$\boxed{\text{Rep}(GL_t)}$

[0] Introduction

We will discuss the construction of Deligne's category $\text{Rep}(GL_t)$. This category is defined for any $t \in \mathbb{C}$, and these categories form a family which interpolates the categories $\underline{\text{Rep}}(GL_d)$ for $d \in \mathbb{N} = \mathbb{Z}_+$ ($\underline{\text{Rep}}(GL_d)$ being the category of finite dimensional polynomial (algebraic) representations$/\mathbb{C}$ of GL_d defined over $\mathbb{C}$). These categories $\text{Rep}(GL_t)$ are pseudo-abelian, symmetric monoidal rigid $\otimes$ categories, which are semisimple (abelian) for $t \notin \mathbb{Z}$.

$\text{Rep}(GL_t)$ is not abelian for $t \in \mathbb{Z}$ ($t = d$), but has $\underline{\text{Rep}}(GL_d)$ and $\underline{\text{Rep}}(gl(m|n))$, $m-n=d$, as quotients.

$\underline{\text{Rep}}$: f.d. representations of $gl(m|n)$

Note that even for $t \in \mathbb{Z}$, $\text{Rep}(GL_t)$ can be embedded into an abelian tensor category (this will be discussed in a future lecture).

[1] Construction of $Rep(GL_t)$

We will now construct $Rep(GL_t)$ for arbitrary $t \in \mathbb{C}$.
Similarly to the construction of $Rep(S_t)$, we have the following "plan":

$$Rep_0 \xrightarrow[\text{add formal finite direct sums}]{} Rep_1 \xrightarrow[\text{add images of idempotents}]{} Rep$$

Rep_0: $\mathbb{C}$-linear symm. monoidal category

Rep_1: additive cat.

Rep: pseudo-abelian category

We will start with the idea behind the construction.

__Prop:__ Every f.d. polynomial representation of $GL(V)$ is completely reducible and is ~~a subquotient of~~ contained in
$\bigoplus_{(r_i,s_i)} V^{\otimes r_i} \otimes V^{*\otimes s_i}$ for some $r, s \in \mathbb{Z}$.

__Proof__: It is easy to see that any polynomial representation can be embedded into $R^m := \mathcal{O}(GL(V))^{\oplus m}$
(ring of poly. functions on $GL(V)$)

Now let $Y \subset R^m$ be a subrepr.
An element of R is a polynomial in e_{ij}, $1/\det$
$\Rightarrow R$ is a quotient of $S^r(V \otimes V^*) \otimes (\Lambda^{\dim V} V^*)^{\otimes s}$
Since $V^* \cong \Lambda^{\dim V - 1} V \otimes \Lambda^{\dim V} V^*$, get that
Y is contained in $\bigoplus_i V^{\otimes r_i} \otimes (\Lambda^{\dim V} V^*)^{\otimes s_i}$ for some r_i, s_i.

It remains to check that $V^{\otimes r_i}$ are completely reducible (follows from Schur-Weyl duality), and that if L_λ is an irr. repr. of $GL(V)$, so is
$L_\lambda \otimes (\Lambda^{\dim V}(V^*))^{\otimes s} := L_{\lambda - (\overbrace{s,s,\dots,s}^{\dim V \text{ times}})}$

This means that if we start with objects of the form $V^{\otimes r} \otimes V^{*\otimes s}$, add formal finite direct sums and then add images of all idempotents, we can hope to obtain all f.d. representations of $GL(V)$.

