On Definability of Types in Dependent Theories

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Outline

φ-Types

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Example.

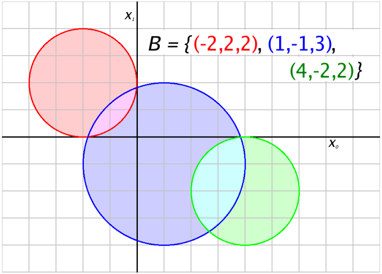
Consider $T = \mathrm{Th}(\mathbb{R}; +, \cdot, <, 0, 1)$ and let

$$\varphi(x_0, x_1; y_0, y_1, y_2) = \left[(x_0 - y_0)^2 + (x_1 - y_1)^2 \leq y_2^2\right].$$

A set $B \subseteq \mathbb{R}^3$ gives a list of parameters corresponding to closed disks in $\mathbb{R}^2$. A φ-type over B is a consistent region of $\mathbb{R}^2$ carved out by the disks.

Example of φ-Types

Let $\varphi(x_0, x_1; y_0, y_1, y_2) = \left[(x_0 - y_0)^2 + (x_1 - y_1)^2 \leq y_2^2\right]$ as before.



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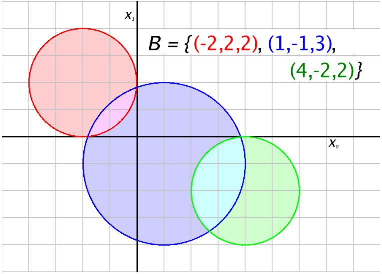

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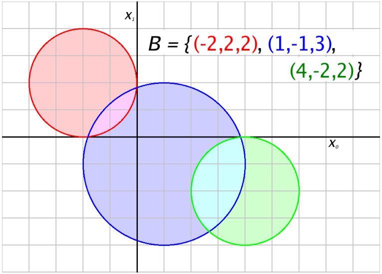

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Definition.

The φ-**Stone Space** over B, denoted $S_\varphi(B)$, is the set of all φ-types over B.

Stability and Dependence

Definition.

We say a partitioned formula $\varphi(\bar{x};\bar{y})$ is **stable** if there do not exist $\langle \bar{a}_i : i < \omega \rangle$ and $\langle \bar{b}_j : j < \omega \rangle$ such that, for all $i, j < \omega$

$$\models \varphi(\bar{a}_i;\bar{b}_j) \text{ if and only if } i < j.$$

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Definition.

We say a partitioned formula $\varphi(\overline{x};\overline{y})$ is **dependent** (or sometimes **NIP**) if there do not exist $\langle \overline{a}_s : s \in \mathcal{P}(\omega)\rangle$ and $\langle \overline{b}_j : j < \omega\rangle$ such that, for all $s \in \mathcal{P}(\omega), j < \omega$

$$\models \varphi(\overline{a}_s;\overline{b}_j) \text{ if and only if } j \in s.$$

Stable vs. Dependent

Let $\varphi(x_0, x_1; y_0, y_1, y_2) = \left[(x_0 - y_0)^2 + (x_1 - y_1)^2 \leq y_2^2\right]$ as before. Then φ is not stable. For example, take $\overline{b}_j = (0, 0, j + 1)$ for $j < \omega$ and $\overline{a}_i = (0, i + 3/2)$ for $i < \omega$.

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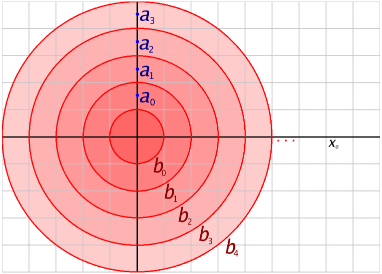

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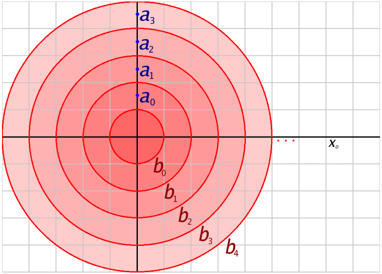

However, φ is dependent. In fact, all formulas in $\mathrm{Th}(\mathbb{R}; +, \cdot, <, 0, 1)$ are dependent.

