\mathbb{1}'}^{(\{1,...,k\}\setminus Im(\iota))}$ means that we apply the map $\phi_{U,\mathbb{1}'}$ only to those factors i of $U^{\otimes k}$ for which $i \notin Im(\iota)$.
- The map res_ι is the map $\Delta_k \to \Delta_l$ given by the diagram $\pi \in \bar{P}_{k,l}$ which connects vertex i in the bottom row with vertes $\iota(i)$ in the top row.

Note that Φ is upper-triangular in terms of the matrix coefficients $\Phi^{l,k}$.

The following lemma is proved in [2, Section 6]:

Lemma 2.0.2. *The functor* $(\cdot)^{\underline{\otimes}\nu} : \mathbf{Uni} \longrightarrow Ind - \underline{Rep}(S_\nu)$ *is well-defined.*

We now prove that given two unital finite-dimensional vector spaces V, V', we have a natural isomorphism

$$(V \otimes V')^{\underline{\otimes}\nu} \cong V^{\underline{\otimes}\nu} \otimes V'^{\underline{\otimes}\nu}$$

Lemma 2.0.3. *The functor* $(\cdot)^{\underline{\otimes}\nu}$ *is a symmetric monoidal functor.*

Proof. Let $(V, \mathbb{1}, U), (V', \mathbb{1}', U') \in \mathbf{Uni}$. The canonical isomorphism of S_n representations

$$\Upsilon_n : V^{\otimes n} \otimes V'^{\otimes n} \longrightarrow (V \otimes V')^{\otimes n}$$

and its inverse Υ_n^{-1} can be rewritten using the isomorphism of $\mathbb{C}[S_n]\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{gl}(V))$-modules

$$V^{\otimes n} \xrightarrow{\sim} \bigoplus_{k=0,...,n} (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

which was defined in [2, Section 6].

The isomorphisms $\Upsilon_n, \Upsilon_n^{-1}$ then interpolate to morphisms in $Ind - \underline{Rep}(S_\nu)$:

$$\underline{\Upsilon}_\nu : V^{\underline{\otimes}\nu} \otimes V'^{\underline{\otimes}\nu} \longrightarrow (V \otimes V')^{\underline{\otimes}\nu}$$

and

$$\underline{\Upsilon}'_\nu : (V \otimes V')^{\underline{\otimes}\nu} \longrightarrow V^{\underline{\otimes}\nu} \otimes V'^{\underline{\otimes}\nu}$$

Below we show how this interpolation is done, and that $\underline{\Upsilon}_\nu \circ \underline{\Upsilon}'_\nu = \text{Id}$, $\underline{\Upsilon}'_\nu \circ \underline{\Upsilon}_\nu = \text{Id}$.

Recall that a patchwork of sets $\{1,...,k\}, \{1,...l\}$ of size s is a pair of injective maps $\iota_k : \{1,...,k\} \hookrightarrow \{1,...,s\}, \iota_l : \{1,...,l\} \hookrightarrow \{1,...,s\}$ such that $Im(\iota_k) \cup Im(\iota_l) = \{1,...,s\}$ (c.f. [1, Section 2]). Denote by $G^s_{k,l}$ the set of all patchworks of sets $\{1,...,k\}, \{1,...l\}$ of size s.

The following statement was proved in [1, 2.16]:

$$\Delta_k \otimes \Delta_l \cong \bigoplus_{s\geq 0} \Delta_s \otimes \mathbb{C}G^s_{k,l}$$

Also,

$$\bigoplus_{k,l\geq 0} U^{\otimes k}\otimes U'^{\otimes l}\otimes \mathbb{C}G^s_{k,l} \cong (\mathbb{C}\mathbb{1}\otimes U' \oplus U\otimes \mathbb{C}\mathbb{1}' \oplus U\otimes U')^{\otimes s}$$

Consider maps

$$\epsilon^s_{k,l} : \Delta_k \otimes \Delta_l \to \Delta_s \otimes \mathbb{C}G^s_{k,l}, \ \ \tilde{\epsilon}^s_{k,l} : \Delta_s \otimes \mathbb{C}G^s_{k,l} \to \Delta_k \otimes \Delta_l$$

(notice that $\epsilon^{s'}_{k,l}\circ \tilde{\epsilon}^s_{k,l} = \delta_{s,s'}$ for any $s, s' \geq 0$),

$$\mu^s_{k,l} : U^{\otimes k}\otimes U'^{\otimes l}\otimes \mathbb{C}G^s_{k,l} \longrightarrow (\mathbb{C}\mathbb{1}\otimes U' \oplus U\otimes \mathbb{C}\mathbb{1}' \oplus U\otimes U')^{\otimes s}$$

and

$$\tilde{\mu}^s_{k,l} : (\mathbb{C}\mathbb{1}\otimes U' \oplus U\otimes \mathbb{C}\mathbb{1}' \oplus U\otimes U')^{\otimes s} \longrightarrow U^{\otimes k}\otimes U'^{\otimes l}\otimes \mathbb{C}G^s_{k,l}$$

(so $\tilde{\mu}^s_{k',l'}\circ \mu^s_{k,l} = \delta_{k,k'}\delta_{l,l'}$ for any $k, k', l, l' \geq 0$).