So we define:

<u>$\mathcal{R}ep_0$</u>: <u>obj</u>: $[(r,s)]$, $r,s \in \mathbb{Z}_+$

What are $GL(V)$-intertwining morphisms between $V^{\otimes r_1} \otimes V^{*\otimes s_1}$ and $V^{\otimes r_2} \otimes V^{*\otimes s_2}$

We have: $\mathrm{Hom}_{GL(V)}(V^{\otimes r_1} \otimes V^{*\otimes s_1}, V^{\otimes r_2} \otimes V^{*\otimes s_2}) =$

$= \mathrm{Hom}_{GL(V)}(V^{\otimes r_1+s_2}, V^{\otimes r_2+s_1})$

A theorem of invariant theory (or Schur-Weyl duality between $GL(V)$ and S_n) tells us that this hom-space is 0 <u>unless</u> $r_1+s_2 = r_2+s_1$, if it is spanned by $\sigma \in S_{r_1+s_2}$

$\Big(\sigma : V^{\otimes r_1+s_2} \to V^{\otimes r_2+s_1} \cong V^{\otimes r_1+s_2}$

$\sigma(v_1 \otimes \cdots \otimes v_{r_1+s_2}) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r_1+s_2)}\Big)$

and these $\sigma \in S_{r_1+s_2}$ form a basis of the above Hom-space if $\dim V \geq r_1+s_2$

So we put

<u>$\mathrm{Mor}(\mathrm{Rep}_0)$</u>: $\mathrm{Hom}_{\mathrm{Rep}_0}([r_1,s_1],[r_2,s_2]) := \begin{cases} 0 & \text{if } r_1+s_2 \neq r_2+s_1 \\ \mathbb{C}S_{r_1+s_2} & \text{else} \end{cases}$

We give the following graphic representation of objects and morphisms:

- an object $[r,s]$ is denoted by r black circles and s ~~whi~~ white circles ~~Ma~~

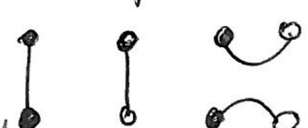
  <u>Example</u>: $[3,2]$: $\bullet\ \bullet\ \bullet\ \circ\ \circ$ (the order doesn't matter)

- a morphism $[r_1,s_1] \xrightarrow{\sigma} [r_2,s_2]$ $(\sigma \in S_{r_1+s_2})$ is denoted by a ~~[illegible]~~ graph whose vertices are written in two rows:

  $[r_1,s_1]$ $\bullet\ \bullet\ \bullet \cdots \bullet\ \circ \ldots \circ$

  $[r_2,s_2]$ $\bullet\ \bullet\ \bullet \cdots \bullet\ \circ \ldots \circ$

  and ~~which~~ the graph itself is a disjoint union of edges of the following forms:

  as in the construction of $\mathrm{Rep}(S_t)$, the edges themselves are of no interest just the partition of the vertices into conn. comp.

Correspondence between elements $\sigma \in S_{r_1+s_2} = S_{r_2+s_1}$ and diagrams:
(<u>idea behind this</u>: each $\bullet$ stands for a factor V in the object $[r,s] \leftrightarrow V^{\otimes r} \otimes V^{*\otimes s}$ and each point $\circ$ stands for a factor V^*). Take the two rows

(Step 1)

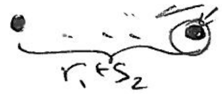
$\bullet \cdots \bullet\ \circ \ldots \circ$

$\bullet \cdots \bullet\ \circ \ldots \circ$

~~Hmms~~ (saying $\mathrm{Hom}_{GL(V)}(V^{\otimes r_1} \otimes V^{*\otimes s_1}, V^{\otimes r_2} \otimes V^{*\otimes s_2}) \cong$

$\cong \mathrm{Hom}_{GL(V)}\left(V^{\otimes r_1+s_2}, V^{\otimes r_2+s_1}\right)$

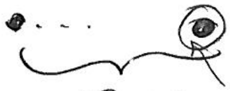
means: we get a new diagram, equivalent to the old one)



came from 2nd row

$\underbrace{}_{r_1+s_2}$ ← contains only black points!

Step 2:

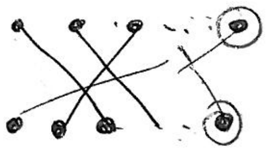


$\underbrace{}_{r_2+s_1}$ came from first row

obtained by moving ∘ points the the other row.