Examples of Stable and Dependent Theories

Examples.

The following theories are stable:

1. The theory of a vector space over a field.
2. $\mathrm{Th}(\mathbb{C}; +, \cdot, 0, 1)$.
3. The theory of differentially closed fields.

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Examples.

The following theories are unstable and dependent:

1. $\mathrm{Th}(\mathbb{R}; +, \cdot, <, 0, 1)$.
2. $\mathrm{Th}(\mathbb{Q}; <)$.
3. $\mathrm{Th}(\mathbb{Z}; <, +)$.

Definability of Types

Definition.

Fix a formula $\varphi(\overline{x};\overline{y})$, a φ-type p, and a formula $\psi(\overline{y})$ defined over $\mathrm{dom}(p)$. We say that ψ **defines** p if, for all $\overline{b} \in \mathrm{dom}(p)$, we have that

$$\varphi(\overline{x};\overline{b}) \in p(\overline{x}) \text{ if and only if } \models \psi(\overline{b}).$$

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$$\varphi(\overline{x};\overline{b}) \in p(\overline{x}) \text{ if and only if } \models \psi(\overline{b}).$$

Definition.

We say that φ has **uniform definability of types** if there exists a finite number of $\emptyset$-definable formulas $\psi_k(\overline{y};\overline{z}_1,...,\overline{z}_n)$ for $k = 1,...,K$ such that, for all non-empty sets $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ and φ-types $p \in S_\varphi(B)$, there exists $\overline{c}_1,...,\overline{c}_n \in B$ and $k_0 \in \{1,...,K\}$ such that

$$\psi_{k_0}(\overline{y};\overline{c}_1,...,\overline{c}_n) \text{ defines } p.$$

Definability of Types for Stable Formulas

Theorem (Shelah).

The following are equivalent for a partitioned formula $\varphi(\overline{x}; \overline{y})$:

1. φ is stable.
2. For all φ-types p, p is definable over $\mathrm{dom}(p)$.
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Goal: Generalize this to dependent formulas.

Stable Theories Have...

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1. (local type space bounds) For any $\varphi(\overline{x};\overline{y})$ and any infinite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq |B|$.
2. (λ-stability) There exists λ such that, for all B with $|B| \leq \lambda$, $|S_1(B)| \leq \lambda$.
3. (stable embeddedness) For any B, every externally definable subset $C \subseteq B$ is definable with parameters in B.
4. (non-forking extensions) For any type p and any $B \supseteq \mathrm{dom}(p)$, there exists a non-forking extension $q \in S(B)$ of p.
5. etc.

Counting Type Spaces

Corollary.

If $\varphi(\overline{x};\overline{y})$ is stable, then there exists $K, n < \omega$ such that, for any non-empty set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

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Theorem (Sauer's Lemma).

If $\varphi(\overline{x};\overline{y})$ is dependent, then there exists $K, n < \omega$ such that, for any non-empty *FINITE* set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

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Corollary.

If $\varphi(\overline{x};\overline{y})$ is stable, then there exists $K, n < \omega$ such that, for any non-empty set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

Theorem (Sauer's Lemma).

If $\varphi(\overline{x};\overline{y})$ is dependent, then there exists $K, n < \omega$ such that, for any non-empty *FINITE* set $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, $|S_\varphi(B)| \leq K \cdot |B|^n$.

Definition.

We say that a dependent formula φ has **VC-density** ℓ if ℓ is the infimum of all $n \in \mathbb{R}_+$ such that the condition in the above theorem holds.

Definability of Types over Finite Sets

For any formula $\varphi(\overline{x};\overline{y})$ and any φ-type p with a finite domain, p is definable.