Recall that

$$V^{\underline{\otimes}\nu}\otimes V'^{\underline{\otimes}\nu} \cong \bigoplus_{k,l\geq 0} (\Delta_k \otimes U^{\otimes k})^{S_k} \otimes (\Delta_l \otimes U'^{\otimes l})^{S_l}$$

and

$$(V\otimes V')^{\underline{\otimes}\nu} = \bigoplus_{s\geq 0}\left(\Delta_s \otimes (\mathbb{C}\mathbb{1}\otimes U' \oplus U\otimes \mathbb{C}\mathbb{1}' \oplus U\otimes U')^{\otimes s}\right)^{S_s}$$

Now, consider the map

$$\Delta_k\otimes\Delta_l\otimes U^{\otimes k}\otimes U'^{\otimes l} \longrightarrow \Delta_s \otimes (\mathbb{C}\mathbb{1}\otimes U' \oplus U\otimes \mathbb{C}\mathbb{1}' \oplus U\otimes U')^{\otimes s}$$

defined as the composition $(\mathrm{Id}_{\Delta_s}\otimes \mu^s_{k,l})\circ(\epsilon^s_{k,l}\otimes \mathrm{Id}_{U^{\otimes k}\otimes U'^{\otimes l}})$.

The map

$$\underline{\Upsilon}_\nu : V^{\underline{\otimes}\nu}\otimes V'^{\underline{\otimes}\nu} \longrightarrow (V\otimes V')^{\underline{\otimes}\nu}$$

is then given by the matrix coefficients

$$\underline{\Upsilon}^s_{\nu,k,l} := e_{S_s}\circ (\mathrm{Id}_{\Delta_s}\otimes \mu^s_{k,l})\circ(\epsilon^s_{k,l}\otimes \mathrm{Id}_{U^{\otimes k}\otimes U'^{\otimes l}})\circ(e_{S_k}\otimes e_{S_l})$$

where e_{S_k} is the symmetrizer with respect to the action of S_k (similarly for S_s, S_l).

The map

$$\underline{\Upsilon}'_\nu : (V\otimes V')^{\underline{\otimes}\nu} \longrightarrow V^{\underline{\otimes}\nu}\otimes V'^{\underline{\otimes}\nu}$$

is similarly given by the matrix coefficients

$$\underline{\Upsilon}'^s_{\nu,k,l} := (e_{S_k}\otimes e_{S_l})\circ(\tilde{\epsilon}^s_{k,l}\otimes \mathrm{Id}_{U^{\otimes k}\otimes U'^{\otimes l}})\circ(\mathrm{Id}_{\Delta_s}\otimes\tilde{\mu}^s_{k,l})\circ e_{S_s}$$

Now, for any $k, k', l, l' \geq 0$, we have:

$$\sum_{s\geq 0}\underline{\Upsilon}'^s_{\nu,k',l'}\circ\underline{\Upsilon}^s_{\nu,k,l} = \delta_{k,k'}\delta_{l,l'}\sum_{s\geq 0}(e_{S_k}\otimes e_{S_l})\circ\tilde{\epsilon}^s_{k,l}\circ\epsilon^s_{k,l}\circ(e_{S_k}\otimes e_{S_l}) =$$
$$= \delta_{k,k'}\delta_{l,l'}\,\mathrm{Id}_{(\Delta_k\otimes U^{\otimes k})^{S_k}\otimes(\Delta_l\otimes U'^{\otimes l})^{S_l}}$$

and similarly for any $s, s' \geq 0$, we have:

$$\sum_{k,l\geq 0}\underline{\Upsilon}^{s'}_{\nu,k,l}\circ\underline{\Upsilon}'^s_{\nu,k,l} = \delta_{s,s'}\sum_{k,l\geq 0} e_{S_s}\circ\mu^s_{k,l}\circ\tilde{\mu}^s_{k,l}e_{S_s} =$$
$$= \delta_{s,s'}\,\mathrm{Id}_{(\Delta_s\otimes(\mathbb{C}\mathbb{1}\otimes U'\oplus U\otimes\mathbb{C}\mathbb{1}'\oplus U\otimes U')^{\otimes s})^{S_s}}$$

Thus we showed that $\underline{\Upsilon}_\nu\circ\underline{\Upsilon}'_\nu = \mathrm{Id}$, $\underline{\Upsilon}'_\nu\circ\underline{\Upsilon}_\nu = \mathrm{Id}$, and the statement of the lemma follows easily from here. □

References

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