Now draw a diagram corresponding to σ ($r_1+s_2=r_2+s_1$)

Step 3:



Step 4:

We now want to go back to our original diagram, so we move back to the other row every pt. which was moved before that:

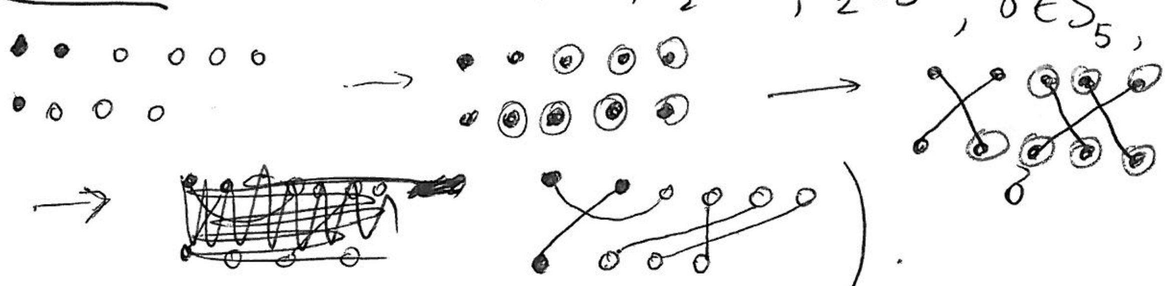
Get:

→  ← if both vertices weren't moved

→  if both vertices were moved (i.e. if it was  at prev. step)

→  or  if one vertex was moved (i.e. if it was  or  at prev. step)

Example: $r_1=2, s_1=4, r_2=1, s_2=3$, $\sigma\in S_5$, $\sigma=(12)(345)$



The isom.

$$\begin{array}{cc} \overline{\varphi} & \mathrm{Hom}_{GL(V)}\left(V^{\otimes r_1}\otimes V^{*\otimes s_1},\ V^{\otimes r_2}\otimes V^{*\otimes s_2}\right) \\ \updownarrow & \cong \\ \varphi & \mathrm{End}_{GL(V)}\left(V^{\otimes r_1+s_2}\right) \end{array} \qquad r_1+s_2=r_2+s_1$$

can be seen as follows: take the 2-colored diagram corresp. to $\sigma\in S_{r_1+s_2}$. ~~Write~~

Then $\overline{\sigma} = (Id_V\otimes\dots\otimes Id_V\otimes ev_V\otimes\dots\otimes ev_V\otimes$

$\otimes\, coev_V\otimes\dots\otimes coev_V\otimes Id_{V^*}\otimes\dots\otimes Id_{V^*})$

where each Id_V corresp. to an edge
each Id_{V^*} corresp. to an edge
each ev_V corresp. to an edge or
each $coev_V$ corr. to an edge or

This explains the correspondence between diagrams as above and elements of the group algebra $\mathbb{C}S_{r_1+s_2}$.

We now need to define composition of morphisms.

~~Let $\sigma \in S_{r_1+s_2}$, $\sigma' \in S_{r_2+s_3}$ $[r_1, s_1] \xrightarrow{\sigma} [r_2, s_2] \xrightarrow{\sigma'} [r_3, s_3]$.~~

~~Claim: $\sigma' \sigma =$~~

Let $\{(r_i, s_i)\}_{i=1}^{3}$, $\sigma \in S_{r_1+s_2}$, $\sigma' \in S_{r_2+s_3}$

$r_i - s_i \equiv \text{const}$

Then $V^{\otimes r_1} \otimes V^{*\otimes r_2} \xrightarrow{\sigma} V^{\otimes r_2} \otimes V^{*\otimes s_2} \xrightarrow{\sigma'} V^{\otimes r_3} \otimes V^{*\otimes s_3}$

Claim: $\sigma' \circ \sigma = (\dim V)^{\ell(\sigma', \sigma)} \sigma * \sigma'$

Expl.: here $\ell(\sigma', \sigma)$, $\sigma * \sigma'$ are obtained as follows:

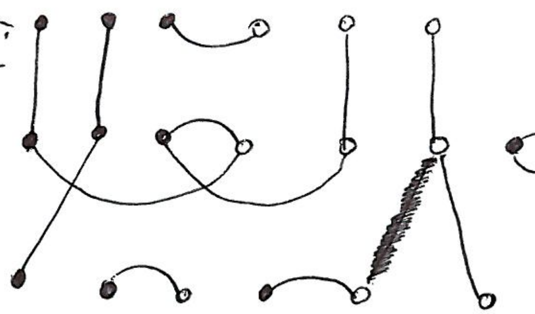
Consider the glued picture

(lower row of σ is identified w/ upper row of σ')

Then $\to$ $\sigma * \sigma'$ is obtained by removing all connected components of the glued picture which lie *entirely* in the middle row ~~and considering only the connected components~~ and replacing any path by a single edge

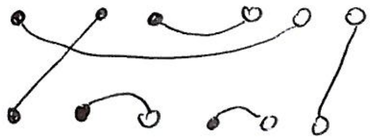
$\to$ $\ell(\sigma', \sigma)$ is the number of conn. comp. of the glued picture lying entirely in the middle row.

example:



σ

σ'

then $\sigma' * \sigma =$  $, \ell(\sigma', \sigma) = 1$

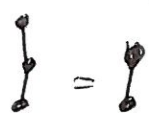
example: $Id_{[r,s]} =$  $\Longleftrightarrow 1 \in \mathbb{C}S_{r+s}$

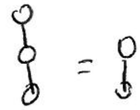
Proof of claim:

This is straightforward from our description of ~~our~~

$$\mathbb{C}S_{r_1+s_2 = r_2+s_1} \twoheadrightarrow Hom_{GL(V)}(V^{\otimes r_1} \otimes V^{* \otimes s_1}, V^{\otimes r_2} \otimes V^{* \otimes s_2})$$

(just need to understand that

$Id_V \circ Id_V = Id_V$ 

$Id_{V^*} \circ Id_{V^*} = Id_{V^*}$ 

$ev_V \circ coev_V = \dim V \cdot Id_{\mathbb{1}}$

any other factors of forms Id_V and Id_{V^*} won't matter...).

So we define composition of morphisms by the formula

$$\boxed{\gamma' \circ \gamma := t^{\ell(\gamma',\gamma)}\, \gamma * \gamma'}$$

This makes $\mathrm{Rep}_0(GL_t)$ a $\mathbb{C}$-linear category, and we have seen that its structure is compatible w/ the structure of the full subcat. of $\underline{\mathrm{Rep}}(GL_n)$ generated by objects of the form $V^{\otimes r} \otimes V^{*\otimes s}$, $V \cong \mathbb{C}^n$ - taut. repr.
Compatible in the sense that $\exists$ a functor

$$\mathrm{Rep}_0(GL_{t=n}) \xrightarrow{F_n} \text{full subcat. of } \underline{\mathrm{Rep}}(GL_n) \text{ described above}$$

Rmk: ~~this is a~~ this functor is, as we have seen, surjective on Hom-spaces.

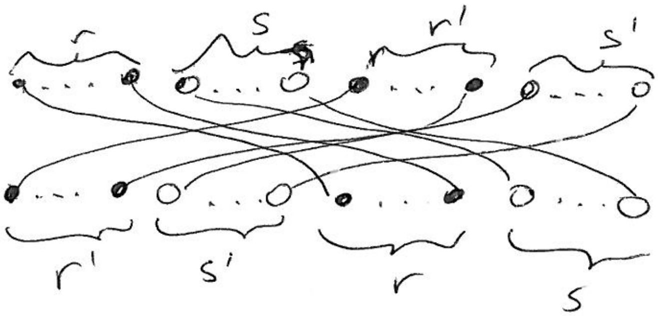
We now define a $\otimes$ structure on $\mathrm{Rep}_0(GL_t)$
(We want F_n to become a $\otimes$ functor).