Let $B_1 = \{\overline{b} \in \mathrm{dom}(p) : \varphi(\overline{x};\overline{b}) \in p(\overline{x})\}$. Then, since $\mathrm{dom}(p)$ is finite, B_1 is finite, so the following is a formula:

$$\psi(\overline{y}) = \left[\bigvee_{\overline{b}\in B_1} \overline{y} = \overline{b}\right].$$

Notice that ψ defines p.

Uniform Definability of Types over Finite Sets

One generalization of definability of types to some dependent theories is UDTFS.

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Definition.

We say that φ has **uniform definability of types over finite sets** (**UDTFS**) if there exists a finite number of $\emptyset$-definable formulas $\psi_k(\overline{y}; \overline{z}_1, ..., \overline{z}_n)$ for $k = 1, ..., K$ such that, for all non-empty *FINITE* sets $B \subseteq \mathfrak{C}^{\mathrm{lg}(\overline{y})}$ and φ-types $p \in S_\varphi(B)$, there exists $\overline{c}_1, ..., \overline{c}_n \in B$ and $k_0 \in \{1, ..., K\}$ such that

$$\psi_{k_0}(\overline{y}; \overline{c}_1, ..., \overline{c}_n) \text{ defines } p.$$

UDTFS Implies VC-Density Bound

Fact.

If φ has UDTFS, witnessed by $\psi_k(\overline{y}; \overline{z}_1, ..., \overline{z}_n)$ for $k = 1, ..., K$, then φ has VC-density $\leq n$.

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Fact.

If φ has UDTFS, witnessed by $\psi_k(\overline{y}; \overline{z}_1, ..., \overline{z}_n)$ for $k = 1, ..., K$, then φ has VC-density $\leq n$.

Why?

For any fixed finite $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$, φ-types over B are determined by:

1. elements $\overline{c}_1, ..., \overline{c}_n \in B$, and
2. some $k_0 \in \{1, ..., K\}$.

So there are, at most, $K \cdot |B^n| = K \cdot |B|^n$ such φ-types.

Facts about UDTFS

Facts.

1. If $\varphi(\overline{x};\overline{y})$ is stable, then φ has UDTFS.
2. If $\varphi(\overline{x};\overline{y})$ has UDTFS, then φ is dependent.

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Theorem (Johnson, Laskowski).

If T is o-minimal, then T has UDTFS.

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Theorem (Johnson, Laskowski).

If T is o-minimal, then T has UDTFS.

$$\begin{array}{ccccc} & & \text{stable} & & \\ & & \Downarrow & & \\ \text{o-minimal} & \Rightarrow & \text{UDTFS} & \Rightarrow & \text{dependent} \end{array}$$

dp-Minimal Theories

Definition.

A theory T is **dp-minimal** if there do not exist $\varphi(x;\overline{y})$, $\psi(x;\overline{z})$, $\langle \overline{b}_i : i < \omega \rangle$, and $\langle \overline{c}_j : j < \omega \rangle$ such that, for all $i_0, j_0 < \omega$, the type

$$\{\neg\varphi(x;\overline{b}_{i_0}), \neg\psi(x;\overline{c}_{j_0})\} \cup \{\varphi(x;\overline{b}_i) : i \neq i_0\} \cup \{\psi(x;\overline{c}_j) : j \neq j_0\}.$$

is consistent.

Examples of dp-Minimal Theories

Examples.

The following theories are dp-minimal:

1. Any o-minimal theory or weakly o-minimal theory,
2. $\mathrm{Th}(\mathbb{Z}; <, +)$,
3. $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |, 0, 1)$ (where $x|y$ iff. $v_p(x) \leq v_p(y)$),
4. Algebraically closed valued fields.
5. In general, any VC-minimal theory is dp-minimal.
6. Any theory with VC-density ≤ 1 is dp-minimal.

dp-Minimal Theories have UDTFS

Theorem (G.).

If T is dp-minimal, then T has UDTFS.