The $\otimes$ structure is as follows:

- $([r,s]) \otimes ([r',s']) := [r+r', s+s']$
- morphisms: $\sigma \otimes \sigma'$ is given by

$$\underbrace{\begin{pmatrix}\text{diagram} \\ \sigma\end{pmatrix} \begin{pmatrix}\text{diagram} \\ \sigma'\end{pmatrix}}_{\text{write one diagram next to the other}}$$

$\mapsto$ symmetry:

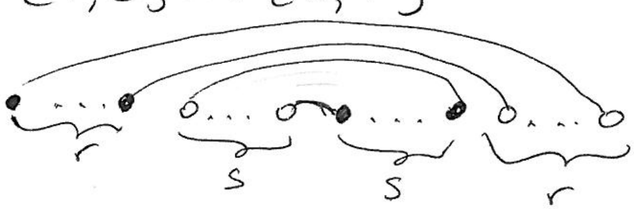


$\mapsto$ monoidal: $\mathbb{1} := [0,0]$ (denoted by empty row)
(obvious)

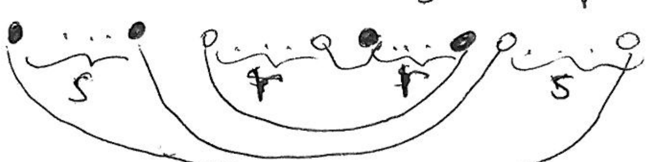
$\mapsto$ associativity – easy to check

$\mapsto$ dual: $[r,s]^* := [s,r]$

coev:



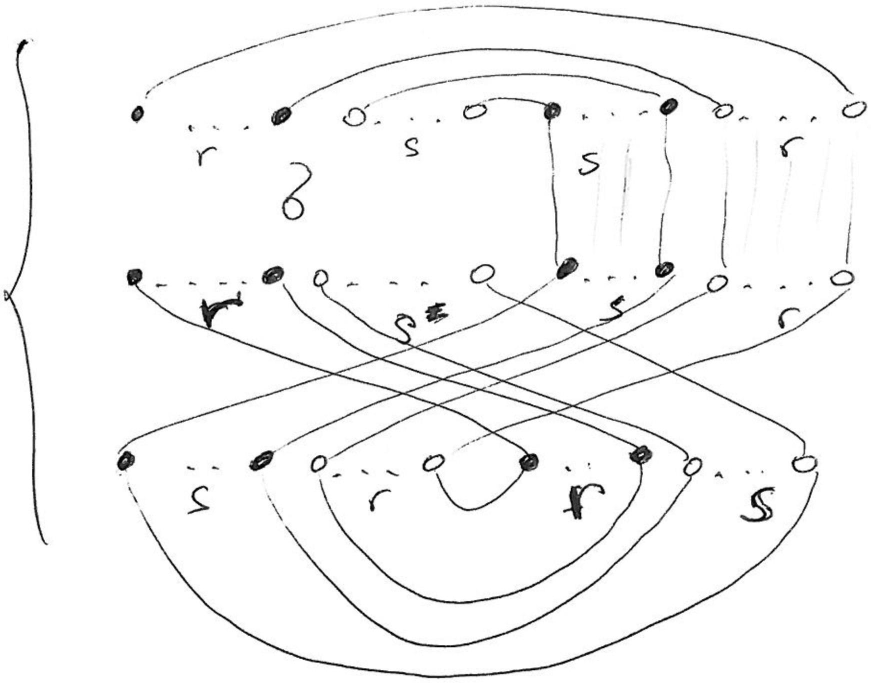
ev:



Rmk: Easy to check that this makes $\mathcal{F}_n$ a $\otimes$ functor.

Can now see what is $\mathrm{tr}(\delta)$ for $\delta \in \mathrm{Hom}_{Rep(GL_t)}([r,s],[r,s])$

$\mathrm{tr}\,\delta$



$\Rightarrow$ $\mathrm{tr}(\delta) = \cancel{\quad} \; t^{\#\text{connected components } \cancel{\text{of}} \text{ diag}}$ 

In particular, for $\delta = \mathrm{Id}_{[r,s]}$, get:

$\dim([r,s]) = \mathrm{tr}(\mathrm{Id}_{[r,s]}) = t^{r+s}$.

$Rep_1(GL_t)$: This is the category defined as

Obj: formal finite direct sums of obj. from Rep_0
$\left(\bigoplus_{i \in I} A_i \right.$, $\left. A_i \in Rep_0 \right)$
(finite set)

Mor: $\mathrm{Hom}_{Rep_1}\left(\bigoplus_{i\in I} A_i, \bigoplus_{j \in J} B_j\right) := \bigoplus_{i,j} \mathrm{Hom}_{Rep_0}(A_i, B_j)$
(considered as ~~___~~ matrix spaces)

Comp: Composition ~~___~~ = mult. of matrices

Tensor structure: obvious (has to be distributive).

$\underline{Rep(GL_t)}$: (Karoubian envelope of Rep_1)

$\underline{Obj}$: (A, e), $A \in Rep_1(GL_t)$, $e \in End_{Rep_1}(A)$ - idempotent

$\underline{Mor}$: $\operatorname{Hom}_{Rep}((A,e),(B,f)) := f \operatorname{Hom}_{Rep_1}(A,B)e$

$\underline{Comp}$: obvious

Theory of Karoubian envelopes gives:

1) Tensor structure (the obvious one)
2) Indecomp. objects (A,e) / isom. $\longleftrightarrow$ (primitive idempotents up to conj), $A = [r,s]$, e - prim. idemp. in $End([r,s])$
3) Krull-Schmidt property: Objects can be decomposed as finite direct sums of indecomposables in a unique way (up to change of order of summands).

$\boxed{3}$ Functor to $\underline{Rep}(GL_d)$

~~We now define the functor~~

Let $d \in \mathbb{Z}_+$.

Consider the $\otimes$-functor

$$\underline{Rep}(GL_{t=d}) \xrightarrow{F_d} \underline{Rep}(GL_d)$$

It is defined since, as we have seen, it is defined on the full subcategory $Rep_0(GL_{t=d})$ of $Rep(GL_{t=d})$, and $Rep(GL_{t=d})$ is the Karoubian envelope of Rep_0.

As in the case of $Rep(S_t)$, this functor:

- takes $[r,s]$ to $V^{\otimes r} \otimes V^{*\otimes s}$ (V - taut. repr. of GL_d)
- surjective on Hom-spaces
- Is equivalent to the functor

$$Rep(GL_{t=d}) \twoheadrightarrow Rep(GL_{t=d})/\mathcal{N}$$

where $\mathcal{N}$ is the tensor ideal of negligible morphisms (i.e.

$$\mathcal{N}(X,Y) = \{f \in Hom(X,Y) \mid \forall g : Y \to X,\ tr(f \circ g) = 0\}$$

. It is easy to check that if $f, f' \in \mathcal{N}$, then

- $f \circ f' \in \mathcal{N}$ (if defined), same for $f+f'$
- $f \otimes g \in \mathcal{N}$ $\forall$ morphism g

  $g \otimes f \in \mathcal{N}$

Reminder: In a s-s category there are no negligible morphisms, since $\begin{pmatrix}\text{semisimplicity}\\ \text{of cat}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{tr defines a non-deg.}\\ \text{pairing}\end{pmatrix}$

So $\underset{f \in \mathcal{N}(X,Y)}{\mathcal{F}_n(f)} = 0$.

It is is also easy to see that if $\mathcal{F}_n(f)=0$, then $\underset{\forall\, g: Y \to X}{\mathcal{F}_n(f \circ g) = 0} \Rightarrow tr(f\circ g)=0$, since $\mathcal{F}_n$, being a $\otimes$ funct, preserves traces. So f is negligible.

Note that unlike the case of $Rep(S_t)$, we have a similar construction for $t = d \in \mathbb{Z}_{<0}$.