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Theorem (G.).

If $\varphi(\overline{x};\overline{y})$ and $N < \omega$ are such that, for all $B \subseteq \mathfrak{C}^{\lg(\overline{y})}$ with $|B| = N$, $|S_\varphi(B)| \leq N(N+1)/2$, then φ has UDTFS (in particular if φ has VC-density < 2, then φ has UDTFS).

Valued Fields and UDTFS

Theorem (G.).

If (K, k, Γ) is a Henselian valued field that has elimination of field quantifiers in the Denef-Pas language, $\mathrm{Th}(k)$ has UDTFS, and $\mathrm{Th}(\Gamma)$ has UDTFS, then the full theory in the Denef-Pas language has UDTFS.

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Examples.

The theories of the following structures in the Denef-Pas language have UDTFS:

1. $\mathbb{Q}_p$,
2. $\mathbb{R}((t))$,
3. $\mathbb{C}((t))$,
4. $\mathbb{C}((t^{\mathbb{Q}}))$.

The UDTFS Conjecture

$$\begin{array}{ccccccc} \text{o-minimal} & \Rightarrow & \text{VC-density} \leq 1 & & \text{stable} & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ \text{VC-minimal} & \Rightarrow & \text{dp-minimal} & \Rightarrow & \text{UDTFS} & \Rightarrow & \text{dependent} \end{array}$$

The UDTFS Conjecture

o-minimal	$\Rightarrow$	VC-density ≤ 1		stable		
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VC-minimal	$\Rightarrow$	dp-minimal	$\Rightarrow$	UDTFS	$\Rightarrow$	dependent

Open Question (Laskowski).

If φ is dependent, then does φ have UDTFS?

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More Open Questions.

1. Is UDTFS closed under reducts?

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More Open Questions.

1. Is UDTFS closed under reducts?
2. If $\varphi(\overline{x}; \overline{y})$ has UDTFS, then does $\varphi^{\text{opp}}(\overline{y}; \overline{x})$ have UDTFS?

VC-Minimal Theories

Definition.

A theory T is VC-minimal if there exists $\Psi = \{\psi_i(x; \overline{y}_i) : i \in I\}$ a collection of formulas so that:

1. For all $i, j \in I$, $\overline{b} \in \mathfrak{C}^{\lg(\overline{y}_i)}$, and $\overline{c} \in \mathfrak{C}^{\lg(\overline{y}_j)}$, one of the four partial types $\{\pm\psi_i(x; \overline{b}), \pm\psi_j(x; \overline{c})\}$ is inconsistent; and
2. For any parameter-definable formula $\theta(x)$, θ is T-equivalent to a boolean combination of instances of $\psi_i(x; \overline{b})$ for various $i \in I$ and $\overline{b} \in \mathfrak{C}^{\lg(\overline{y}_i)}$.

Examples of VC-Minimal Theories

Examples.

1. If T is o-minimal, then T is VC-minimal.
2. If T is strongly minimal, then T is VC-minimal.
3. In particular, $\mathrm{Th}(\mathbb{C}; +, \cdot, 0, 1)$ is VC-minimal.
4. The theory of algebraically closed valued fields is VC-minimal.

The Kueker Conjecture

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If T is a theory in a countable language such that every uncountable model of T is $\aleph_0$-saturated, then T is $\aleph_0$-categorical or $\aleph_1$-categorical.

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If T is a theory in a countable language such that every uncountable model of T is $\aleph_0$-saturated, then T is $\aleph_0$-categorical or $\aleph_1$-categorical.

Theorem (G.).

If T is VC-minimal, then T satisfies the Kueker Conjecture.

Proof Sketch.

- First we show that a VC-minimal theory T is either stable, or interprets an infinite linear order.
- By the work of Hrushovski, if T is stable, then T satisfies the Kueker Conjecture.
- By that same paper of Hrushovski, if T interprets a linear order, then T satisfies the Kueker Conjecture.

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