Indeed, consider the category of representations of $GL_{|d|}$ with $\mathbb{Z}_2$-gradation given by the action of $\begin{pmatrix}-1 & & \\ & \ddots & \\ & & -1\end{pmatrix} \in GL_{|d|}$ (in particular, the taut. repr. is purely odd).

Then we have

$$Rep(GL_{t=d}) \xrightarrow{\mathcal{F}_d} \underline{Rep}(GL_{|d|}, \mathbb{Z}_2\text{-graded})$$

~~which~~ which induces an equivalence of categories

$$Rep(GL_{t=d}) \Big/ \underset{\text{negl. morphisms}}{\mathcal{N}} \xrightarrow[\sim]{\mathcal{F}_d} \underline{Rep}(GL_{|d|}, \mathbb{Z}_2\text{-graded})$$

We also have a functor

$$\mathrm{Rep}(GL_{t=m-n}) \xrightarrow{\overline{\mathcal{F}}_{m,n}} \underline{\mathrm{Rep}}\,(gl(m|n))$$

category of f.d. (integrable) modules over the superalgebra $gl(m|n)$

(this is a $\otimes$ category, with $\mathbb{1}$ = purely even 1-dim repr.
a taut. repr. $V \cong \mathbb{C}^m \oplus \mathbb{C}^n$
__dual__: V^* is dual of V w/
$(a \cdot f)(u) = -(-1)^{p(a)p(f)} f(a.u)$
$a \in gl(m|n)$
$f \in V^*$
this category is __not__ semisimple if $m>0$ or $n>0$)

Similarly to the functors $\mathcal{F}_{d}$, this functor $\overline{\mathcal{F}}_{m,n}$ is full, since it is known that

$$\mathrm{Hom}_{gl(m|n)}(V^{\otimes r_1} \otimes V^{*\otimes s_1}, V^{\otimes r_2} \otimes V^{*\otimes s_2}) = \begin{cases} 0 & \text{if } r_1+s_2 \neq r_2+s_1 \\ \text{quotient of } \mathbb{C}S_{r_1+s_2} & \text{else} \end{cases}$$

~~The case of $r_1+s_2 \neq r_2+s_1$ can be checked using the central element $x \cdot \mathrm{Id} \in gl(m|n)$, $x \in \mathbb{C}$, acts on $V^{\otimes r} \otimes V^{*\otimes s}$ by scalar x^{r-s}).~~

($r_1+s_2 \neq r_2+s_1$ case: Can be seen by taking a purely odd central element $\begin{pmatrix} \ddots & 0 \\ 0 & x \ddots \end{pmatrix}$ $x \in \mathbb{C}\setminus\{1,0\}$ and considering its action on $V^{\otimes r} \otimes V^{*\otimes s}$)

Rmk: All of these functors $\mathcal{F}_d$, $d \in \mathbb{Z}$, $\overline{\mathcal{F}_{m,n}}$, $m,n \in \mathbb{Z}_{\geq 0}$ are special cases of the following universal property of $Rep(GL_t)$:

Prop: Let $t \in \mathbb{C}$. Let T be a tensor category which is Karoubian, and let $T_t :=$ t-dim. objects admitting a dual & their isom.

Then $Hom^{\otimes}_{strict}(Rep(GL_t), T) \underset{\text{eq. of categories}}{\xrightarrow[\sim]{\Phi}} T_t$

(strict) tensor functors + monoidal transformations

$$F \longmapsto F([1,0])$$
$$(F_1 \xrightarrow{\eta} F_2) \longmapsto \eta([1,0])$$

~~The proof of this prop. is as follows:~~

~~This proposition follows fro~~

Proof of proposition:

Properties of Karoubi envelope imply that

$$Hom^{\otimes}(Rep(GL_t), T) \xrightarrow{\sim} Hom^{\otimes}(Rep_0(GL_t), T)$$

So we only need to check that the latter is equivalent to T_t.

Indeed, given a functor $F : Rep_0(GL_t) \to T$, $\dim F([1,0]) = t$, $F([0,1])$ is dual to $F([1,0])$, and so $F([1,0]) \in T_t$. ~~Easy to check that~~ Φ is also well defined on morphisms (obvious).

To show that Φ is essentially surjective, note that given an object $X \in T_t$, X defines a $\otimes$functor

$$Rep_0(GL_t) \xrightarrow{F_X} T$$
$$[r,s] \longmapsto X^{\otimes r} \otimes X^{*\otimes s}$$

~~and this functor~~ To check that

can be ~~well~~ defined on morphisms, ~~note that~~ it is in fact enough to check that $F_X(ev_{[1,0]}, coev_{[1,0]}, Id_{[0,1]}, Id_{[1,0]})$ are well-defined (since other diagrams are $\otimes$ products of diagrams corresp. to these morphisms). ~~But these~~

But we know how to define the ~~image~~ images under F_X of these (ev_X, $coev_X$, etc), and in fact, this is the <u>only</u> ~~way~~ possible images. So F_X ~~is unique~~ can be defined, and there is only one way to define it.

<u>Why Φ is faithful?</u> If $\eta: F_1 \to F_2$, then η is completely determined by $\eta([1,0])$, $\eta([0,1])$. But, in fact, $\eta([0,1]) = (\eta([1,0])^{-1})^*$, so η is determined by $\eta([1,0]) = \Phi(\eta)$. Thus Φ is faithful.

<u>Why Φ is full?</u> Similarly, given $\varphi: X \to Y$ in T_t (φ - isom.) we can put $\eta([1,0]) := \varphi : F_X([1,0]) \to F_Y([1,0])$, and this will determine η completely ($\eta: F_X \to F_Y$) (one needs to check some axioms, of course).

<u>Rmk:</u> Comes and Wilson proved a useful criterion for determining whether $F: Rep(GL_t) \to T$ as above is full. The criterion says that a sufficient condition for this to happen is that for $X := F([0,1])$, we have:

$$Hom_T(X^{\otimes r_1} \otimes X^{*\otimes s_1}, X^{\otimes r_2} \otimes X^{*\otimes s_2}) = \begin{cases} 0 & \text{if } r_1+s_2 \neq r_2+s_1 \\ \text{quotient of } \mathbb{C}S_{r_1+s_2} & \text{else} \end{cases}$$

Remark:

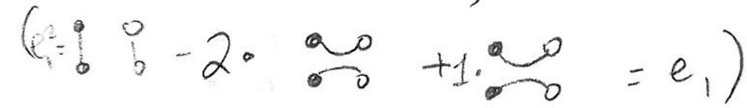
It was shown that $\mathbb{C}B_{r,s}(t)$ are semisimple algebras for $t \notin \mathbb{Z} \Rightarrow Rep(GL_t)$ is a semisimple (abelian) category for $t \notin \mathbb{Z}$.

In the case when $t \in \mathbb{Z}$, it is certainly non-s.s., since it contains negligible morphisms (which are not allowed in semi-simple categories).

Example: • $t=0$, $r,s=1$. Then $Id_{[r,s]}$ is a negligible morphism.

• $t=1$, $r=s=1$. Then take $e_1 :=$ 

$tr(e_1)$ is 0 $(1^2 - 1^1 = 0)$, and e_1 is an idempotent



Rmk: For $t \in \mathbb{Z}$, the category $Rep(GL_t)$ is not semisimple, in particular, it has tensor ideals $\mathcal{J}(m|n)$ s.t. $m,n \in \mathbb{Z}_{\geq 0}, m-n=t$

$C_{m,n} := Rep(GL_t) / \mathcal{J}(m|n) \xhookrightarrow{\text{fully faithful functor}} Rep(gl(m|n))$.

It was proved by J. Comes that those $\mathcal{J}(m|n)$ are, in fact, the only tensor ideals in $Rep(GL_t)$, so $C_{m,n}$ ~~Rep(gl(m|n))~~ are the only possible quotients of $Rep(GL_t)$ by a tensor ideal (up to equiv. of categories).